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ON A NON-LOCAL PARABOLIC PROBLEM

Abstract. The aim of this paper is to investigate the existence of solutions of a nonlocal parabolic problem. The method of upper and lower solutions and the classical maximum principle are used to obtain our results.

1. Introduction

In [4], Deng studied the following nonlocal boundary-value problem

$$(P_1) \qquad \begin{cases} u_t = \Delta u + f(x,u), & x \in \Omega, t > 0; \\ u(t,x) = \int_\Omega \Phi(x,y)u(y,t)dy, & x \in \partial\Omega, t > 0; \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $\partial\Omega \in C^2$, $f(x,u)$ is C^0 in x and C^1 in u with $f(x,0) = 0$, and $\Phi(x,y)$ is a continuous function defined for $x \in \partial\Omega$, $y \in \overline{\Omega}$.

He first established the comparison principle for (P_1). Then he showed the local existence of the solution and he discussed its long time behavior, assuming

$$(1.1) \qquad \int_\Omega \Phi^2(x,y)dy \le 1/|\Omega|.$$

The results obtained in [4] generalize the result of [6].

In [10], Yin considered a problem similar to (P_1), namely,

$$(P_2) \qquad \begin{cases} u_t - Lu = f(t,x,u), & \text{in } D_T, \\ u(t,x) = \int_\Omega \Phi(x,y)u(t,y)dy, & (t,x) \in \Gamma_T, \\ u(0,x) = u_0(x), & \text{for all } x \in \Omega; \end{cases}$$

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where $D_T = (0,T) \times \Omega$, $\Gamma_T = (0,T) \times \partial\Omega$, $\overline{D}_T = (0,T) \times \overline{\Omega}$, $T > 0$, $\Phi \in C[\partial\Omega \times \overline{\Omega}, \mathbb{R}]$,

$$L := \sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(t,x)\frac{\partial}{\partial x_i},$$

$a_{ij}, b_i \in C^\alpha (0 < \alpha < 1)$ and L is a uniformly elliptic operator in D_T.

He proved the uniqueness and the global existence of a solution of (P_2) under the conditions

(1.2) $\Phi(t,x) \geq 0$ and $\int_\Omega \Phi(x,y)dy \leq \rho < 1.$

In general the condition (1.1) is stronger than (1.2).

This result is based on the comparison principle with nonlocal conditions combined with the existence and uniqueness of a solution for the problem

(P_3) $\begin{cases} u_t - Lu = g(t,x), & \text{in } D_T, \\ u(t,x) = h(t,x), & \text{in } \Gamma_T, \\ u(0,x) = u_0(x), & \text{for } x \in \Omega. \end{cases}$

It has been pointed out in [10] without proof that problem (P_3) has a unique solution under condition (1.2). He assumed only that h is continuous in $\overline{D}_T$.

In this paper, we consider the problem

(P) $\begin{cases} u_t - \Delta u = f(t,x,u), & (t,x) \in D_T, \\ \dfrac{\partial}{\partial \eta}u(t,x) + \sigma u(t,x) = \int_\Omega \Phi(x,y)u(t,y)d\mu(y), & (t,x) \in \Gamma_T, \\ u(0,x) = u_0(x), & \text{for all } x \in \Omega, \end{cases}$

where $f : [0,+\infty[\times \Omega \times \mathbb{R} \to \mathbb{R}$ is Hölder continuous, Φ is Hölder continuous and μ denotes a Radon measure, $\sigma \in \mathbb{R}^+$ and $\frac{\partial}{\partial \eta}$ is the normal derivative.

We establish the global existence of solutions under the classical maximum principle and the lower and upper solutions for a linear parabolic problem. The study of the problem with nonlocal conditions is of significance. Such problems have applications in physics and other areas of applied mathematics. For example, nonlocal conditions can be applied in the theory of elasticity with better effect than the initial or Darboux conditions. The nonlocal conditions were introduced in [3] for studying of linear parabolic problems. Nonlinear differential problems of parabolic type with nonlocal conditions together with their physical interpretations were considered by Byszewski in [2]. For other results on parabolic differential equations, we refer to [5], [8] and [11]; and for parabolic systems with time delays, [9] and [11].

2. Maximum principle for linear problems and lower and upper solutions

Let Ω be a smooth open bounded domain in $\mathbb{R}^n$. We denote by $(C^m(D_T), \|.\|_m)$ the Banach space of m-the continuously differentiable functions in D_T, and by $C^{2,1}(D_T)$ the class of functions $u \in C(D_T)$ such that the derivative of the form $D_x^2 \in C^0(D_T)$. For $k \in \mathbb{N}$, $(C^{k+\alpha, \frac{k+\alpha}{2}}(\overline{D}_T))$ denotes the Hölder space of exponent $\alpha \in (0,1)$, with the norm $\|u\|_{k+\alpha}$.

In this paper, we will assume that Ω is sufficiently smooth, the function $f(t,x,u)$ is continuous in $\overline{D}_T \times \mathbb{R}$, and locally Lipschitz with respect to u. We suppose f satisfy:

there exists a function $\theta : \overline{D}_T \to \mathbb{R}_+$ which is bounded for all $(t,x) \in \overline{D}_T$ such that

$$(2.1) \qquad f(t,x,u) - f(t,x,v) \geq -\theta(t,x)(u-v), \text{ if } u \geq v, \; u,v \in \mathbb{R}.$$

Also, we assume that the density Φ is in $C^{1+\alpha, \frac{1+\alpha}{2}}(\overline{D}_T)$ satisfying the compatibility condition:

$$(2.2) \qquad \frac{\partial}{\partial \eta} u_0(x) + \sigma u_0(x) = \int_\Omega \Phi(0,x) u_0(y) d\mu(y), \; x \in \partial\Omega.$$

We first define the upper and lower solutions.

DEFINITION. A function $U^* \in C^{2,1}(D_T) \cap C^0(\overline{D}_T)$ is called an upper solution of problem (P) if

$$
\begin{cases}
U_t^* - \Delta U^* \geq f(t,x,U^*) & \text{in } D_T, \\
\dfrac{\partial}{\partial \eta} U^*(t,x) + \sigma U^*(t,x) \geq \int_\Omega \Phi(x,y) U^*(y,t) d\mu(y) & \text{on } \Gamma_T, \\
U^*(0,x) \geq u_0(x) & \text{for } x \in \Omega.
\end{cases}
$$

A lower solution U_*, is defined similarly by reversing the above inequalities.

Throughout this paper, we assume that $\Phi(t,x) \geq 0$ and $\int_\Omega \Phi(x,y) dy \neq 0$ for $x \in \partial\Omega$. First, we give the following fundamental maximum principle.

LEMMA 2.1. (Lemma 2.1, p. 54, [7]). *Let* $u \in C^{2,1}(D_T) \cap C^0(\overline{D}_T)$ *be such that*

$$
\begin{cases}
u_t - \Delta u + Mu \geq 0 & \text{in } D_T, \\
\dfrac{\partial}{\partial t} w - \Delta w + Mw \geq 0 & \text{on } \Gamma_T, \\
u(0,x) \geq 0 & \text{for } x \in \Omega,
\end{cases}
$$

where $\sigma \geq 0$ *and* $M \equiv M(t,x)$ *is a bounded function in* D_T. *Then* $u(t,x) \geq 0$ *in* $\overline{D}_T$. *Moreover* $u(t,x) > 0$ *in* D_T *unless it is identically zero.*

Now, we consider the following linear parabolic problem:

$$(L) \quad \begin{cases} u_t - \Delta u + Mu = g(t,x) & \text{in } D_T, \\ \dfrac{\partial}{\partial \eta} u(t,x) + \sigma u(t,x) = h(t,x) & \text{on } \Gamma_T, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $\sigma \geq 0$ and M is a positive constant.

THEOREM 2.2. (Theorem 1.3-1, p. 31, [11]). *Assume* $g \in C^{\alpha, \frac{\alpha}{2}}(\overline{D}_T)$, $u_0 \in C^{2+\alpha}(\overline{\Omega})$ *and* $h \in C^{1+\alpha, \frac{1+\alpha}{2}}(\overline{D}_T)$ *verifying condition* (2.2). *Then the problem* (L) *has a unique solution* u *in* $C^{2+\alpha, \frac{2+\alpha}{2}}(\overline{D}_T)$. *In addition, there exists a positive constant independent of* u *such that* $\|u\|_{2+\alpha} \leq C$.

3. Global existence: lower and upper solutions

In [10], Yin established the existence of solution of (P_2) under the condition (1.2) by using comparison principle and monotone iterative method. In this section, our purpose is to obtain the global existence of (P). We employ the method of lower and upper solutions, the maximum principle and the integral representation of the solution. The uniqueness of the solution is also proved. Then under some additional conditions, we construct the lower and upper solutions of the problem (P) and as a result, the global existence is obtained.

THEOREM 3.1. *Assume that condition* (2.1) *and* (2.2) *hold. Let* $U_*, U^* \in C^{2,1}(D_T) \cap C(\overline{D}_T)$, *be respectively the lower and upper solutions of* (P) *such that* $U_* \leq U^*$. *Then* (P) *has at least a solution* u *such that* $U_* \leq u \leq U^*$.

Proof of Theorem 3.1. From hypothesis (2.1), there exists a positive constant $M > 0$ such that the function $s \to f(t,x,s) + Ms$ is increasing in $\mathbb{R}$ for all $(t,x) \in \overline{D}_T$.

Let $(u_n)_{n \geq 0}$ be a sequence defined by:

$$(3.1) \quad \begin{cases} u_0 = U^*, \\ \dfrac{\partial}{\partial t} u_{n+1} - \Delta u_{n+1} + Mu_{n+1} = g_n(t,x) & \text{in } D_T, \\ \dfrac{\partial}{\partial \eta} u_{n+1}(t,x) + \sigma u_{n+1}(t,x) = h_n(t,x) & \text{on } \Gamma_T, \\ u_{n+1}(0,x) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $g_n(t,x) = f(t,x,u_n) + Mu_n$ and $h_n(x,y) = \int_\Omega \Phi(x,y) u_n(t,y) d\mu(y)$.

The sequence (u_n) is well defined in $C^{2,1}(D_T) \cap C^0(\overline{D}_T)$. In fact, the problem (3.1) is a linear problem with respect u_{n+1}. Hence using Theorem 2.2 in section 2, we will obtain for each n, a unique solution:

$$u_{n+1} \in C^{2+\alpha, \frac{2+\alpha}{2}}(\overline{D}_T) \subset C^{2,1}(D_T) \cap C^0(\overline{D}_T).$$

First, we will show that $U_* \leq u_n \leq U^*$, for all n.

It suffices to show that $u_n \leq U^*$, for all n. The other inequality will be obtained similarly.

From problem (3.1) and the definition of U^*, we will show that $u_n - U^*$ verify

$$\begin{cases} \dfrac{\partial}{\partial t}(u_{n+1} - U^*) - \Delta(u_{n+1} - U^*) + M(u_{n+1} - U^*) \\ \qquad\qquad \leq (f(t,x,u_n) + Mu_n) - (f(t,x,U^*) + MU^*) \quad \text{in } D_T, \\ \dfrac{\partial}{\partial \eta}(u_{n+1} - U^*)(t,x) + \sigma(u_{n+1} - U^*)(t,x) \\ \qquad\qquad \leq \displaystyle\int_\Omega \Phi(x,y)(u_n - U^*)(t,y)d\mu(y) \qquad\qquad \text{on } \Gamma_T, \\ (u_{n+1} - U^*)(0,x) \leq 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{in } \Omega. \end{cases}$$

If we suppose $u_n \leq U^*$, using (2.1), we will see that $w = u_{n+1} - U^*$ verify

$$\begin{cases} \dfrac{\partial}{\partial t}w - \Delta w + Mw \leq 0 & \text{in } D_T, \\ \dfrac{\partial}{\partial \eta}w(t,x) + \sigma w(t,x) \leq 0 & \text{on } \Gamma_T, \\ w(x,0) \leq 0 & \text{in } \Omega. \end{cases}$$

Now, by Lemma 2.1 in section 2, we deduce that $w \leq 0$ in $\overline{D}_T$ and the proof can be easily completed by induction arguments.

Now, the solution u_{n+1} of problem (3.1) has the following integral representation (Lemma 4.2, p. 63, [7]):

$$u_{n+1}(t,x) = \int_\Omega \Gamma(t,x;0,\xi)u_0(\xi)d\xi + \int_0^t d\tau \int_\Omega \Gamma(x,t;\tau,\xi)g_n(\tau,\xi)d\xi$$

$$+ \int_0^t d\tau \int_{\partial\Omega} \Gamma(t,x;\tau,\xi)\Psi_n(\tau,\xi)d\xi,$$

where Γ is the fundamental solution of the parabolic operator

$$L_M = (u_t - \Delta + M), \text{ in } D_T,$$

and Ψ_n is the density of single larger potential. The density Ψ_n verify the integral equation:

$$\Psi_n(t,x) = 2\int_0^t d\tau \int_{\partial\Omega} \left[\frac{\partial\Gamma}{\partial\eta}(t,x;\tau,\xi) + \sigma\Gamma(t,x;\tau,\xi)\right] \Psi_n(\tau,\xi)d\xi - 2h_n(t,x).$$

We note $H(t,x;\xi,\tau) = (\frac{\partial\Gamma}{\partial\eta} + \sigma\Gamma)(t,x;\tau,\xi)$.

Hence, we conclude that u_{n+1} and Ψ_{n+1} are respectively bounded in $C(\overline{D}_T)$ and $C(\partial\Omega)$.

By Ascoli-Arzela Theorem, we will see that (u_n) and (Ψ_n) have respectively subsequences (u_{nk}) and (Ψ_{nk}) such that

$$u_{nk} \to u \text{ in } C(\overline{D}_T) \text{ and } \Psi_{nk} \to \Psi \text{ in } C(\partial\Omega).$$

After passing to the limit in the integral representation, we obtain that

$$u(t,x) = \int_\Omega \Gamma(t,x;0,\xi)u_0(x)(\xi)d\xi + \int_0^t d\tau \int_\Omega \Gamma(t,x;\xi,\tau)g(\tau,\xi)d\xi$$
$$+ \int_0^t d\tau \int_{\partial\Omega} \Gamma(t,x;\tau,\xi)\Psi(\tau,\xi)d\xi,$$

and

$$\Psi(t,x) = 2\int_0^t d\tau \int_{\partial\Omega} H(t,x;\tau,\xi)\Psi(\tau,\xi)d\xi - 2h(t,x),$$

where $g(\tau,\xi) = f(\xi,\tau) + Mu(\tau,\xi)$, and $h(x,y) = \int_\Omega \Phi(x,y)u(t,y)d\mu(y)$ which shows that u is a solution of (P).

Before proving the uniqueness of the solution, under some additional conditions, it is necessary to construct a lower and upper solutions of (P) by iterative method. Staring from a suitable initial iteration u_0 it is possible to construct a sequence $\{u_n\}$ successively from the modified nonlocal problem (3.1). Denote the sequences with $u_0 = U_*$ and $u_0 = U^*$ by $\{\underline{u}_n\}$ and $\{\overline{u}_n\}$ respectively, and refer to them as upper and lower sequences. We show that under some conditions each of the two sequences converges monotonically to a unique solution of (P). To achieve this goal we prove the following theorem:

THEOREM 3.2. *Let the assumptions of Theorem 3.1 hold and the condition (1.2) be satisfied. Then the two sequences $\{\underline{u}_n\}$ and $\{\overline{u}_n\}$ are well defined in $C^{2,1}(D_T) \cap C^0(\overline{D}_T)$ for each n and converges monotonically to $\underline{u}$ and $\overline{u}$, respectively. Moreover, $\underline{u}$ and $\overline{u}$ are minimal and maximal solutions of (P).*

Proof of Theorem 3.2. Clearly, Theorem 3.1 guarantees that there exists a unique solution $u \in C^{2,1}(D_T) \cap C^0(\overline{D}_T)$ of (3.1). We claim that

$$U_*(t, x) \leq u(t, x) \leq U^*(t, x), \text{ in } D_T,$$

whenever $u(x, t)$ is a solution of (3.1).

Let $w = \overline{u}_0 - \overline{u}_1 = U^* - \overline{u}_1$. By (3.1),

$$
\begin{cases}
L_M w = w_t - \Delta w + Mw = \dfrac{\partial}{\partial t}(U^* - \overline{u}_1) - \Delta(U^* - \overline{u}_1) + M(U^* - \overline{u}_1) \\
\qquad\qquad = (f(t, x, U^*) + MU^*)) - (f(t, x, \overline{u}_1) + M\overline{u}_1)) \geq 0 \quad \text{in } D_T, \\
\dfrac{\partial}{\partial \eta} w(t, x) + \sigma w(t, x) = \displaystyle\int_\Omega \Phi(x, y)(U^* - \overline{u}_1)(t, y)d\mu(y) \geq 0 \qquad\qquad \text{on } \Gamma_T, \\
w(0, x) = U^*(0, x) - u_0(x) \geq 0 \qquad\qquad\qquad\qquad\qquad \text{in } \Omega.
\end{cases}
$$

In view of Lemma 2.1, $w(t, x) \geq 0$ for all $(t, x) \in D_T$, which shows that $\overline{u}_1 \leq \overline{u}_0$.

A similar argument, using the property of a lower solution, gives $\underline{u}_1 \geq \underline{u}_0$.

Let $w^{(1)} = \overline{u}_1 - \underline{u}_1$, it follows again from Lemma 2.1, that $w^{(1)} \geq 0$. The above conclusions show that

$$\underline{u}_0 \leq \underline{u}_1 \leq \overline{u}_1 \leq \overline{u}_0.$$

The above conclusion of the Lemma 2.1 follows by the principle of induction. We conclude that $\{\underline{u}_n\}$ and $\{\overline{u}_n\}$ are monotonic and uniformly bounded on $\overline{D}_T$ such that

$$U_* \leq \underline{u}_{n+1} \leq \underline{u}_n \leq \overline{u}_n \leq \overline{u}_{n+1} \leq U^*, \text{ in } \overline{D}_T.$$

By standard argument, we claim that there exist $\underline{u}$ and $\overline{u}$ such that $\underline{u}, \overline{u} \in C^{2,1}(D_T) \cap C^0(\overline{D}_T)$ and

$$\lim_{n \to +\infty} \overline{u}_n(t, x) = \overline{u}(t, x) \text{ and } \lim_{n \to +\infty} \underline{u}_n(t, x) = \underline{u}(t, x).$$

Furthermore $\underline{u}$, and $\overline{u}$ satisfy (P) with $\underline{u} = \overline{u} = u$.

To complete the proof, we need to show that $\overline{u}$ *and* $\underline{u}$ are respectively maximal and minimal solutions of (P), these can be easily proved by induction.

THEOREM 3.3. *Assume that* (1.2) *and* (2.1) *hold, and further*

$$\overline{\lim}_{u \to -\infty} f(t, x, u) > 0 \text{ and } \underline{\lim}_{u \to +\infty} f(t, x, u) < 0.$$

Then there exists a unique solution of (P) *for any* $T > 0$.

Proof of Theorem 3.3. To prove the existence of the solution, we only need to find the lower and upper solutions of (P). Let $a < 0 < b$ be constant

such that

$$f(t,x,a) \geq 0, f(t,x,b) \leq 0 \text{ in } D_T,$$

and

$$a \leq u_0(x) \leq b \text{ on } \Omega.$$

Then a and b are lower and upper solution of (P)respectively.

To prove the uniqueness, it is enough to show $\underline{u} = \overline{u}$ on $\overline{D_T}$.

Set $w = (\underline{u} - \overline{u})^2$, since $\underline{u} \leq \overline{u}$, $f(t,x,u)$ satisfies (2.1), we get

$$(3.2) \qquad w_t - \Delta w = 2(\underline{u} - \overline{u})(\underline{u} - \overline{u})_t - \Delta(\underline{u} - \overline{u})$$

$$= 2(\underline{u} - \overline{u})(\underline{u}_t - \overline{u}_t)(\underline{u}_t - \Delta \underline{u}_t - (\overline{u}_t - \Delta \overline{u}_t))$$

$$- 2\nabla(\underline{u} - \overline{u})(\nabla \underline{u} - \nabla \overline{u})$$

$$\leq 2\theta(t,x)w \leq 2Nw,$$

such that $\exists N \in \mathbb{R}_+; \theta(t,x) \leq N$, for all $(t,x) \in D_T$

$$(3.3) \qquad\qquad w(0,x) = 0, \text{ for all } x \in \Omega,$$

and on Γ_T,

$$(3.4) \qquad \frac{\partial w(t,x)}{\eta} + \sigma w(t,x) = \frac{\partial(\underline{u} - \overline{u})^2}{\eta} + \sigma(\underline{u} - \overline{u})^2$$

$$= 2(\underline{u} - \overline{u})\frac{\partial(\underline{u} - \overline{u})}{\eta} + \sigma(\underline{u} - \overline{u})^2$$

$$= 2(\underline{u} - \overline{u})[\frac{\partial(\underline{u} - \overline{u})}{\eta} + \frac{\sigma}{2}(\underline{u} - \overline{u})]$$

$$\leq 2(\underline{u} - \overline{u}) \int_\Omega \Phi(x,y)(\underline{u}(t,y) - \overline{u}(t,y))d\mu(y)$$

(3.2) and (3.3) imply $w(t,x) \leq 0$ in D_T and from (3.4) $w(t,x) = 0$ on Γ_T, which implies $\underline{u} = \overline{u}$ on $\overline{D_T}$, the proof is thus complete.

As a consequence of Theorem 3.3, we can obtain the invariance properties of the solution of (P).

THEOREM 3.4. *Assume that (2.1) and (2.2) hold. Let there exist positive constants a, b with $a \leq 0 \leq b$ such that*

$$f(t,x,a) \geq 0, \qquad f(t,x,b) \leq 0 \text{ in } D_T$$

and

$$\sigma a \leq \int_\Omega \Phi(x,y)dy \leq \sigma b \text{ on } \Gamma_T.$$

Then for any u_0 with $a \leq u_0(x) \leq b$, problem (P) has a unique solution $u(t,x)$ such that $a \leq u(t,x) \leq b$ in $\overline{D_T}$.

Proof of Theorem 3.4. It is easy to verify that a, b are respectively the lower and upper solutions of (P). The conclusion follows from Theorem 3.3.

4. Long time behavior of solution

In [6], Friedman showed that if $\int_\Omega |\Phi(x,y)| dy < 1$, for any $x \in \Omega$ and $f(x,u) = c(x)u$, $(c \leq 0)$, then the solution of (P_1) decays. Moreover [10] proved that, under condition (1.1) with $f(x,u)$ is decreasing in u and for $C > 0$, $\alpha > 0$,

$$(4.1) \qquad U(t) \equiv max_{x \in \overline{\Omega}} |u(t,x)| \leq Ce^{-\alpha t}, \text{ for } t > 0.$$

Under the assumption (1.2) and $uf(t,x,u) \leq 0$ for all $(t,x) \in \overline{D}_T$, [10] showed that (4.1) is also true for the solution of (P_2). In this section, we also show that (4.1) is true for any solution $u(t,x)$ of (P), under the same condition (1.2) we employ the same method used in the proof of [6, Theorem 2.3].

Assume that

$$(4.2) \qquad uf(t,x,u) \leq 0, \text{ for all } (t,x) \in \overline{D}_T.$$

THEOREM 4.1. *Assume that (4.2) holds. If condition (1.2) is satisfied, then for any solution u of (P), (4.1) is true.*

Proof of Theorem 4.1. Under the same method used in the proof of [6, Theorem 2.3], we only need to prove

$$(4.3) \qquad \overline{\lim}_{t \to +\infty} |u(t,x)| \leq \rho M, \text{ for all } x \in \Omega,$$

where $M > 0$ is such that

$$(4.4) \qquad |u_0(x)| \leq M, \text{ for all } x \in \Omega.$$

Theorem 3.4, implies that $|u(t,x)| \leq M$ in D_T and we have

$$(4.5) \qquad |u(t,x)| \leq \left| \int_\Omega |\Phi(x,y)u(t,y)dy \right| \leq \rho M.$$

Consider the function $\varphi(x) = e^{\lambda R} - e^{\lambda x_1}$, where R is any positive number satisfying $R > 2x_1$, for all $x \in \overline{\Omega}$ and λ is a positive constant to be determined later. The function φ satisfies

$$\varphi_t - \Delta\varphi = \lambda^2 e^{\lambda x_1} > e^{\lambda x_1} \text{ in } D_T,$$

we can take λ sufficiently large so that

$$(4.6) \qquad \varphi_t - \Delta\varphi > \alpha,$$

where $\alpha = inf_{x \in \overline{\Omega}} e^{\lambda x_1}$. Moreover, set

$$\alpha_0 = inf_{x \in \overline{\Omega}} \varphi(x), \quad \text{and} \quad \alpha_1 = sup_{x \in \overline{\Omega}} \varphi(x)$$

and consider the function

$$(4.7) \qquad \Psi(t,x) = M\left(\frac{\varphi(x)}{\alpha_0}\right)e^{-\gamma t} + \rho M,$$

where γ is a positive constant to be determined later. By (4.6),

$$(4.8) \quad \Psi_t - \Delta\Psi - f(t,x,\Psi) > -\gamma M\left(\frac{\Psi(x)}{\alpha_0}\right)e^{-\gamma t} + M\left(\frac{\alpha}{\alpha_0}\right)e^{-\gamma t} - f(t,x,\Psi).$$

If we take $\gamma = \frac{\alpha}{\alpha_0}$, we obtain

$$\Psi_t - \Delta\Psi - f(t,x,\Psi) > 0, \quad \text{in } D_T,$$

$$\Psi(0,x) \geq \rho + M > M, \qquad \forall x \in \Omega,$$

$$\frac{\partial}{\eta}\Psi(t,x) + \Psi(t,x) > \rho M, \quad \text{on } \Gamma_T.$$

By standard results on the asymptotic behavior of the solution of parabolic equations [5, Chapter 6] it follows that

$$|u(t,x)| < \Psi(t,x) \leq \rho M + M\frac{\alpha_1}{\alpha_0}e^{-\gamma t}, \text{ on } \overline{D}_T$$

from which (4.3) follows, and the proof is therefore complete.

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