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ON Ψ-INSTABILITY OF NON-LINEAR MATRIX LYAPUNOV SYSTEMS

Abstract. We prove necessary and sufficient conditions for Ψ-instability of trivial solutions of linear matrix Lyapunov systems and also sufficient conditions for Ψ-instability of trivial solutions of non-linear matrix Lyapunov systems.

1. Introduction

The importance of Matrix Lyapunov systems, which arise in a number of areas of control engineering problems, dynamical systems, and feedback systems are well known. In this paper we focus our attention to the first order non-linear matrix Lyapunov systems of the form

$$X'(t) = A(t)X(t) + X(t)B(t) + F(t, X(t)), \tag{1.1}$$

where $A(t), B(t)$ are square matrices of order n, whose elements a_{ij}, b_{ij}, are real valued continuous functions of t on the interval $R_+ = [0, \infty)$, and $F(t, X(t))$ is a continuous square matrix of order n defined on $(R_+ \times \mathbb{R}^{n\times n})$,such that$F(t, O) = O$, where $\mathbb{R}^{n\times n}$ denote the space of all $n \times n$ real valued matrices. The continuity of A, B, and F ensures the existence of a solution of (1.1).

Akinyele [1] introduced the notion of Ψ-stability, and this concept was extended to solutions of ordinary differential equations by Constantin [3]. Later Morchalo [9] introduced the concepts of Ψ-(uniform) stability, Ψ-asmyptotic stability of trivial solutions of linear and non-linear systems of differential equations. The study of instability of solutions of system of differential equations was motivated by Coppel [4]. Further, the concepts of Ψ-(uniform) stability and Ψ-instability to non-linear Volterra integro-differential equatons were studied by Diamandescu [[5], [6]]. Recently, Murty and Suresh Ku-

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mar [12] extended the concepts of Ψ-boundedness and Ψ-stability to matrix Lyapunov systems.

The purpose of our paper is to provide conditions for Ψ-instability of trivial solutions of the Kronecker product system associated with the linear as well as non-linear matrix Lyapunov system (1.1).

This paper is well organized as follows. In section 2 we present some basic definitions, notations and properties relating to Ψ-stability and Kronecker products and obtain the general solution of corresponding non-linear Kronecker product system associated with (1.1). Further, we prove two lemmas which are useful for latter discussion. In section 3 we obtain necessary and sufficient conditions for Ψ-instability of trivial solutions of the corresponding linear Kronecker product system and the results are illustrated with suitable examples. In section 4 we study Ψ-instability of trivial solutions of non-linear Kronecker product system.

This paper extends some of the results (Theorem 1. and Theorem 2.) of Diamandescu [6] developed for linear equations to matrix Lyapunov systems.

2. Preliminaries

In this section we present some basic definitions and results which are useful for later discussion.

Let $\mathbb{R}^n$ be the Euclidean n-dimensional space. Elements in this space are column vectors, denoted by $u = (u_1, u_2, \dots u_n)^*$ (* denotes transpose) and their norm defined by

$$\|u\| = \max\{|u_1|, |u_2|, \dots |u_n|\}.$$

For a $n \times n$ real matrix, we define the norm

$$|A| = \sup_{\|x\| \leq 1} \|Ax\|.$$

Let $\Psi_k : R_+ \to (0, \infty)$, $k = 1, 2, \dots n, \dots n^2$, be continuous functions, and let

$$\Psi = \text{diag}[\Psi_1, \Psi_2, \dots, \Psi_{n^2}].$$

Then the matrix $\Psi(t)$ is an invertible square matrix of order n^2, for each $t \geq 0$.

Definition 2.1. [12] Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ then the Kronecker product of A and B written $A \otimes B$ is defined to be the partitioned matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & . & . & . & a_{1n}B \\ a_{21}B & a_{22}B & . & . & . & a_{2n}B \\ . & . & . & . & . & . \\ a_{m1}B & a_{m2}B & . & . & . & a_{mn}B \end{bmatrix}$$

is an $mp \times nq$ matrix and is in $\mathbb{R}^{mp\times nq}$.

DEFINITION 2.2. [12] Let $A = [a_{ij}] \in \mathbb{R}^{m \times n}$, we denote

$$\hat{A} = VecA = \begin{bmatrix} A_{.1} \\ A_{.2} \\ . \\ . \\ A_{.n} \end{bmatrix}, \text{ where } A_{.j} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ . \\ . \\ a_{mj} \end{bmatrix} (1 \leq j \leq n).$$

Regarding properties and rules for Kronecker product of matrices we refer to Murty and Suresh Kumar [12].

Now by applying the Vec operator to the non-linear matrix Lyapunov system (1.1) and using the above properties, we have

$$\hat{X}'(t) = H(t)\hat{X}(t) + G(t, \hat{X}(t)), \tag{2.1}$$

where $H(t) = (B^* \otimes I_n) + (I_n \otimes A)$ is a $n^2 \times n^2$ matrix and $G(t, \hat{X}(t)) = Vec(F(t, X(t)))$ is a column matrix of order n^2.
The corresponding linear system of (2.1) is

$$\hat{X}'(t) = H(t)\hat{X}(t). \tag{2.2}$$

DEFINITION 2.3. The trivial solution of (2.1) is said to be Ψ-stable on R_+ if for every $\varepsilon > 0$ and every t_0 in R_+, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that any solution $\hat{X}(t)$ of (2.1) which satisfies the inequality $\|\Psi(t_0)\hat{X}(t_0)\| < \delta$, also satisfies the inequality $\|\Psi(t)\hat{X}(t)\| < \varepsilon$ for all $t \geq t_0$. Otherwise, the system is said to be Ψ-unstable.

LEMMA 2.1. *Let $Y(t)$ and $Z(t)$ be the fundamental matrices for the systems*

$$X'(t) = A(t)X(t), \tag{2.3}$$

and

$$[X^*(t)]' = B^*(t)X^*(t) \tag{2.4}$$

respectively. Then the matrix $Z(t) \otimes Y(t)$ is a fundamental matrix of (2.2) *and every solution of* (2.2) *is of the form $\hat{X}(t) = (Z(t) \otimes Y(t))c$, where c is a n^2-column vector.*

Proof. For proof, we refer to Lemma 1 of [12]. ∎

THEOREM 2.1. *Let $Y(t)$ and $Z(t)$ be the fundamental matrices for the systems* (2.3) *and* (2.4)*, then any solution of* (2.1)*, satisfying the initial con-*

dition $\hat{X}(t_0) = \hat{X}_0$, *is given by*

$$\hat{X}(t) = (Z(t) \otimes Y(t))(Z^{-1}(t_0) \otimes Y^{-1}(t_0))\hat{X}_0 + \int_{t_0}^{t} (Z(t) \otimes Y(t))(Z^{-1}(s) \otimes Y^{-1}(s))G(s, \hat{X}(s))ds. \tag{2.5}$$

Proof. First we show that any solution of (2.1) is of the form $\hat{X}(t) = (Z(t) \otimes Y(t))c + \widetilde{X}(t)$, where $\widetilde{X}(t)$ is a particular solution of (2.1) and is given by

$$\widetilde{X}(t) = \int_{t_0}^{t} (Z(t) \otimes Y(t))(Z^{-1}(s) \otimes Y^{-1}(s))G(s, \hat{X}(s))ds.$$

Here we observe that, $\hat{X}(t_0) = (Z(t_0) \otimes Y(t_0))c = \hat{X}_0$, $c = (Z^{-1}(t_0) \otimes Y^{-1}(t_0))\hat{X}_0$. Let $u(t)$ be any other solution of (2.1), write $w(t) = u(t) - \widetilde{X}(t)$, then w satisfies (2.2), hence $w = (Z(t) \otimes Y(t))c$, $u(t) = (Z(t) \otimes Y(t))c + \widetilde{X}(t)$.

Next we consider the vector $\widetilde{X}(t) = (Z(t) \otimes Y(t))v(t)$, where $v(t)$ is an arbitrary vector to be determined, so as to satisfy equation (2.1). Consider

$$\begin{aligned}
\widetilde{X}'(t) &= (Z(t) \otimes Y(t))'v(t) + (Z(t) \otimes Y(t))v'(t) \\
&\Rightarrow H(t)\widetilde{X}(t) + G(t, \hat{X}(t)) = H(t)(Z(t) \otimes Y(t))v(t) + (Z(t) \otimes Y(t))v'(t) \\
&\Rightarrow (Z(t) \otimes Y(t))v'(t) = G(t, \hat{X}(t)) \\
&\Rightarrow v'(t) = (Z^{-1}(t) \otimes Y^{-1}(t))G(t, \hat{X}(t)) \\
&\Rightarrow v(t) = \int_{t_0}^{t} (Z^{-1}(s) \otimes Y^{-1}(s))G(s, \hat{X}(s))ds.
\end{aligned}$$

Hence the desired expression follows immediately. ∎

Now we prove two lemmas which are useful in the proof of main theorem.

Lemma 2.2. Let $Y(t)$ and $Z(t)$ be invertible matrices which are continuous functions of t on R_+ and let P be a projection. If there exists a continuous function $\zeta : R_+ \to (0, \infty)$ and a positive constant K such that

$$\int_0^t \zeta(s)|\Psi(t)(Z(t) \otimes Y(t))P(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s)|ds \leq K, \quad \forall \quad t \geq 0,$$

and

$$\int_0^\infty \zeta(s)ds = \infty,$$

then there exists a constant $L > 0$ such that

$$|\Psi(t)(Z(t) \otimes Y(t))P| \leq Le^{-\frac{1}{K}\int_0^t \zeta(s)ds}, \quad \forall \quad t \geq 0,$$

and hence

$$\lim_{t\to\infty} |\Psi(t)(Z(t)\otimes Y(t))P| = 0.$$

Proof. The result is obvious, when $P = 0$.

For $P \neq 0$, let $\xi(t) = |\Psi(t)(Z(t)\otimes Y(t))P|^{-1}$, for $t \geq 0$. From the identity

$$\Big(\int_0^t \zeta(s)\xi(s)ds\Big)\Psi(t)(Z(t)\otimes Y(t))P$$
$$= \int_0^t \zeta(s)\Psi(t)(Z(t)\otimes Y(t))P(Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s) \cdot \Psi(s)(Z(s)\otimes Y(s))P\xi(s)ds,$$

for $t \geq 0$, it follows that

$$\Big(\int_0^t \zeta(s)\xi(s)ds\Big)|\Psi(t)(Z(t)\otimes Y(t))P|$$
$$\leq \int_0^t \zeta(s)|\Psi(t)(Z(t)\otimes Y(t))P(Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)| \cdot |\Psi(s)(Z(s)\otimes Y(s))P|\xi(s)ds,$$

for $t \geq 0$. Here the scalar function $g(t) = \int_0^t \zeta(s)\xi(s)ds$ satisfies the inequality

$$g(t)\xi^{-1}(t) \leq K, \ \ \forall\ t \geq 0,$$

and also

$$g'(t) = \zeta(t)\xi(t) \geq \frac{1}{K}g(t)\zeta(t), \ \ \forall\ t \geq 0.$$

It follows that

$$g(t) \geq g(t_1)e^{\frac{1}{K}\int_{t_1}^t \zeta(s)ds}, \ \text{ for } \ t \geq t_1 > 0$$

and hence

$$|\Psi(t)(Z(t)\otimes Y(t))P| = \xi^{-1}(t) \leq Kg^{-1}(t) \leq Kg^{-1}(t_1)e^{-\frac{1}{K}\int_{t_1}^t \zeta(s)ds},$$

for all $t \geq t_1 > 0$. Since $|\Psi(t)(Z(t)\otimes Y(t))P|$ is a continuous function on $[0, t_1]$, and $\int_0^\infty \zeta(s)ds = \infty$, it follows that there exists a positive constant L such that

$$|\Psi(t)(Z(t)\otimes Y(t))P| \leq Le^{-\frac{1}{K}\int_0^t \zeta(s)ds}, \ \ \forall\ t \geq 0,$$

and hence

$$\lim_{t\to\infty} |\Psi(t)(Z(t)\otimes Y(t))P| = 0. \ \blacksquare$$

Lemma 2.3. *Let $Y(t)$ and $Z(t)$ be invertible matrices which are continuous functions of t on R_+ and let P be a projection. If there exists a constant*

$K > 0$ such that

$$\int_0^t |\Psi(t)(Z(t) \otimes Y(t))P(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s)|ds \leq K, \ \ \forall\ t \geq 0,$$

then for any vector $u \in R^{n^2}$ such that $Pu \neq 0$,

$$\limsup_{t\to\infty} \|\Psi(t)(Z(t) \otimes Y(t))Pu\| = \infty.$$

Proof. Let $g(t) = \|\Psi(t)(Z(t) \otimes Y(t))Pu\|^{-1}$, for $t \geq 0$. For $0 \leq t \leq T$, consider

$$\Big(\int_t^T g(s)ds\Big)\Psi(t)(Z(t) \otimes Y(t))Pu$$

$$= \int_t^T \Psi(t)(Z(t)\otimes Y(t))P(Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)\Psi(s)(Z(s)\otimes Y(s))Pug(s)ds,$$

it follows that

$$\Big(\int_t^T g(s)ds\Big)\|\Psi(t)(Z(t) \otimes Y(t))Pu\|$$

$$\leq \int_t^T |\Psi(t)(Z(t) \otimes Y(t))P(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s)|$$

$$\cdot \|\Psi(s)(Z(s) \otimes Y(s))Pu\|g(s)ds.$$

The scalar function g satisfies the inequality

$$g^{-1}(t)\int_t^T g(s)ds \leq K, \ \ \text{for all } \ 0 \leq t \leq T,$$

it follows that $\int_t^\infty g(s)ds$ exists. Hence $\liminf_{t\to\infty} g(t) = 0$, which implies

$$\limsup_{t\to\infty} \|\Psi(t)(Z(t) \otimes Y(t))Pu\| = \infty. \ ∎$$

3. Ψ-instability of linear systems

In this section we study Ψ-instability of trivial solution of the linear system (2.2). The conditions for Ψ-instability of the trivial solution of (2.2) can be expressed in terms of the fundamental matrices of (2.3) and (2.4).

Theorem 3.1. *Let $Y(t)$ and $Z(t)$ be the fundamental matrices of (2.3) and (2.4) respectively. Then the trivial solution of (2.2) is Ψ-unstable on R_+ if and only if there is a projection P such that $|\Psi(t)(Z(t) \otimes Y(t))P|$ is unbounded on R_+.*

Proof. Suppose that the trivial solution of (2.2) is Ψ-unstable on R_+. Then $|\Psi(t)(Z(t) \otimes Y(t))|$ is unbounded on R_+ (Theorem 3 of [12]) and consequently there exists a projection P such that $|\Psi(t)(Z(t) \otimes Y(t))P|$ is unbounded on R_+.

Conversely, suppose that $|\Psi(t)(Z(t) \otimes Y(t))P|$ is unbounded on R_+. To the contrary, let us assume that the trivial solution of (2.2) is Ψ-stable, then for every $\epsilon > 0$ and $t_0 \geq 0$, there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that any solution $\hat{X}(t)$ of (2.2) which satisfies the inequality $\|\Psi(t)\hat{X}(t_0)\| < \delta(\epsilon, t_0)$ exists and satisfies the inequality $\|\Psi(t)\hat{X}(t)\| < \epsilon$, for all $t \geq t_0$.

Let $t_0 \geq 0$ and $\hat{X}_0 \in R^{n^2}$ such that $|\Psi(t_0)(Z(t_0) \otimes Y(t_0))P| \neq 0$ and

$$\|\hat{X}_0\| < \delta |\Psi(t_0)(Z(t_0) \otimes Y(t_0))P|^{-1} = \delta_0,$$

we have

$$\|\Psi(t_0)(Z(t_0) \otimes Y(t_0))P\hat{X}_0\| < \delta.$$

It follows that

$$\|\Psi(t)(Z(t) \otimes Y(t))P\hat{X}_0\| < \epsilon,$$

for all $t \geq t_0$. Thus, for $v \in R^{n^2}$, $\|v\| \leq 1$, we have

$$\|\Psi(t)(Z(t) \otimes Y(t))P\delta_0 v\| < \epsilon, \quad \text{for all} \quad t \geq t_0,$$

and also

$$|\Psi(t)(Z(t) \otimes Y(t))P| < \epsilon\delta_0^{-1}, \quad \text{for all} \quad t \geq t_0,$$

which is a contradiction. Hence the trivial solution of (2.2) is Ψ-unstable on R_+. ∎

EXAMPLE 3.1. Consider the linear matrix Lyapunov system corresponding to (1.1) with

$$A(t) = \begin{bmatrix} \frac{1}{t+1} & 0 \\ 0 & \frac{-1}{t+1} \end{bmatrix}, \quad B(t) = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}.$$

Then the fundamental matrices of (2.3), (2.4) are

$$Y(t) = \begin{bmatrix} t+1 & 0 \\ 0 & \frac{1}{t+1} \end{bmatrix}, \quad Z(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-2t} \end{bmatrix}.$$

Now the fundamental matrix of (2.2) is

$$Z(t) \otimes Y(t) = \begin{bmatrix} e^t(t+1) & 0 & 0 & 0 \\ 0 & \frac{e^t}{t+1} & 0 & 0 \\ 0 & 0 & (t+1)e^{-2t} & 0 \\ 0 & 0 & 0 & \frac{e^{-2t}}{t+1} \end{bmatrix}.$$

Let

$$\Psi(t) = \begin{bmatrix} \frac{e^{-t}}{\sqrt{t+1}} & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & \frac{e^{2t}}{t+1} & 0 \\ 0 & 0 & 0 & e^{2t} \end{bmatrix} \text{ and } P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, we have

$$\Psi(t)(Z(t) \otimes Y(t))P = \begin{bmatrix} \sqrt{t+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly $|\Psi(t)(Z(t) \otimes Y(t))P| = \sqrt{t+1}$ is unbounded on R_+, from Theorem 3.1 the trivial solutions of linear system (2.2) is Ψ-unstable on R_+.

Theorem 3.2. *If there exists a projection $P \neq 0$ and a positive constant K such that the fundamental matrices $Y(t)$ and $Z(t)$ of* (2.3) *and* (2.4) *respectively, satisfies the inequality*

$$\int_t^\infty |\Psi(t)(Z(t) \otimes Y(t))P(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s)|ds \leq K, \quad \textit{for all } t \geq 0,$$

then the trivial solution of (2.2) *is Ψ-unstable on R_+.*

Proof. Suppose to the contrary, assume that the trivial solution of (2.2) is Ψ-stable on R_+. Therefore, for $\epsilon > 0$ and $t_0 \geq 0$, there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that any solution $\hat{X}(t)$ of (2.2) which satisfies the inequality $\|\Psi(t)\hat{X}(t_0)\| < \delta(\epsilon, t_0)$ exists and satisfies the inequality $\|\Psi(t)\hat{X}(t)\| < \epsilon$, for all $t \geq t_0$.

Without loss of generality, we may take $Y(0) = I_n$, $Z(0) = I_n$ (else, we replace $Y(t)$ with $Y(t)Y^{-1}(0)$, $Z(t)$ with $Z(t)Z^{-1}(0)$ and P with $(Z(0) \otimes Y(0))P(Z^{-1}(0) \otimes Y^{-1}(0)))$. Clearly $Z(0) \otimes Y(0) = I_{n^2}$.

For $t_0 = 0$, we can choose $\hat{X}_0 \in R^{n^2}$ such that $\hat{X}_0 = P\hat{X}_0$ and $0 < \|\Psi(0)\hat{X}_0\| \leq \delta(\epsilon, 0)$. Then, $\|\Psi(t)\hat{X}(t)\| < \epsilon$, for all $t \geq 0$. On the other hand,

$$\Psi(t)\hat{X}(t) = \Psi(t)(Z(t) \otimes Y(t))(Z^{-1}(0) \otimes Y^{-1}(0))\hat{X}_0 = \Psi(t)(Z(t) \otimes Y(t))P\hat{X}_0.$$

From Lemma 2.3, it follows that

$$\limsup_{t \to \infty} \|\Psi(t)(Z(t) \otimes Y(t))P\hat{X}_0\| = \infty,$$

which is a contradiction. Thus, the trivial solution of (2.2) is Ψ-unstable on R_+. ∎

Example 3.2. In Example 3.1, if we take $A = I_2$ and $B = -I_2$, then the fundamental matrix of (2.2) is $(Z(t) \otimes Y(t)) = I_4$. Let

$$\Psi(t) = \begin{bmatrix} t+1 & 0 & 0 & 0 \\ 0 & 1+t & 0 & 0 \\ 0 & 0 & \frac{1}{t+1} & 0 \\ 0 & 0 & 0 & \frac{1}{t+1} \end{bmatrix} \text{ and } P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then we have

$$\Psi(t)(Z(t) \otimes Y(t))P(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s) = \begin{bmatrix} \frac{t+1}{s+1} & 0 & 0 & 0 \\ 0 & \frac{t+1}{s+1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore

$$\int_t^\infty |\Psi(t)(Z(t) \otimes Y(t))P(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s)|ds = \frac{1}{t+1} \leq 1, \quad \text{for all } t \geq 0.$$

Thus, from Theorem 3.2 the trivial solution of (2.2) is Ψ-unstable on R_+.

4. Ψ-instability of non-linear systems

In this section we obtain sufficient conditions for Ψ-instability of non-linear matrix Lyapunov systems.

THEOREM 4.1. *Suppose that:*

(i) *There exist supplementary projections P_1 and P_2, $P_2 \neq 0$ and a constant $K > 0$ such that the fundamental matrices $Y(t)$, $Z(t)$ of* (2.3) *and* (2.4) *respectively, satisfies the condition*

$$\int_0^t |\Psi(t)(Z(t) \otimes Y(t))P_1(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s)|ds$$
$$+ \int_t^\infty |\Psi(t)(Z(t) \otimes Y(t))P_2(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s)|ds \leq K,$$

for all $t \geq 0$.

(ii) *The function $G(t, \hat{X}(t))$ satisfies the inequality*

$$\|\Psi(t)G(t, \hat{X}(t))\| \leq \phi(t)\|\Psi(t)\hat{X}(t)\|,$$

for every continuous $\hat{X} : R_+ \to R^{n^2}$, where $\phi(t)$ is a continuous non-

negative bounded function on R_+ such that

$$|\phi(t)| \leq M, \quad \text{for all} \quad t \geq 0,$$

where M is a positive constant.

(iii) $MK < 1$.

Then, the trivial solution of (2.1) *is* $\Psi-$*unstable on* R_+.

Proof. Suppose to the contrary, assume that the trivial solution of (2.1) is Ψ-stable on R_+. Therefore, for every $\epsilon > 0$ and any $t_0 \geq 0$, there exists $\delta = \delta(\epsilon, t_0) > 0$ such that any solution $\hat{X}(t)$ of (2.1) which satisfies the inequality $\|\Psi(t_0)\hat{X}(t_0)\| < \delta$ exists and satisfies the inequality $\|\Psi(t)\hat{X}(t)\| < \epsilon$, for all $t \geq 0$.

Without loss of generality, we assume that $Y(0) = I_n$, $Z(0) = I_n$, then $Z(0) \otimes Y(0) = I_{n^2}$. For $t_0 = 0$, we can choose $\hat{X}(0) \in R^{n^2}$ such that $P_1\hat{X}(0) = 0$ and $\|\Psi(0)\hat{X}(0)\| < \delta(\epsilon, 0)$. Then $\|\Psi(t)\hat{X}(t)\| < \epsilon$, for all $t \geq 0$.

Consider the function

$$u(t) = \hat{X}(t) - \int_0^t (Z(t) \otimes Y(t))P_1(Z^{-1}(s) \otimes Y^{-1}(s))G(s, \hat{X}(s))ds$$
$$+ \int_t^\infty (Z(t) \otimes Y(t))P_2(Z^{-1}(s) \otimes Y^{-1}(s))G(s, \hat{X}(s))ds, \quad t \geq 0.$$

For $0 \leq t \leq \tau$, we have

$$\left\| \int_t^\tau (Z(t) \otimes Y(t))P_2(Z^{-1}(s) \otimes Y^{-1}(s))G(s, \hat{X}(s))ds \right\|$$
$$\leq |\Psi^{-1}(t)| \int_t^\tau |\Psi(t)(Z(t) \otimes Y(t))P_2(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s)| \cdot \|\Psi(s)G(s, \hat{X}(s))\|ds$$
$$\leq |\Psi^{-1}(t)| \int_t^\tau |\Psi(t)(Z(t) \otimes Y(t))P_2(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s)| \cdot \phi(s)\|\Psi(s)\hat{X}(s))\|ds$$
$$< M\epsilon|\Psi^{-1}(t)| \int_t^\tau |\Psi(t)(Z(t) \otimes Y(t))P_2(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s)|ds.$$

From hypothesis (i), it follows that

$$\int_t^\infty (Z(t) \otimes Y(t))P_2(Z^{-1}(s) \otimes Y^{-1}(s))G(s, \hat{X}(s))ds$$

is convergent. Clearly, the function $u(t)$ exists and is also continuously differentiable on R_+. Consider

$$
\begin{aligned}
u'(t) &= \hat{X}'(t) - \int_0^t (Z(t)\otimes Y(t))' P_1 (Z^{-1}(s)\otimes Y^{-1}(s)) G(s,\hat{X}(s))ds \\
&\quad - (Z(t)\otimes Y(t)) P_1 (Z^{-1}(t)\otimes Y^{-1}(t)) G(t,\hat{X}(t)) \\
&\quad + \int_t^\infty (Z(t)\otimes Y(t))' P_2 (Z^{-1}(s)\otimes Y^{-1}(s)) G(s,\hat{X}(s))ds \\
&\quad - (Z(t)\otimes Y(t)) P_2 (Z^{-1}(t)\otimes Y^{-1}(t)) G(t,\hat{X}(t)) \\
&= H(t)\hat{X}(t) + G(t,\hat{X}(t)) \\
&\quad - (Z(t)\otimes Y(t))\,[P_1 + P_2]\,(Z^{-1}(t)\otimes Y^{-1}(t)) G(t,\hat{X}(t)) \\
&\quad - H(t)\Big(\int_0^t (Z(t)\otimes Y(t)) P_1 (Z^{-1}(s)\otimes Y^{-1}(s)) G(s,\hat{X}(s))ds \\
&\quad - \int_t^\infty (Z(t)\otimes Y(t)) P_1 (Z^{-1}(s)\otimes Y^{-1}(s)) G(s,\hat{X}(s))ds\Big) \\
&= H(t)u(t), \quad \text{for all } t \ge 0.
\end{aligned}
$$

Therefore, $u(t)$ is a solution of the linear system (2.2) on R_+. Since

$$
\begin{aligned}
&\|\Psi(t)u(t)\| \le \|\Psi(t)\hat{X}(t)\| \\
&\quad + \int_0^t |\Psi(t)(Z(t)\otimes Y(t)) P_1 (Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)| \|\Psi(s)G(s,\hat{X}(s))\| ds \\
&\quad + \int_t^\infty |\Psi(t)(Z(t)\otimes Y(t)) P_2 (Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)| \|\Psi(s)G(s,\hat{X}(s))\| ds \\
&\le \epsilon + \int_0^t |\Psi(t)(Z(t)\otimes Y(t)) P_1 (Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)| \phi(s) \|\Psi(s)\hat{X}(s)\| ds \\
&\quad + \int_t^\infty |\Psi(t)(Z(t)\otimes Y(t)) P_2 (Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)| \phi(s) \|\Psi(s)\hat{X}(s)\| ds \\
&\le \epsilon(1 + MK), \quad \text{for } t \ge 0,
\end{aligned}
$$

it follows that the solution $u(t)$ is Ψ-bounded on R_+.

On the other hand,

$$
\begin{aligned}
u(t) &= (Z(t)\otimes Y(t))(Z^{-1}(0)\otimes Y^{-1}(0))u(0) \\
&= (Z(t)\otimes Y(t))\,[P_1u(0) + P_2u(0)] = (Z(t)\otimes Y(t))P_2u(0).
\end{aligned}
$$

Let $P_2u(0) \neq 0$, then from Lemma 2.3, we have

$$\limsup_{t\to\infty} \|\Psi(t)u(t)\| = \infty,$$

which is a contradiction. Thus, $P_2 u(0) = 0$ and hence $u(t) = 0$ on R_+. Therefore

$$\hat{X}(t) = \int_0^t (Z(t) \otimes Y(t))P_1(Z^{-1}(s) \otimes Y^{-1}(s))G(s, \hat{X}(s))ds$$
$$- \int_t^\infty (Z(t) \otimes Y(t))P_2(Z^{-1}(s) \otimes Y^{-1}(s))G(s, \hat{X}(s))ds, \quad t \geq 0.$$

Consider

$$\begin{aligned}
&\|\Psi(t)\hat{X}(t)\| \\
&\leq \int_0^t |\Psi(t)(Z(t) \otimes Y(t))P_1(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s)| \|\Psi(s)G(s, \hat{X}(s))\| ds \\
&\quad + \int_t^\infty |\Psi(t)(Z(t) \otimes Y(t))P_2(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s)| \|\Psi(s)G(s, \hat{X}(s))\| ds \\
&\leq \int_0^t |\Psi(t)(Z(t) \otimes Y(t))P_1(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s)|\phi(s)\|\Psi(s)\hat{X}(s)\| ds \\
&\quad + \int_t^\infty |\Psi(t)(Z(t) \otimes Y(t))P_2(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s)| \ \phi(s) \ \|\Psi(s)\hat{X}(s)\| ds \\
&\leq MK\|\Psi(t)\hat{X}(t)\|, \quad \text{for } t \geq 0,
\end{aligned}$$

which leads to a contradiction. Hence, the trivial solution of (2.1) is Ψ-unstable on R_+. ∎

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