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ON A BOUNDARY VALUE PROBLEM FOR A THIRD ORDER DIFFERENTIAL INCLUSION

Abstract. We consider a boundary value problem for third order nonconvex differential inclusion and we obtain some existence results by using the set-valued contraction principle.

1. Introduction

This paper is concerned with the following boundary value problem

$$(1.1) \qquad x''' + k^2 x' \in F(t,x), \quad a.e.\ ([-1,1]), \quad x(-1) = x(1) = x'(1) = 0$$

where $F(.,.) : [-1,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a set-valued map and $k \in [-\pi, \pi]$.

The present note is motivated by a recent paper of Bartuzel and Fryszkowski ([1]), where it is considered problem (1.1) and a version of the Filippov Lemma for this problem is obtained. The aim of our paper is to present two additional results obtained by the application of the set-valued contraction principle due to Covitz and Nadler ([10]).

The first result follows a classical idea by applying the set-valued contraction principle in the space of solutions of the problem. The second result is also a Filippov type theorem concerning the existence of solutions to problem (1.1). Recall that for a differential inclusion defined by a Lipschitzian set-valued map with nonconvex values, Filippov's theorem consists in proving the existence of a solution starting from a given "quasi" solution. This time we apply the contraction principle in the space of derivatives of solutions instead of the space of solutions. In addition, as usual at a Filippov existence type theorem, we obtain an estimate between the starting "quasi" solution and the solution of the differential inclusion. The idea of applying the set-valued contraction principle in the space of derivatives of the solutions belongs to Bressan, Cellina and Fryszkowski ([2]) and it was used for

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the first time by Tallos ([11, 13]) in deriving Filippov type results. Other similar results concerning differential inclusions may be found in [4–9] etc..

The Filppov type result we propose in our approach is an alternative to the one in [1]. The two results are not comparable since the hypotheses concerning the quasi solution are different. Moreover, the methods used in their proofs are also different: the proof of the result in [1] follows Filippov's construction, while in our approach we obtain a "pointwise" estimate from a norm estimate.

For the motivation of the study of problem (1.1) we refer to [1] and references therein.

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main results.

2. Preliminaries

In this short section we sum up some basic facts that we are going to use later.

Let (X,d) be a metric space and consider a set valued map T on X with nonempty values in X. T is said to be a λ-contraction if there exists $0<\lambda<1$ such that:

$$d_H(T(x),T(y)) \leq \lambda d(x,y) \quad \forall x,y \in X,$$

where $d_H(.,.)$ denotes the Pompeiu-Hausdorff distance. Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A,B)=\max\{d^*(A,B),d^*(B,A)\}, \quad d^*(A,B)=\sup\{d(a,B); a\in A\},$$

where $d(x,B)=\inf_{y\in B} d(x,y)$.

The set-valued contraction principle ([10]) states that if X is complete, and $T: X \to \mathcal{P}(X)$ is a set valued contraction with nonempty closed values, then $T(.)$ has a fixed point, i.e. a point $z\in X$ such that $z\in T(z)$.

We denote by $Fix(T)$ the set of all fixed points of the set-valued map T. Obviously, $Fix(T)$ is closed.

Proposition 2.1. ([12]) *Let X be a complete metric space and suppose that T_1, T_2 are λ-contractions with closed values in X. Then*

$$d_H(Fix(T_1),Fix(T_2)) \leq \frac{1}{1-\lambda} \sup_{z\in X} d(T_1(z),T_2(z)).$$

Let $I=[-1,1]$. By a solution of problem (1.1) we mean a function $x(.) \in W := W^{1,3}(I)\cap H_0^1(I)$ satisfying (1.1).

As usual, we denote by $C(I,\mathbb{R})$ the Banach space of all continuous functions from I to $\mathbb{R}$ with the norm $||x(.)||_C=\sup_{t\in I}|x(t)|$ and $L^1(I,\mathbb{R})$ the Banach space of integrable functions $u(.): I\to\mathbb{R}$ endowed with the norm $||u(.)||_1=\int_{-1}^1 |u(t)|dt$.

For each $x(.) \in W$ define

$$S_{F,x} := \{f(.) \in L^1(I, \mathbb{R}); \quad f(t) \in F(t, x(t)) \quad a.e. \ (I)\}.$$

LEMMA 2.2. ([1]) *If $f(.) : [-1, 1] \to \mathbb{R}$ is an integrable function and $k \in [-\pi, \pi]$ then the equation*

$$x''' + k^2 x' = f(t) \quad a.e. \ (I),$$

with the boundary conditions $x(-1) = x(1) = x'(1) = 0$ has a unique solution given by

$$x(t) = \int_{-1}^{1} G(t,s) f(s) ds,$$

where $G(.,.)$ is the associated Green function. Namely,

$$G(t,x) = \begin{cases} \frac{(1-cosk(1+x))(1-cosk(1-t))}{k^2(1-cos2k)} & \text{if } \quad -1 \le x \le t \le 1, \\ \frac{(1-cosk(1+x))(1-cosk(1-t))-(1-cosk(x-t))(1-cos2k)}{k^2(1-cos2k)} & \\ & \text{if } -1 \le t \le x \le 1. \end{cases}$$

Moreover, if $k \neq 0$

$$0 \le G(t,x) \le G_0 := \frac{k^2(5\sqrt{5} - 11)}{sin^2 k} \quad \forall \ (t,x) \in I \times \mathbb{R}.$$

In order to study problem (1.1) we introduce the following hypothesis on F.

HYPOTHESIS 2.3. (i) *$F(.,.) : I \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ has nonempty closed values and for every $x \in \mathbb{R}$ $F(., x)$ is measurable.*

(ii) *There exists $L(.) \in L^1(I, \mathbb{R}_+)$ such that for almost all $t \in I, F(t, \cdot)$ is $L(t)$-Lipschitz in the sense that*

$$d_H(F(t,x), F(t,y)) \le L(t)|x - y| \quad \forall \ x, y \in \mathbb{R}$$

and $d(0, F(t, 0)) \le L(t) \quad a.e. \ (I)$.

Denote $L_0 := \int_{-1}^{1} L(s)ds$ and assume that $k \neq 0$.

3. The main results

We are able now to present a first existence result for problem. (1.1).

THEOREM 3.1. *Assume that Hypothesis 2.3 is satisfied, $F(.,.)$ has compact values and $G_0 L_0 < 1$. Then the problem (1.1) has a solution.*

Proof. We transform the problem (1.1) in a fixed point problem. Consider the set-valued map $T : C(I, \mathbb{R}) \to \mathcal{P}(C(I, \mathbb{R}))$ defined by

$$T(x) := \left\{ v(.) \in C(I, \mathbb{R}); \quad v(t) := \int_{-1}^{1} G(t,s) f(s) ds, \quad f \in S_{F,x} \right\}.$$

Note that since the set-valued map $F(.,x(.))$ is measurable with the measurable selection theorem (e.g., Theorem III. 6 in [3]) it admits a measurable selection $f(.):I\to\mathbb{R}$. Moreover, from Hypothesis 2.3

$$|f(t)|\leq L(t)+L(t)|x(t)|,$$

i.e., $f(.)\in L^1(I,\mathbb{R})$. Therefore, $S_{F,x}\neq\emptyset$.

It is clear that the fixed points of $T(.)$ are solutions of problem (1.1). We shall prove that $T(.)$ fulfills the assumptions of Covitz Nadler contraction principle.

First, we note that since $S_{F,x}\neq\emptyset$, $T(x)\neq\emptyset$ for any $x(.)\in C(I,\mathbb{R})$.

Secondly, we prove that $T(x)$ is closed for any $x(.)\in C(I,\mathbb{R})$.

Let $\{x_n\}_{n\geq 0}\in T(x)$ such that $x_n(.)\to x^*(.)$ in $C(I,\mathbb{R})$. Then $x^*(.)\in C(I,\mathbb{R})$ and there exists $f_n\in S_{F,x}$ such that

$$x_n(t)=\int_{-1}^{1}G(t,s)f_n(s)ds.$$

Since $F(.,.)$ has compact values and Hypothesis 2.3 is satisfied we may pass to a subsequence (if necessary) to get that $f_n(.)$ converges to $f(.)\in L^1(I,\mathbb{R})$ in $L^1(I,\mathbb{R})$.

In particular, $f\in S_{F,x}$ and for any $t\in I$ we have

$$x_n(t)\to x^*(t)=\int_{-1}^{1}G(t,s)f(s)ds,$$

i.e., $x^*\in T(x)$ and $T(x)$ is closed.

Finally, we show that $T(.)$ is a contraction on $C(I,\mathbb{R})$.

Let $x_1(.),x_2(.)\in C(I,\mathbb{R})$ and $v_1\in T(x_1)$. Then there exists $f_1\in S_{F,x_1}$ such that

$$v_1(t)=\int_{-1}^{1}G(t,s)f_1(s)ds,\quad t\in I.$$

Consider the set-valued map

$$G(t):=F(t,x_1(t))\cap\{x\in\mathbb{R};\quad |f_1(t)-x|\leq L(t)|x_1(t)-x_2(t)|\},\quad t\in I.$$

From Hypothesis 2.3 one has

$$d_H(F(t,x_1(t)),F(t,x_2(t)))\leq L(t)|x_1(t)-x_2(t)|,$$

hence $G(.)$ has nonempty closed values. Moreover, since $G(.)$ is measurable, there exists $f_2(.)$ a measurable selection of $G(.)$. It follows that $f_2\in S_{F,x_2}$ and for any $t\in I$

$$|f_1(t)-f_2(t)|\leq L(t)|x_1(t)-x_2(t)|.$$

Define

$$v_2(t) = \int_{-1}^{1} G(t,s)f_2(s)ds, \quad t \in I,$$

and we have

$$|v_1(t) - v_2(t)| \leq \int_{-1}^{1} |G(t,s)|.|f_1(s) - f_2(s)|ds \leq G_0 \int_{-1}^{1} |f_1(s) - f_2(s)|ds$$

$$\leq G_0 \int_{-1}^{1} L(s)|x_1(s) - x_2(s)|ds \leq G_0L_0||x_1 - x_2||_C.$$

So, $||v_1 - v_2||_C \leq G_0L_0||x_1 - x_2||_C$.

From an analogous reasoning by interchanging the roles of x_1 and x_2 it follows

$$d_H(T(x_1), T(x_2)) \leq G_0L_0||x_1 - x_2||_C.$$

Therefore, $T(.)$ admits a fixed point which is a solution to problem (1.1).

The next theorem is the main result of this paper. As one can see it is, in fact, no necessary to assume that $F(.,.)$ has compact values as in Theorem 3.1.

THEOREM 3.2. *Assume that Hypothesis* 2.3 *is satisfied and* $G_0L_0 < 1$. *Let* $y(.) \in W$ *be such that there exists* $q(.) \in L^1(I, \mathbb{R}_+)$ *with*

$$d(y'''(t) + k^2y'(t), F(t, y(t))) \leq q(t) \quad a.e.\ (I).$$

Then for every $\varepsilon > 0$ *there exists* $x(.)$ *a solution of problem* (1.1) *satisfying for all* $t \in I$

$$|x(t) - y(t)| \leq \frac{G_0}{1 - G_0L_0} \int_{-1}^{1} q(t)dt + \varepsilon.$$

Proof. For $u(.) \in L^1(I, \mathbb{R})$ define the following set valued maps

$$M_u(t) = F(t, \int_{-1}^{1} G(t,s)u(s)ds), \quad t \in I,$$

$$T(u) = \{\phi(.) \in L^1(I, \mathbb{R}); \quad \phi(t) \in M_u(t) \quad a.e.\ (I)\}.$$

It follows from Lemma 2.2 that $x(.)$ is a solution of problem (1.1) if and only if $x'''(.) + k^2x'(.)$ is a fixed point of $T(.)$.

We shall prove first that $T(u)$ is nonempty and closed for every $u \in L^1(I, \mathbb{R})$. The fact that the set valued map $M_u(.)$ is measurable is well known. For example the map $t \to \int_{-1}^{1} G(t,s)u(s)ds$ can be approximated by step functions and we can apply Theorem III. 40 in [3]. Since the values of

F are closed with the measurable selection theorem (Theorem III.6 in [3]) we infer that $M_u(.)$ admits a measurable selection ϕ. One has

$$|\phi(t)| \leq d(0, F(t,0)) + d_H(F(t,0), F(t, \int_{-1}^{1} G(t,s)u(s)ds))$$

$$\leq L(t)(1 + G_0 \int_{-1}^{1} |u(s)|ds),$$

which shows that $\phi \in L^1(I, \mathbb{R})$ and $T(u)$ is nonempty.

On the other hand, the set $T(u)$ is also closed. Indeed, if $\phi_n \in T(u)$ and $||\phi_n - \phi||_1 \to 0$ then we can pass to a subsequence ϕ_{n_k} such that $\phi_{n_k}(t) \to \phi(t)$ for a.e. $t \in I$, and we find that $\phi \in T(u)$.

We show next that $T(.)$ is a contraction on $L^1(I, \mathbb{R})$.

Let $u, v \in L^1(I, \mathbb{R})$ be given and $\phi \in T(u)$. Consider the following set-valued map:

$$H(t) = M_v(t) \cap \{x \in \mathbb{R}; \quad |\phi(t) - x| \leq L(t) |\int_{-1}^{1} G(t,s)(u(s) - v(s))ds|\}.$$

From Proposition III.4 in [3], $H(.)$ is measurable and from Hypothesis 2.3 ii) $H(.)$ has nonempty closed values. Therefore, there exists $\psi(.)$ a measurable selection of $H(.)$. It follows that $\psi \in T(v)$ and according with the definition of the norm we have

$$||\phi - \psi||_1 = \int_{-1}^{1} |\phi(t) - \psi(t)|dt \leq \int_{-1}^{1} L(t)(\int_{-1}^{1} |G(t,s)|.|u(s) - v(s)|ds)dt$$

$$= \int_{-1}^{1} \Big(\int_{-1}^{1} L(t)|G(t,s)|dt\Big)|u(s) - v(s)|ds \leq G_0 L_0 ||u - v||_1.$$

We deduce that

$$d(\phi, T(v)) \leq G_0 L_0 ||u - v||_1.$$

Replacing u by v we obtain

$$d_H(T(u), T(v)) \leq G_0 L_0 ||u - v||_1,$$

thus $T(.)$ is a contraction on $L^1(I, \mathbb{R})$.

We consider next the following set-valued maps

$$F_1(t,x) = F(t,x) + q(t)[-1,1], \quad (t,x) \in I \times \mathbb{R},$$

$$M_u^1(t) = F_1(t, \int_{-1}^{1} G(t,s)u(s)ds),$$

$$T_1(u) = \{\psi(.) \in L^1(I, \mathbb{R}); \quad \psi(t) \in M_u^1(t) \quad a.e.\ (I)\}, \quad u(.) \in L^1(I, \mathbb{R}).$$

Obviously, $F_1(.,.)$ satisfies Hypothesis 2.3.

Repeating the previous step of the proof we obtain that T_1 is also a G_0L_0-contraction on $L^1(I,\mathbb{R})$ with closed nonempty values.

We prove next the following estimate

$$d_H(T(u),T_1(u)) \leq \int_{-1}^{1} q(t)dt. \tag{3.1}$$

Let $\phi \in T(u)$ and define

$$H_1(t) = M_u^1(t) \cap \{z \in \mathbb{R}; \quad |\phi(t) - z| \leq q(t)\}.$$

With the same arguments used for the set valued map $H(.)$, we deduce that $H_1(.)$ is measurable with nonempty closed values. Hence let $\psi(.)$ be a measurable selection of $H_1(.)$. It follows that $\psi \in T_1(u)$ and one has

$$||\phi - \psi||_1 = \int_{-1}^{1} |\phi(t) - \psi(t)|dt \leq \int_{-1}^{1} q(t).$$

As above we obtain (3.1).

We apply Proposition 2.1 and we infer that

$$d_H(Fix(T), Fix(T_1)) \leq \frac{1}{1 - G_0L_0} \int_{-1}^{1} q(t)dt.$$

Since $v(.) = y'''(.) + k^2y'(.) \in Fix(T_1)$ it follows that for any $\varepsilon > 0$ there exists $u(.) \in Fix(T)$ such that

$$||v - u||_1 \leq \frac{1}{1 - G_0L_0} \int_{-1}^{1} q(t)dt + \frac{\varepsilon}{G_0}.$$

We define $x(t) = \int_{-1}^{1} G(t,s)u(s)ds$, $t \in I$ and we have

$$|x(t) - y(t)| \leq \int_{-1}^{1} |G(t,s)|.|u(s) - v(s)|ds \leq \frac{G_0}{1 - G_0L_0} \int_{-1}^{1} q(t)dt + \varepsilon$$

which completes the proof.

Remark 3.3. The assumption in Theorem 3.2 is satisfied, in particular, for $y(.) = 0$ and therefore, via Hypothesis 2.3, with $q(.) = L(.)$. In this case, Theorem 3.1 provides an existence result for problem (1.1) together with a priori bounds for the solution.

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