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UNIVALENT HARMONIC MAPPINGS WITH TWO PREASSIGNED VALUES

Abstract. In this paper we consider a class $H^*(a)$ of normalized harmonic functions which map the unit disk onto starlike domains. We give necessary and sufficient condition for $f \in H^*(a)$, distortion bounds and extreme points. We partially solve the problem of the Koebe set for $H^*(a)$. We also consider the Schild's Conjecture for the class TH^* of harmonic mappings.

In 1984 J. Clunie and T. Sheil-Small ([2]) initiated a systematic study of univalent sense-preserving and harmonic mappings of the unit disk $\mathbb{D} = \{z : |z| < 1\}$. Such mappings can be written in the form

$$f(z) = h(z) + \overline{g(z)}$$

where h, g are analytic in $\mathbb{D}$ and

$$h(z) = \sum_{n=1}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n.$$

If in addition $h'(0) = 1$, $g'(0) = 0$ then such functions form a compact family usually denoted by SH^0. Various extremal questions within the class SH^0 have been considered and solved. Moreover, some interesting problems remain open. The theory being developed along the line similar to that of the geometric function theory.

There exists a relatively rich literature concerning functions F holomorphic and univalent in $\mathbb{D}$ that satisfy the so called Montel normalization, i.e. $F(0) = 0$, $F(a) = a$ for a given a, $0 < |a| < 1$, while harmonic mappings subject to the Montel type normalization have not been considered.

Let $H(a)$ denote the class of functions F harmonic, sense-preserving and univalent in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ that have the form

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$$F(z) = \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{n=2}^{\infty} b_n z^n}$$

and satisfy the condition $F(a) = a$ for a fixed a, $0 < |a| < 1$. It is easy to see that

$$F(z) = a_1(F) \cdot G(z),$$

where $G(z) \in SH^0$. It follows that $a_1 = \frac{a}{G(a)}$ and in view of the known bounds on $|G(a)|$ (see [3], p. 81) we conclude that the class $H(a)$ is compact.

Suppose that a function f is of the form

$$f(z) = a_1 z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=2}^{\infty} \overline{b}_n \overline{z}^n, \quad z \in \mathbb{D} \tag{1}$$

and the condition

$$\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \leq |a_1| \tag{2}$$

is satisfied. Then ([1]) f is univalent sense-preserving harmonic mapping of $\mathbb{D}$ onto a domain starlike with respect to the origin. If in addition $-a_n \geq 0$, $-b_n \geq 0$, for $n \geq 2$ then (2) is also necessary for univalence and starlikness of f.

In this note we shall be concerned with a class $H^*(a)$ that consists of those harmonic functions that for a given a, $0 < |a| < 1$ satisfy the conditions

$$f(a) = a, \quad \sum_{n=2}^{\infty} n(a_n + b_n) \leq a_1, \quad a_n \geq 0,\ b_n \geq 0 \tag{3}$$

we shall discuss covering properties and extend a result due to Z. Lewandowski ([5]).

Using (3) we see that

$$\begin{aligned} f(z) &= \Big(1 + \sum_{n=2}^{\infty} (a_n + b_n) a^{n-1}\Big) z - \sum_{n=2}^{\infty} a_n z^n - \sum_{n=2}^{\infty} b_n \overline{z}^n \\ &= z - \sum_{n=2}^{\infty} a_n (z^n - a^{n-1} z) - \sum_{n=2}^{\infty} b_n (\overline{z}^n - a^{n-1} z). \end{aligned}$$

Hence $H^*(a)$ is the class of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n (z^n - a^{n-1} z) - \sum_{n=2}^{\infty} b_n (\overline{z}^n - a^{n-1} z)$$

that are harmonic univalent and sense preserving in the unit disc $\mathbb{D}$ and satisfy (3).

We give a condition that characterizes the class $H^*(a)$. It is established by our next theorem.

Theorem 1. *For*

$$f(z) = z - \sum_{n=2}^{\infty} a_n(z^n - a^{n-1}z) - \sum_{n=2}^{\infty} b_n(\overline{z}^n - a^{n-1}z),$$

$f \in H^*(a)$ *if and only if*

$$\sum_{n=2}^{\infty}(a_n + b_n)(n - a^{n-1}) \leq 1.$$

Proof. Let $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n - \sum_{n=2}^{\infty} b_n \overline{z}^n$ satisfies $f(a) = a$. Hence we have $1 = a_1 - \sum_{n=2}^{\infty}(a_n + b_n)a^{n-1}$. Combining this and $0 \leq a_1 - \sum_{n=2}^{\infty} n(a_n + b_n)$ together we obtain

$$-1 \leq a_1 - \sum_{n=2}^{\infty}(a_n + b_n)n - a_1 + \sum_{n=2}^{\infty}(a_n + b_n)a^{n-1}$$

and finally

$$\sum_{n=2}^{\infty}(a_n + b_n)(n - a^{n-1}) \leq 1.$$

This completes the proof. ∎

Let us show that the class $H^*(a)$ is convex. To see that, let us suppose that $\alpha \in [0,1]$ is fixed and $f_1(z), f_2(z)$ belong to $H^*(a)$ where

$$f_1(z) = z - \sum_{n=2}^{\infty} a_n(z^n - a^{n-1}z) - \sum_{n=2}^{\infty} b_n(\overline{z}^n - a^{n-1}z),$$

$$f_2(z) = z - \sum_{n=2}^{\infty} a'_n(z^n - a^{n-1}z) - \sum_{n=2}^{\infty} b'_n(\overline{z}^n - a^{n-1}z).$$

Then the function

$$f_\alpha(z) = \alpha f_1(z) + (1-\alpha) f_2(z)$$
$$= z - \sum_{n=2}^{\infty}(\lambda a_n + (1-\lambda)a'_n)(z^n - a^{n-1}z) - \sum_{n=2}^{\infty}(\lambda b_n + (1-\lambda)b'_n)(\overline{z}^n - a^{n-1}z)$$

satisfies $f_\alpha(a) = a$ and

$$\sum_{n=2}^{\infty}(n - a^{n-1})(\lambda a_n + (1-\lambda)a'_n + \lambda b_n + (1-\lambda)b'_n) \leq 1.$$

Recall that a function is an extremal point of a family $H^*(a)$ if and only if it can't be written as a proper convex combination of different functions which belong to $H^*(a)$.

We now determine extreme points for $H^*(a)$.

Theorem 2. *The extreme points of $H^*(a)$ are $h_1(z) = z$ and functions of the form $h_n(z) = \frac{n}{n-a^{n-1}}z - \frac{1}{n-a^{n-1}}z^n$ or $g_n(z) = \frac{n}{n-a^{n-1}}z - \frac{1}{n-a^{n-1}}\overline{z}^n$ for $n = 2, 3, \ldots$.*

Proof. Let $h_1(z) = z$, $h_n(z) = \frac{n}{n-a^{n-1}}z - \frac{1}{n-a^{n-1}}z^n$ and $g_n(z) = \frac{n}{n-a^{n-1}}z - \frac{1}{n-a^{n-1}}\overline{z}^n$ for $n \geq 2$. We show that every convex combination of functions of the form $\lambda_1 h_1(z) + \sum\limits_{n=2}^{\infty}(\lambda_n h_n(z) + \gamma_n g_n(z))$ where $\lambda_n, \gamma_n \geq 0$, $n \geq 2$ and $\lambda_1 = 1 - \sum\limits_{n=2}^{\infty}(\lambda_n + \gamma_n) \geq 0$ is a function from the family $H^*(a)$. Let

$$
\begin{aligned}
f(z) &= \lambda_1 h_1(z) + \sum_{n=2}^{\infty}(\lambda_n h_n(z) + \gamma_n g_n(z)) \\
&= z - z\sum_{n=2}^{\infty}\lambda_n - z\sum_{n=2}^{\infty}\gamma_n \\
&\quad + \sum_{n=2}^{\infty}\left[\lambda_n\left(\frac{n}{n-a^{n-1}}z - \frac{1}{n-a^{n-1}}z^n\right) + \gamma_n\left(\frac{n}{n-a^{n-1}}z - \frac{1}{n-a^{n-1}}\overline{z}^n\right)\right] \\
&= z + z\sum_{n=2}^{\infty}\lambda_n\frac{a^{n-1}}{n-a^{n-1}} - \sum_{n=2}^{\infty}\lambda_n\frac{1}{n-a^{n-1}}z^n \\
&\quad + z\sum_{n=2}^{\infty}\gamma_n\frac{a^{n-1}}{n-a^{n-1}} - \sum_{n=2}^{\infty}\gamma_n\frac{1}{n-a^{n-1}}\overline{z}^n \\
&= z - \sum_{n=2}^{\infty}\lambda_n\frac{1}{n-a^{n-1}}(z^n - a^{n-1}z) - \sum_{n=2}^{\infty}\gamma_n\frac{1}{n-a^{n-1}}(\overline{z}^n - a^{n-1}z).
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\sum_{n=2}^{\infty}(a_n + b_n)(n - a^{n-1}) &= \sum_{n=2}^{\infty}\left(\frac{\lambda_n}{n-a^{n-1}} + \frac{\gamma_n}{n-a^{n-1}}\right)(n - a^{n-1}) \\
&= \sum_{n=2}^{\infty}(\lambda_n + \gamma_n) = 1 - \lambda_1 \leq 1
\end{aligned}
$$

and $f(z) \in H^*(a)$. Conversely if $f \in H^*(a)$ then $a_n \leq \frac{1}{n-a^{n-1}}$ and $b_n \leq \frac{1}{n-a^{n-1}}$. Let $\lambda_n = a_n(n - a^{n-1})$, $\gamma_n = b_n(n - a^{n-1})$, $n \geq 2$ and $\lambda_1 =$

$1 - \sum\limits_{n=2}^{\infty} (\lambda_n + \gamma_n)$. Then

$$\begin{aligned}
f(z) &= z - \sum_{n=2}^{\infty} a_n(z^n - a^{n-1}z) - \sum_{n=2}^{\infty} b_n(\overline{z}^n - a^{n-1}z) \\
&= z - \sum_{n=2}^{\infty} \frac{\lambda_n}{n - a^{n-1}}(z^n - a^{n-1}z) - \sum_{n=2}^{\infty} \frac{\gamma_n}{n - a^{n-1}}(\overline{z}^n - a^{n-1}z) \\
&= z\lambda_1 + z\sum_{n=2}^{\infty} \lambda_n \frac{n}{n - a^{n-1}} + z\sum_{n=2}^{\infty} \gamma_n \frac{n}{n - a^{n-1}} \\
&\quad - \sum_{n=2}^{\infty} \frac{\lambda_n}{n - a^{n-1}} z^n - \sum_{n=2}^{\infty} \frac{\gamma_n}{n - a^{n-1}} \overline{z}^n \\
&= z\lambda_1 + \sum_{n=2}^{\infty} \lambda_n \left(\frac{n}{n - a^{n-1}} z - \frac{1}{n - a^{n-1}} z^n \right) \\
&\quad + \sum_{n=2}^{\infty} \gamma_n \left(\frac{n}{n - a^{n-1}} z - \frac{1}{n - a^{n-1}} \overline{z}^n \right) \\
&= z\lambda_1 + \sum_{n=2}^{\infty} (\lambda_n h_n(z) + \gamma_n g_n(z)) . \quad \blacksquare
\end{aligned}$$

In the sequel we shall need the following

Lemma. *For $0 < a \le x < 1$ there holds*

$$\frac{x^{n-1} - a^{n-1}}{x - a} \le \frac{n - a^{n-1}}{2 - a}.$$

Proof. The inequality can be written in the form

$$x^{n-1}(2 - a) + x(a^{n-1} - n) - 2a^{n-1} + na \le 0$$

for $x \ge a$. Denote the left-hand side of this inequality by

$$\varphi(x) = x^{n-1}(2 - a) + x(a^{n-1} - n) - 2a^{n-1} + na.$$

We find that

$$\varphi''(x) = (2 - a)(n - 1)(n - 2)x^{n-3} \ge 0$$

which shows that $\varphi(x)$ is convex for $x \ge a$. Since

$$\varphi(a) = a^{n-1}(2 - a) + a(a^{n-1} - n) - 2a^{n-1} + na = 0$$

it is sufficient to show that $\varphi(1) \le 0$. To do this we bring the inequality

$$\varphi(1) = 2 - a + a^{n-1} - n - 2a^{n-1} + na \le 0$$

to the form

$$n \geq \frac{2 - a^{n-1} - a}{1 - a} = 1 + (1 + a + a^2 + \ldots + a^{n-2}).$$

It is obvious that the maximal value of the right-hand side is at most n which ends the proof. ■

The class $H^*(a)$ does not posses rational symmetry, that is if $f(z) \in H^*(a)$ then $e^{i\alpha} f(e^{-i\alpha} z) \notin H^*(a)$. It would be interesting to determinate the so called Koebe set for $H^*(a)$ i.e. the set $\bigcap_{f \in H^*(a)} f(\mathbb{D})$. Unfortunately we are able to get a partial result only.

THEOREM 3. *The Koebe set of $H^*(a)$ is a connected set and there holds the following inclusion*

$$\left\{ w : |w| \leq \frac{1}{2-a} \right\} \cup \left\{ w : |w - a| \leq \frac{(1-a)^2}{2-a} \right\} \subset \bigcap_{f \in H^*(a)} f(\mathbb{D}).$$

Proof. If $w_1 \in \bigcap f(\mathbb{D})$ then in view of starlikeness of f the segment $[0, w_1]$ is contained in $f(\mathbb{D})$ for each $f \in H^*(a)$. It follows, that $[0, w_1] \in \bigcap f(\mathbb{D})$. Hence, the set $\bigcap f(\mathbb{D})$ is connected and starshaped with respect to the origin. Observe that

$$\begin{aligned} |f(z)| &= \left| z - \sum_{n=2}^{\infty} a_n z^n + \sum_{n=2}^{\infty} a_n a^{n-1} z - \sum_{n=2}^{\infty} b_n \bar{z}^n + \sum_{n=2}^{\infty} b_n a^{n-1} z \right| \\ &\geq |z| \Big(1 + \sum_{n=2}^{\infty} a_n a^{n-1} + \sum_{n=2}^{\infty} b_n a^{n-1} - \sum_{n=2}^{\infty} a_n |z|^{n-1} - \sum_{n=2}^{\infty} b_n |z|^{n-1} \Big) \\ &= |z| \Big(1 + \sum_{n=2}^{\infty} (a_n + b_n) a^{n-1} - \sum_{n=2}^{\infty} (a_n + b_n) |z|^{n-1} \Big). \end{aligned}$$

Taking $z = e^{i\theta}$ for $0 \leq \theta < 2\theta$ and applying the lemma with $x = 1$ we get

$$\begin{aligned} |f(e^{i\theta})| &\geq 1 - \sum_{n=2}^{\infty} (a_n + b_n)(1 - a^{n-1}) \\ &\geq 1 - \left(\frac{1-a}{2-a} \right) \sum_{n=2}^{\infty} (a_n + b_n)(n - a^{n-1}) \geq 1 - \frac{1-a}{2-a} = \frac{1}{2-a}. \end{aligned}$$

Next

$$\begin{aligned} |f(z) - a| = \Big| (z - a) &- \sum_{n=2}^{\infty} a_n (z^n - a^{n-1} z) \\ &+ \sum_{n=2}^{\infty} b_n a^{n-1} z - \sum_{n=2}^{\infty} b_n a^n + \sum_{n=2}^{\infty} b_n a^n - \sum_{n=2}^{\infty} b_n \bar{z}^n \Big| \end{aligned}$$

$$= \Big|(z-a) - \sum_{n=2}^{\infty} a_n(z^n - a^{n-1}z) + \sum_{n=2}^{\infty} b_n a^{n-1}(z-a) - \sum_{n=2}^{\infty} b_n(\overline{z}^n - a^n)\Big|$$

$$\geq |z-a|\bigg(\Big|1 - z\sum_{n=2}^{\infty} a_n \frac{z^{n-1}-a^{n-1}}{z-a} + \sum_{n=2}^{\infty} b_n a^{n-1}\Big| - \sum_{n=2}^{\infty} b_n \Big|\frac{\overline{z}^n - a^n}{z-a}\Big|\bigg)$$

$$= |z-a|\Big(\Big|1 - z\sum_{n=2}^{\infty} a_n(z^{n-2} + \ldots + a^{n-2}) + \sum_{n=2}^{\infty} b_n a^{n-1}\Big|\Big)$$

$$- \sum_{n=2}^{\infty} b_n(|z|^{n-1} + \ldots + a^{n-1})$$

$$\geq \big||z| - |a|\big|\Big(1 + \sum_{n=2}^{\infty} b_n a^{n-1} - |z|\sum_{n=2}^{\infty} a_n(|z|^{n-2} + \ldots + a^{n-2})$$

$$- \sum_{n=2}^{\infty} b_n(|z|^{n-1} + \ldots + a^{n-1})\Big).$$

Setting $|z| = 1$ we get

$$|f(z) - a| \geq (1-a)\Big(1 + \sum_{n=2}^{\infty} b_n a^{n-1} - \sum_{n=2}^{\infty} a_n(1 + a + \ldots + a^{n-2})$$

$$- \sum_{n=2}^{\infty} b_n(1 + a + \ldots + a^{n-1})\Big)$$

$$= (1-a)\Big(1 + \sum_{n=2}^{\infty} b_n a^{n-1} - \sum_{n=2}^{\infty} a_n(1 + a + \ldots + a^{n-2})$$

$$- \sum_{n=2}^{\infty} b_n(1 + a + \ldots + a^{n-2}) - \sum_{n=2}^{\infty} b_n a^{n-1}\Big) =$$

$$(1-a)\Big(1 - \sum_{n=2}^{\infty}(a_n + b_n)(1 + a + \ldots + a^{n-2})\Big)$$

$$= (1-a)\Big(1 - \sum_{n=2}^{\infty}(a_n + b_n)\frac{1-a^{n-1}}{1-a}\Big)$$

$$\geq (1-a)\Big(1 - \sum_{n=2}^{\infty}(a_n + b_n)\frac{n-a^{n-1}}{2-a}\Big)$$

$$\geq (1-a)\Big(1 - \frac{1}{2-a}\sum_{n=2}^{\infty}(a_n + b_n)(n - a^{n-1})\Big) \geq \frac{(1-a)^2}{2-a}$$

and the result follows. ∎

Our next theorem gives the distortion bounds for $f \in H^*(a)$.

THEOREM 4. *If* $f(z) \in H^*(a)$, *then*

$$|f(z)| \geq |z|, \quad for \quad |z| \leq a,$$

$$|f(z)| \geq \frac{2}{2-a}|z| - \frac{1}{2-a}|z|^2, \quad for \quad |z| \geq a,$$

$$|f(z)| \leq \frac{2}{2-a}|z| + \frac{1}{2-a}|z|^2, \quad for \quad |z| < 1.$$

Proof. By the proof of the previous theorem we know that

$$|f(z)| \geq |z|\Big(1 + \sum_{n=2}^{\infty}(a_n + b_n)a^{n-1} - \sum_{n=2}^{\infty}(a_n + b_n)|z|^{n-1}\Big).$$

Since $-|z| \geq -a$, hence

$$|f(z)| \geq |z|\Big(1 + \sum_{n=2}^{\infty}(a_n + b_n)a^{n-1} - \sum_{n=2}^{\infty}(a_n + b_n)a^{n-1}\Big) = |z|.$$

For $|z| \geq a$ using the inequality from the lemma we obtain

$$\begin{aligned}
|f(z)| &\geq |z|\Big(1 - \sum_{n=2}^{\infty}(a_n + b_n)(|z|^{n-1} - a^{n-1})\Big) \\
&\geq |z|\left(1 - \frac{|z| - a}{2-a}\sum_{n=2}^{\infty}(a_n + b_n)(n - a^{n-1})\right) \\
&\geq |z|\left(1 - \frac{|z| - a}{2-a}\right) = \frac{2}{2-a}|z| - \frac{1}{2-a}|z|^2.
\end{aligned}$$

Equality holds for functions $f(z) = z$, $f_2(z) = \frac{2z-z^2}{2-a}$ and $g_2(z) = \frac{2z-\bar{z}^2}{2-a}$ and positive values of z.

To obtain the last inequality we use the triangle inequality and few easy facts. This completes the proof. ■

Now we determine the maximal value of the modulus on the unit circle.

THEOREM 5. *If* $f(z) \in H^*(a)$, *then*

$$\max_{z=e^{i\theta}} |f(z)| = \frac{3}{2-a}.$$

Proof. Setting $z = e^{i\theta}$ for $f(z) \in H^*(a)$ we obtain

$$\begin{aligned}
|f(e^{i\theta})| &\le 1 + \sum_{n=2}^{\infty} a_n \frac{|e^{in\theta} - a^{n-1}e^{i\theta}|}{n - a^{n-1}}(n - a^{n-1}) \\
&\quad + \sum_{n=2}^{\infty} b_n \frac{|e^{-in\theta} - a^{n-1}e^{i\theta}|}{n - a^{n-1}}(n - a^{n-1}) \\
&\le 1 + \frac{1}{2-a}\sum_{n=2}^{\infty} a_n(1 + a^{n-1})(n - a^{n-1}) \\
&\quad + \frac{1}{2-a}\sum_{n=2}^{\infty} b_n(1 + a^{n-1})(n - a^{n-1}) \\
&\le 1 + \frac{1+a}{2-a}\sum_{n=2}^{\infty}(a_n + b_n)(n - a^{n-1}) \le 1 + \frac{1+a}{2-a} = \frac{3}{2-a}.
\end{aligned}$$

Equality holds for functions $f_2(z) = \frac{2z-z^2}{2-a}$ and $g_2(z) = \frac{2z-\bar{z}^2}{2-a}$. ∎

In 1956 Z. Lewandowski ([5]) considered the class $\mathcal{A}$ consisting of all polynomials of the form

$$f(z) = z - \sum_{n=2}^{N} a_n z^n, \quad a_n \ge 0$$

that satisfy the condition

$$\sum_{n=2}^{N} n a_n = 1.$$

He proved the following

Theorem 6. *Let d_0 and d^* denote radii of largest disks centered at the origin covered by images of circles $|z| \le r_0$, where r_0 is the radius of convexity of function $f(z) \in \mathcal{A}$, and $|z| < 1$, respectively, then for each $f(z) \in \mathcal{A}$ there holds*

$$\frac{d_0}{d^*} \ge \frac{3}{4}$$

and the constant is best possible.

In this way he settled a conjecture of A. Schild ([6]). A few years later Gray and Schild ([4]) gave a new and much shorter proof of this conjecture. H. Silverman ([7]) remarked that the above statement is valid for the whole class of functions having the above form and satisfying the condition

$$\sum_{n=2}^{\infty} n a_n \le 1.$$

Here we extend this for a class of harmonic mappings.

Let TH^* denote the class of harmonic mappings of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n - \sum_{n=2}^{\infty} b_n \overline{z}^n$$

that satisfy the conditions a_n, $b_n \geq 0$, $b_1 = 0$. Suppose that $f(z)$ satisfies

$$\sum_{n=2}^{\infty} n(a_n + b_n) \leq 1. \tag{4}$$

They are univalent in $\mathbb{D}$, not univalent in any larger disk centered at $z = 0$ and they map the unit disk onto domains starlike with respect to the origin. For such functions we have

$$|f(z)| = \left| z - \sum_{n=2}^{\infty} a_n z^n - \sum_{n=2}^{\infty} b_n \overline{z}^n \right| \geq |z| - \sum_{n=2}^{\infty} a_n |z|^n - \sum_{n=2}^{\infty} b_n |z|^n = f(|z|)$$

and

$$d_0 = f(r) - f(0) = f(r),$$
$$d^* = f(1) - f(0) = f(1),$$

where r satisfy the equality

$$\frac{\partial}{\partial \theta}\left(\arg\left(\frac{\partial}{\partial \theta} f\left(re^{i\theta}\right)\right)\right)$$
$$= \mathfrak{Re}\left(1 + \frac{-\sum_{n=2}^{\infty} a_n r^n n(n-1)e^{i(n-1)\theta} - \sum_{n=2}^{\infty} b_n r^n n(n+1)e^{-i(n+1)\theta}}{r - \sum_{n=2}^{\infty} a_n r^n n e^{i(n-1)\theta} + \sum_{n=2}^{\infty} b_n r^n n e^{-i(n+1)\theta}}\right) = 0.$$

For $\theta = 0$ we have

$$1 + \frac{-\sum_{n=2}^{\infty} a_n r^n n(n-1) - \sum_{n=2}^{\infty} b_n r^n n(n+1)}{r - \sum_{n=2}^{\infty} a_n r^n n + \sum_{n=2}^{\infty} b_n r^n n} = 0.$$

This is equivalent to

$$r - \sum_{n=2}^{\infty} a_n r^n n + \sum_{n=2}^{\infty} b_n r^n n - \sum_{n=2}^{\infty} a_n r^n (n^2 - n) - \sum_{n=2}^{\infty} b_n r^n (n^2 + n) = 0.$$

Hence

$$\sum_{n=2}^{\infty} (a_n + b_n) n^2 r^{n-1} = 1. \tag{5}$$

We prove the Schild's Conjecture for the family of harmonic mappings. Therefore for $f \in TH^*$,

$$\frac{d_0}{d^*} = \frac{f(r)}{f(1)} = \frac{r - \sum_{n=2}^{\infty}(a_n + b_n)r^n}{1 - \sum_{n=2}^{\infty}(a_n + b_n)} \geq \frac{3}{4}.$$

Since expressions in the nominator and the denominator are positive, the inequality $\frac{d_0}{d^*} \geq \frac{3}{4}$ is equivalent to

$$(6)\quad \varphi(\{a_n\}, \{b_n\}, r) = 4r - 4\sum_{n=2}^{\infty}(a_n + b_n)r^n - 3 + 3\sum_{n=2}^{\infty}(a_n + b_n) \geq 0.$$

In view of (4) and (5), if the function φ is nonnegative, then $\frac{d_0}{d^*} \geq \frac{3}{4}$. Because φ is increasing with respect to the variable r and the radius of convexity for the family TH^* equals $\frac{1}{2}$, it is sufficient to justify (6) for $r \in [\frac{1}{2}, 1)$. For $r = \frac{1}{2}$ we have

$$\begin{aligned}\varphi\left(\frac{1}{2}\right) &= 2 - 4\sum_{n=2}^{\infty}(a_n + b_n)\left(\frac{1}{2}\right)^n - 3 + 3\sum_{n=2}^{\infty}(a_n + b_n)\\ &= 3\sum_{n=2}^{\infty}(a_n + b_n) - 4\sum_{n=2}^{\infty}(a_n + b_n)\left(\frac{1}{2}\right)^n - 1.\end{aligned}$$

Due to

$$\sum_{n=2}^{\infty}(a_n + b_n)n^2\left(\frac{1}{2}\right)^{n-1} = 1,$$

we have

$$\begin{aligned}\varphi\left(\frac{1}{2}\right) &= 3\sum_{n=2}^{\infty}(a_n + b_n) - 4\sum_{n=2}^{\infty}(a_n + b_n)\left(\frac{1}{2}\right)^n - \sum_{n=2}^{\infty}(a_n + b_n)n^2\left(\frac{1}{2}\right)^{n-1}\\ &= \sum_{n=2}^{\infty}(a_n + b_n)\left(3 - 4\left(\frac{1}{2}\right)^n - n^2\left(\frac{1}{2}\right)^{n-1}\right)\\ &= \sum_{n=2}^{\infty}(a_n + b_n)\left(3 - 4\left(\frac{1}{2}\right)^n - 2n^2\left(\frac{1}{2}\right)^n\right).\end{aligned}$$

Since $a_n + b_n \geq 0$ and $3 - \left(\frac{1}{2}\right)^n(4 + 2n^2) \geq 0$ for $n \geq 2$ ([4]) we conclude that $\varphi\left(\frac{1}{2}\right) \geq 0$ which ends the proof.

The extremal functions of the class TH^* is the family of functions

$$f^*(z) = z - a_2 z^2 - \left(\frac{1}{2} - a_2\right)\bar{z}^2, \quad 0 \leq a_2 \leq \frac{1}{2}.$$

Notice that for $a_2 = 0$ we get $f^*(z) = z - \frac{1}{2}\bar{z}^2$ and for $a_2 = \frac{1}{2}$ we have $f^*(z) = z - \frac{1}{2}z^2$. It is obvious that $\frac{d_0}{d^*} = \frac{3}{4}$ for $f^*(z)$.

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References

[1] Y. Avci, E. Złotkiewicz, *On harmonic univalent mappings*, Ann. Univ. Mariae Curie-Skłodowska Sect. A 44 (1990), 1–7.
[2] J Clunie, T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A I. Math. 9 (1984), 3–25.
[3] P. Duren, *Harmonic Mappings*, Cambridge University Press, 2004.
[4] E. Gray, A. Schild, *A new proof of a conjecture of Schild*, Proc. Amer. Math. Soc. 16 (1963), 76–77.
[5] Z. Lewandowski, *Nouvelles remarques sur les theoremes de Schild relatifs a une classe de fonctions univalentes (Demonstration d'une hypothese de Schild)*, Ann. Univ. Mariae Curie-Skłodowska Sect. A 8 (1956), 81–94.
[6] A. Schild, *On a class od functions schlicht in the unit circle*, Proc. Amer. Math. Soc. 5 (1953), 115–120.
[7] H. Silverman, *Survey with open problems on univalent functions whose coefficients are negative*, Rocky Mountain J. Math. 3 (1991), 1099–1125.

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