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AN EMBEDDING THEOREM FOR UNBOUNDED CONVEX SETS IN A BANACH SPACE

Abstract. Let V be a closed convex cone and C_V be a space of nonempty closed convex subsets of a Banach space such that their Hausdorff distance form the cone V is finite. In this paper we embed the space C_V isometrically and isomorphically into a Banach space.

1. Introduction

The aim of this paper is to obtain an embedding theorem for a certain class of closed convex subsets of a Banach space. Since that class does not have a linear space structure, embedding theorem is a method of defining differentiation and integration of multivalued functions with both bounded and unbounded images.

In 1952 Rådström [2] showed embedding theorem for a class of closed convex bounded subsets of a normed space. Before we recall this theorem let us quote the following definition.

Definition 1. Commutative semigroup S with zero is called an abstract cone if there is a bilinear mapping

$$m : \mathbb{R}_+ \times S \ni (\lambda, s) \to \lambda \cdot s \in S,$$

such that $1 \cdot s = s \ \forall s \in S$.

Theorem 1. [2] *Let S be an abstract convex cone with cancellation law. Let d be a metric on S such that:*

1. $\forall x, y, z \in S : d(x + z, y + z) = d(x, y)$;
2. $\forall x, y \in S, \ s \geqslant 0 : d(sx, sy) = sd(x, y)$;

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3. *balls* $\{x : d(x,0) < \epsilon\}$ *are absorbing.*

Then S can be embedded isometrically and isomorphically as a convex cone into a normed space.

Obviously a class of closed convex bounded subsets of a normed space is an abstract convex cone.

Drewnowski [1] generalized this theorem for locally convex spaces and Urbański [5] for topological-linear spaces.

However the class of nonempty closed convex subsets of a Banach space cannot be embeded into a Banach space. Indeed assume there is an injective embedding f. A nonempty closed convex cone $K \neq \{0\}$ is an element of this class and $K + K = K$. Hence $f(K) = f(K) + f(K) \neq 0$. Then there must be an inverse element l of $f(K)$. It follows $f(K) = f(K) + f(K) + l = f(K) + l = 0$, a contradiction.

Although a class of closed convex subsets of a normed space is not a linear space nor can it be embedded into a linear space, there are subclasses for which the embedding theorems occur. In 1973 Robinson [3, Theorem 3] showed that the class of closed unbounded convex subsets of $\mathbb{R}^n$ sharing the same recession cone and endowed with the Hausdorff distance d can be embedded isometrically into a real vector space. An embedding into a normed linear space was not possible, because of the lack of continuity of nonnegative scalar multiplication [3, Theorem 2]. To eliminate this disadvantage Robinson introduced a subclass C_V consisting of all nonempty closed unbounded convex subsets of $\mathbb{R}^n$ such that their Hausdorff distance form the cone V is finite. From [3, Theorem 3] it follows:

Theorem. *The class C_V, metrized by d, can be embedded isometrically as a convex cone in a normed linear space.*

S. Robinson proved this theorem using finite dimensional methods. In this paper we generalize his result to infinite dimensional Banach spaces using linear functionals. The main tools are lemmas 2 and 3.

This article is organized as follows. First we introduce a space C_V as a class of nonempty closed convex subset of a Banach space, having finite distance from a certain convex cone. This space endowed with Hausdorff distance is a complete metric space. Next we define operations, addition of two element of C_V and multiplication of an element of C_V by a nonnegative scalar. Subsequently we prove that the cancellation law, metric's positive homogeneity and invariance under translation holds. As a consequence, we are finally able to embed C_V isometrically and isomorphically as a cone into a Banach space.

2. Properties of the space C_V

Let X be a Banach space with the norm $\|\cdot\|$, V a closed convex cone in X (where by a cone we understand a subset of X such that $\alpha x \in X$ for $x \in V$, $\alpha \in \mathbb{R}_+$). By d we understand a metric on X generated by the norm $\|\cdot\|$. Let A, B be nonempty closed convex subsets of X. By A_r we denote $\{x \in X : d(x, A) \leqslant r\}$. We define

$$\rho(A, B) := \inf\{r : A \subset B_r\}.$$

Now we can recall the definition of a Hausdorff distance [3]

$$d_H(A, B) := \max\{\rho(A, B), \rho(B, A)\}.$$

By C_V we understand the class of all nonempty closed convex subsets of X such that their Hausdorff distance form V is finite. The space C_V endowed with the Hausdorff distance is a complete metric space.

The following operations in C_V are well-known:

$$\forall\ W_1,\ W_2 \in C_V : W_1 \dot{+} W_2 := \operatorname{cl}\{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\},$$
$$\forall\ \alpha > 0,\ W \in C_V : \alpha \cdot W := \{\alpha w : w \in W\}.$$

If $\alpha = 0$ then we additionally define

$$\alpha \cdot W := V.$$

We see that $(C_V, \dot{+})$, with cone V as an zero element, is an abstract convex cone. One can easily verify that the scalar multiplication is continuous, see for example [3, Theorem 2].

We denote by V' the family of all $\xi \in X' = \{\zeta : X \to \mathbb{R} : \zeta$ linear and continuous $\}$ such that $\sup\{\xi(v) : v \in V\} < \infty$.

For $\xi \in V'$ and $W \in C_V$ we define

$$\xi(W) := \sup\{\xi(w) : w \in W\}.$$

Observe that $\xi(V) = 0$.

Lemma 1. *Let W, W_1, $W_2 \in C_V$, $\xi \in V'$, $\alpha \geqslant 0$. Then*

$$\xi(W_1 \dot{+} W_2) = \xi(W_1) + \xi(W_2)$$

and

$$\xi(\alpha \cdot W) = \alpha \cdot \xi(W).$$

Proof. 1. First we prove that $\xi(W_1 \dot{+} W_2) = \xi(W_1) + \xi(W_2)$.

$$\begin{aligned}\xi(W_1 + W_2) &= \sup\{\xi(x) : x = w_1 + w_2, w_1 \in W_1, w_2 \in W_2\}\\ &= \sup\{\xi(w_1 + w_2) : w_1 \in W_1, w_2 \in W_2\}\\ &= \sup\{\xi(w_1) + \xi(w_2) : w_1 \in W_1, w_2 \in W_2\}\\ &= \sup\{\xi(w_1) : w_1 \in W_1\} + \sup\{\xi(w_2) : w_2 \in W_2\}\\ &= \xi(W_1) + \xi(W_2).\end{aligned}$$

The fact that ξ is continuous implies

$$\sup\{\xi(x) : x \in W_1 \dot{+} W_2\} = \sup\{\xi(x) : x \in W_1 + W_2\}.$$

We have proven that

$$\xi(W_1 \dot{+} W_2) = \xi(W_1 + W_2) = \xi(W_1) + \xi(W_2).$$

2. Now we will show that $\xi(\alpha \cdot W) = \alpha \cdot \xi(W)$. If $\alpha > 0$ then:

$$\xi(\alpha \cdot W) = \sup\{\xi(\alpha x) : x \in W\} = \sup\{\alpha \cdot \xi(x) : x \in W\} = \alpha \cdot \xi(W).$$

If $\alpha = 0$ then:

$$\xi(\alpha \cdot W) = \xi(V) = 0 = \alpha \cdot \xi(W). \ \blacksquare$$

Now we want to show that the cancellation law holds for addition of elements of C_V. We need the following lemma.

Lemma 2. *Let W_1, $W_2 \in C_V$. Then $W_1 \subset W_2$ iff*

$$\xi(W_1) \leqslant \xi(W_2) \quad \textit{for all}\ \xi \in V'.$$

As a consequence $W_1 = W_2$ iff

$$\xi(W_1) = \xi(W_2) \quad \textit{for all}\ \xi \in V'.$$

Proof. ($\Rightarrow$) Obviously

$$\sup\{\xi(x) : x \in W_1\} \leqslant \sup\{\xi(x) : x \in W_2\} \text{ for all } \xi \in V'.$$

($\Leftarrow$) Suppose that $W_1 \not\subset W_2$. Then there is a point $x_0 \in W_1$ such that $x_0 \notin W_2$. We know that W_2 is convex and closed whereas $\{x_0\}$ is convex and compact. Then, by the geometric version of Hahn–Banach Theorem [4, Theorem 3.4] there exists $\zeta \in X'$ such that

$$\sup\{\zeta(y) : y \in W_2\} < \zeta(x_0).$$

The above ineqality implies that $\sup\{\zeta(y) : y \in V\} < \infty$ so $\zeta \in V'$. Therefore

$$\zeta(W_2) = \sup\{\zeta(y) : y \in W_2\} < \zeta(x_0) \leqslant \sup\{\zeta(y) : y \in W_1\} = \zeta(W_1).$$

We have a contradiction. $\blacksquare$

As a consequence we obtain.

Proposition 1. *Let W_1, W_2, V_1, $V_2 \in C_V$, $V_2 \subset V_1$. If $W_1 + V_1 \subset W_2 + V_2$ then*

$$W_1 \subset W_2.$$

Proof. Let $\xi \in V'$. Then

$$\xi(W_1) + \xi(V_1) = \xi(W_1 + V_1) \leqslant \xi(W_2 + V_2) = \xi(W_2) + \xi(V_2).$$

Since $\xi(V_2) \leqslant \xi(V_1)$, we have that

$$\xi(W_1) - \xi(W_2) \leqslant \xi(V_2) - \xi(V_1) \leqslant 0.$$

Thus $W_1 \subset W_2$. ■

The cancellation law is a particular case of the last result.

Proposition 2 (Cancellation law). *If $W_1 + W = W_2 + W$, then*

$$W_1 = W_2.$$

The next proposition states that Hausdorff metric is invariant under translation. First we have to prove the following lemma.

Lemma 3. *Let W_1, $W_2 \in C_V$. Then*

$$d_H(W_1, W_2) = \sup\,\{|\xi(W_1) - \xi(W_2)| : \xi \in V', \|\xi\| = 1\}.$$

Proof. We first show that

$$(1) \qquad d_H(W_1, W_2) \geq \sup\,\{|\xi(W_1) - \xi(W_2)| : \xi \in V', \|\xi\| = 1\}.$$

Let $d_H(W_1, W_2) = r$. By the definition of Hausdorff metric we have that

$$W_1 \subset (W_2)_r \quad W_2 \subset (W_1)_r.$$

Then, for $\xi \in X'$, such that $\|\xi\| = 1$, we can estimate the value of $\xi((W_2)_r)$, by the following

$$\begin{aligned}\xi((W_2)_r) &= \sup\{\xi(w + rv) : w \in W_2, \|v\| \leqslant 1\}\\ &\leqslant \sup\{\xi(w) : w \in W_2\} + r \cdot \sup\{\xi(v) : \|v\| \leqslant 1\} = \xi(W_2) + r.\end{aligned}$$

Thus

$$\xi(W_1) - \xi(W_2) \leqslant \xi((W_2)_r) - \xi(W_2) \leqslant r.$$

Similarly we can prove that

$$\xi(W_2) - \xi(W_1) \leqslant r.$$

As a consequence we get

$$|\xi(W_1) - \xi(W_2)| \leqslant r \text{ for all } \xi \in \mathrm{V}', \|\xi\| = 1,$$

which yields (1).

To prove the opposite inequality let $\varepsilon > 0$ be arbitrary. Let $w \in W_2$ be such that

$$d(w, W_1) \geqslant d_H(W_1, W_2) - \varepsilon.$$

We denote $s := d(w, W_1)$. Then by the geometric version of Hahn–Banach Theorem [4, Theorem 3.4] we have that there exists a nontrivial functional $\xi \in X'$, such that

$$\sup\{\xi(x) : x \in W_1\} \leqslant \inf\{\xi(y) : y \in K(w, s)\}.$$

The above ineqality implies that $\sup\{\xi(y) : y \in V\} < \infty$ so $\xi \in V'$. We denote

$$\zeta(x) := \frac{1}{\|\xi\|}\xi(x) \quad \text{for all } x \in X.$$

Then functional $\zeta \in V'$ is of norm 1 and

$$\sup\{\zeta(x) : x \in W_1\} \leqslant \inf\{\zeta(y) : y \in K(w,s)\} = \zeta(w) - s.$$

Thus

$$\begin{aligned} d_H(W_1, W_2) - \varepsilon \leqslant s \leqslant \zeta(w) - \sup\{\zeta(x) : x \in W_1\} \\ \leqslant \sup\{\zeta(y) : y \in W_2\} - \sup\{\zeta(x) : x \in W_1\} \\ = \zeta(W_1) - \zeta(W_2). \end{aligned}$$

By letting ε converge to 0 we obtain that

$$d_H(W_1, W_2) \leqslant \zeta(W_1) - \zeta(W_2).$$

Similarly we can prove that

$$d_H(W_1, W_2) \leqslant \zeta(W_2) - \zeta(W_1).$$

Therefore

$$d_H(W_1, W_2) \leqslant |\zeta(W_1) - \zeta(W_2)|.$$

Finally

$$d_H(W_1, W_2) \leqslant \sup\ \{|\xi(W_1) - \xi(W_2)| : \xi \in V', \|\xi\| = 1\}. \ \blacksquare$$

As a simple consequence of the previous lemma we obtain Hausdorff metric's invariance.

Proposition 3. *Let W, V_1, $V_2 \in C_V$. Then*

$$d_H(V_1 + W, V_2 + W) = d_H(V_1, V_2).$$

Additionally, by the previous lemma, we have these propositions

Proposition 4. *Let W_1, $W_2 \in C_V$, $\alpha \geqslant 0$. Then*

$$d_H(\alpha \cdot W_1, \alpha \cdot W_2) = \alpha \cdot d_H(W_1, W_2).$$

Proposition 5. *Let W_1, W_2, V_1, $V_2 \in C_V$. Then*

$$d_H(W_1 + V_1, W_2 + V_2) \leqslant d_H(W_1, W_2) + d_H(V_1, V_2).$$

Proof. We have

$$\begin{aligned} d_H(W_1, W_2) + d(V_1, V_2) &= d_H(W_1 + V_1, W_2 + V_1) + d_H(V_1 + W_2, V_2 + W_2) \\ &\geqslant d_H(W_1 + V_1, W_2 + V_2). \ \blacksquare \end{aligned}$$

To summarize:

Remark 1. C_V is a complete metric space; C_V is a commutative semigroup with cancellation law, V is a zero element; Hausdorff metric d_H is positively homogeneous and invariant under translation.

By the formerly mentioned result, we obtain the following

Theorem 2. *Let X be a Banach space, V a closed convex cone in X, d_H a Hausdorff distance. Let*

$$C_V = \{A \subset X \,:\, A \text{ nonempty closed and convex}, \, d_H(A, V) < \infty\}.$$

Then C_V can be embedded isometrically and isomorphically as a closed convex cone into a Banach space.

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