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ON PRESERVERS OF SINGULARITY AND NONSINGULARITY OF MATRICES

Abstract. Operators preserving singularity and nonsingularity of matrices were studied in paper of P. Botta under the assumption that operators are linear. In the present paper the linearity of operators is not assumed: we only assume that operators are of the form $T = (f_{i,j})$, where $f_{i,j} : K \longrightarrow K$ and K is a field, $i, j \in \{1, 2, \ldots, n\}$. If $n \geq 3$, then in the matrix space $M_n(K)$ operators preserving singularity and nonsingularity of matrices must be as in paper of P. Botta. If $n \leq 2$, operators may be nonlinear. In this case the forms of the operators are presented.

Let $\mathbb{R}, \mathbb{N}$ denote the set of real numbers or positive integer numbers, respectively. Let $M_n(K)$ be the set of $n \times n$ matrices over a field K, i.e. $M_n(K) \in K^{n \times n}$, where $n \in \mathbb{N}$. We denote by $E_{j,k}$ the matrix whose j, k entry is 1 and the remaining entries of which are 0.

First of all let us introduce

Definition 1. An operator T from $M_n(K)$ into itself is an operator preserving singularity of matrices from $M_n(K)$ if and only if for every singular matrix $A \in M_n(K)$ the matrix $T(A)$ is singular.

Definition 2. An operator T from $M_n(K)$ into itself is an operator preserving nonsingularity of matrices from $M_n(K)$ if and only if for every nonsingular matrix $A \in M_n(K)$ the matrix $T(A)$ is nonsingular.

Let S and NS denote the sets of singular and nonsingular matrices from $M_n(K)$, respectively.

In the paper we consider the operators T from $M_n(K)$ into itself of the form

$$(1) \qquad T = (f_{i,j}), \quad \text{where } f_{i,j} : K \longrightarrow K, \quad i, j = 1, 2, \ldots, n$$

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for any matrix $A \in M_n(K)$, where the matrix $T(A) := (f_{i,j}(a_{i,j}))$ for $i, j = 1, 2 \ldots, n$.

REMARK 1. In case of $n = 1$ an operator T of the form (1) is an operator preserving singularity and nonsingularity of matrices from $M_1(K)$ if and only if for $x \in K$ the equivalence $x = 0 \iff f_{1,1}(x) = 0$ holds.

Let us consider the case $n \geq 2$. We prove the following

LEMMA. *If an operator T of the form* (1), *from $M_n(K)$ into itself for $n \geq 2$, where K is a field, preserves singularity and nonsingularity of matrices in the space $M_n(K)$, then the equivalence*

$$x = 0 \iff f_{i,j}(x) = 0 \tag{2}$$

holds for all $x \in K$, $i, j \in \{1, 2, \ldots, n\}$.

Proof. Let us assume that $n \geq 2$. Let T be an operator preserving singularity and nonsingularity of matrices. Let indices $i, j \in \{1, 2, \ldots, n\}$ be arbitrary and fixed. Let us consider a permutation σ of the numbers $\{1, 2, \ldots, n\}$ such that $\sigma(i) = j$. Let us consider the generalized permutation matrix $B_1 \in NS$ with entries

$$b_{k,l} = \begin{cases} x & \text{for } k = i,\ l = j \\ 1 & \text{for } k \in \{1, 2, \ldots, n\}, k \neq i, l = \sigma(k) \\ 0 & \text{for other indices,} \end{cases}$$

for $x \neq 0$. From Laplace's formula with respect to the i-th row we have

$$\det T(B_1) = f_{i,j}(x) \cdot D_{i,j} + \sum_{k=1,\ k \neq j}^{n} f_{i,k}(0) \cdot D_{i,k} \neq 0, \tag{3}$$

where $D_{i,j}, D_{i,k} \in K$ are some coefficients.

Let $B_2 \in S$ be the matrix obtained from B_1 for $x = 0$. Then we have

$$\det T(B_2) = \sum_{k=1}^{n} f_{i,k}(0) \cdot D_{i,k} = 0 \tag{4}$$

with the same coefficients $D_{i,k}$. From the difference

$$\det T(B_1) - \det T(B_2) = (f_{i,j}(x) - f_{i,j}(0)) \cdot D_{i,j} \neq 0$$

we obtain

$$f_{i,j}(x) - f_{i,j}(0) \neq 0 \text{ for } x \neq 0. \tag{5}$$

Let $r \in \{1, 2, \ldots, n\}$, $r \neq j$. Let us consider the matrices $B_{3,r} = B_2 + xE_{i,r}$. Since in the matrices $B_{3,r}$ all entries in the j-th column are equal to zero, then $B_{3,r} \in S$. Using the Laplace's formula with respect to the i-th row we obtain $\det T(B_{3,r}) = f_{i,r}(x) \cdot D_{i,r} + \sum_{k=1,\ k\neq r}^{n} f_{i,k}(0) \cdot D_{i,k} = 0$.

Let us consider the difference $\det T(B_{3,r}) - \det T(B_2) = (f_{i,r}(x) - f_{i,r}(0)) \cdot D_{i,r} = 0$ for $r \neq j$. By (5) we obtain

$$D_{i,r} = 0 \text{ for } r \neq j. \tag{6}$$

By (3) and (6) we have $\det T(B_1) = f_{i,j}(x) \cdot D_{i,j} \neq 0$, we get $D_{i,j} \neq 0$.

By (4) and (6) we have $\det T(B_2) = f_{i,j}(0) \cdot D_{i,j} = 0$. As $D_{i,j} \neq 0$, then $f_{i,j}(0) = 0$. Then by (5) we have $f_{i,j}(x) \neq 0$ for $x \neq 0$. This completes the proof of Lemma. □

Theorem. (a) *If $n = 2$, then T of the form (1) preserves singularity and nonsingularity of matrices on $M_n(K)$ if and only if there exist nonzero $u_1, u_2, v_1, v_2 \in K$ and an injective function $g : K \longrightarrow K$ satisfying $g(0) = 0$ and $g(xy) = g(x)g(y)$ for all $x, y \in K$ such that $f_{i,j}(x) = u_i v_j g(x)$ for all $x \in K$.*
(b) *If $n \geq 3$, then T of the form (1) preserves the singularity and nonsingularity of matrices on $M_n(K)$ if and only if there are nonzero $u_1, u_2, \ldots, u_n$, $v_1, v_2, \ldots, v_n \in K$ and an injective function $g : K \longrightarrow K$ satisfying $g(xy) = g(x)g(y)$ and $g(x + y) = g(x) + g(y)$ for all $x, y \in K$ such that $f_{i,j}(x) = u_i v_j g(x)$ for all $x \in K$.*

Thus, for $n \geq 3$ the singularity and nonsingularity preserving maps T on $M_n(K)$ may be written in the form $T(A) = U[g(a_{i,j})]V$, where $U = \operatorname{diag}(u_1, u_2, \ldots, u_m)$ and $V = \operatorname{diag}(v_1, v_2, \ldots, v_n)$ are invertible diagonal matrices and g is an injective endomorphism of K. For $n = 2$, the additivity of g may be even reduced to the sole requirement that $g(0) = 0$. Note that the maps of part (a) may be nonlinear: for example, one can take $g(x) = x^3$.

Proof. Let $n \geq 2$ and suppose T is an operator of the form (1) preserving singularity and nonsingularity on $M_n(K)$. By Lemma $f_{i,j}(0) = 0$ and $f_{i,j}(x) \neq 0$ for all i, j and all $x \neq 0$. Denote $c_{i,j} = f_{i,j}(1)$. As $f_{i,j}(1) \neq 0$, we obtain that $c_{i,j} \neq 0$ for all i, j. Put $g_{i,j}(x) = c_{i,j}^{-1} f_{i,j}(x)$. Clearly, $g_{i,j}(0) = 0$ and $g_{i,j}(1) = 1$ for all i, j. Define the matrix $C = (c_{i,j})$. By Lemma rank $C \geq 1$. We prove, that rank $C = 1$.

In case $n = 2$, the rank $C = 1$, because C is the singular the matrix which all entries not equal to zero.

In case $n \geq 3$, suppose for contradiction that the rank $C = k > 1$. Without a loss of generality we may assume that the determinant of the upper-left submatrix of C of the order k is not equal to zero. Let us consider the matrix $B_4 = \sum_{i=1}^{k} \sum_{j=1}^{k} E_{i,j} + \sum_{l=k+1}^{n} E_{l,l}$. Note that matrix $B_4 \in S$. Then the matrix $T(B_4) = \sum_{i=1}^{k} \sum_{j=1}^{k} E_{i,j} c_{i,j} + \sum_{l=k+1}^{n} c_{l,l} E_{l,l}$. From properties of determinants det $F(B_4) \neq 0$ and $T(B_4) \in NS$. Then the rank C can not be greater than 1, it must be equal to one.

Then in cases $n = 2$ and $n \geq 3$ the equality rank $C = 1$ holds. This implies that there are $u_i, v_j \in K$ such that $f_{i,j}(1) = u_i v_j$ for all i, j.

For $1 \leq i \neq r \leq m$ and $1 \leq k \neq l \leq n$, let σ be a permutation of the set $\{1, 2, \ldots, n\}$ such that $\sigma(i) = k$ and $\sigma(r) = l$. We define $B_5 = xE_{i,l} + E_{r,k} + xE_{r,l} + \sum_{m=1, m\neq r} E_{n,\sigma(m)}$. Matrices $B_5, T(B_5) \in S$, we get

$$0 = f_{i,k}(1)f_{r,l}(x) - f_{r,k}(1)f_{i,l}(x) = \sum_{m=1}^{n} u_m v_m g_{r,l}(x) - \sum_{m=1}^{n} u_m v_m g_{i,l}(x),$$

that is $g_{r,l}(x) = g_{i,l}(x)$ for all $x \in K$. Consequently, the matrix $G = (g_{i,j})$ is constant along its column. Analogously, one can show that G is constant along the rows. This implies that all $g_{i,j}$ are one and the same function g, therefore $f_{i,j}(x) = u_i v_j g(x)$ for all i, j and all x.

To prove that g is injective, suppose $x \neq y$ and consider $B_6 = E_{1,1} + E_{1,2} + xE_{2,1} + yE_{2,2} + \sum_{n=3}^{n} E_{m,m}$. We have $B_6, T(B_6) \in NS$ and hence

$$0 \neq f_{1,1}(1)f_{2,2}(y) - f_{1,2}(1)f_{2,1}(x) = \sum_{m=1}^{n} u_m v_m g(y) - \sum_{m=1}^{n} u_m v_m g(x),$$

which implies that $g(x) \neq g(y)$, as desired.

To show that $g(xy) = g(x)g(y)$, take $B_7 = E_{1,1} + xE_{1,2} + yE_{2,1} + xyE_{2,1} + \sum_{n=3}^{n} E_{m,m}$. Since $B_7, T(B_7) \in S$, we obtain that

$$0 = f_{1,1}(1)f_{2,2}(xy) - f_{1,2}(x)f_{2,1}(y) = \sum_{m=1}^{n} u_m v_m g(xy) - \sum_{m=1}^{n} u_m v_m g(x)g(y),$$

so we arrive at the equality $g(xy) = g(x)g(y)$.

At this point we have proved the necessary condition on T part of (a). To get the necessary condition on T part of (b), assume $n \geq 3$ and consider

$$B_8 = xE_{1,1} + E_{1,2} + yE_{2,1} + E_{2,3} + (x+y)E_{3,1} + E_{3,2} + E_{3,3} + \sum_{m=4}^{n} E_{m,m}.$$

As $B_8 \in S$, we conclude that the determinant of $T(B_8)$ must be zero, which means that

$$0 = \sum_{m=1}^{n} u_m v_m (-g(x) - g(y) + g(xy)).$$

Thus, $g(x + y) = g(x) + g(y)$. The proof of the necessary condition on T part of (b) is also complete.

We now prove the sufficient condition on T parts (a) and (b). By Lemma T maps the zero matrix to itself. By the Theorem from [3] it follows that T is an operator preserving ranks of matrices from $M_n(K)$ in parts (a) and (b). Then it also preserves the singularity and nonsingularity of matrices from $M_n(K)$.

This completes the proof of the Theorem. □

In the paper [1] a similar problem is studied under the assumption that operator T is linear. Linear preservers problems and results are presented in [2].

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