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PLANAR PACKING OF CYCLES AND UNICYCLIC GRAPHS

Abstract. We say that a graph G is packable into a complete graph K_n if there are two edge-disjoint subgraphs of K_n both isomorphic to G. It is equivalent to the existence of a permutation σ of a vertex set in G such that if an edge xy belongs to $E(G)$, then $\sigma(x)\sigma(y)$ does not belong to $E(G)$. In 2002 Garcia et al. have shown that a non-star tree T is planary packable into a complete graph K_n.

In this paper we show that for any packable cycle C_n except of the case $n = 5$ and $n = 7$ there exists a planar packing into K_n. We also generalize this result to certain classes of unicyclic graphs.

1. Introduction

In this paper we use standard graph theory notation. We deal with finite, simple graphs without loops and multiple edges. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. The degree of a vertex x is denoted by $\deg(x)$. We say that G is *planar* if it can be drawn on a plane so that the vertices are located in distinct points and the edges are represented by nonintersecting segments of curves joining their endpoints. A *plane graph* is a planar graph with a fixed plane embedding.

Let G be a graph of order n. A *packing* of G into a complete graph K_n is a permutation $\sigma : V(G) \to V(G)$ such that if an edge xy belongs to $E(G)$ then $\sigma(x)\sigma(y)$ does not belong to $E(G)$. If there exists a packing of G into K_n we say that G is a *packable* graph. A *cyclic packing of G* is a cyclic permutation $\sigma : V(G) \to V(G)$ (so σ has exactly one cycle in its decomposition into cycles). A basic result concerning packing problem [1] is:

THEOREM 1. *If $|E(G)| \leq n-2$ then there exists a packing of G into K_n.*

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Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two graphs with the same set of vertices and disjoint sets of edges. The *edge-sum* of G_1 and G_2, denoted by $G_1 \oplus G_2$, is a graph with the vertex set V and the edge set $E_1 \cup E_2$. If there exists a packing σ of G such that $G \oplus \sigma(G)$ is a planar graph we say that σ is *a planar packing* of G.

It is known [6] that a non-star tree of order n can be packed into K_n. Garcia et al. [3] considered planar packing of trees. They proved that there exists a planar packing for any non-star tree T. Woźniak [7] improved this result by showing the existence of a cyclic permutation σ such that $T \oplus \sigma(T)$ is a planar graph.

In [2] the authors characterized packable unicyclic graphs. In this paper we consider the existence of planar packings for packable unicyclic graphs.

The paper is organized as follows. In Section 2 we give a characterization of planary packable cycles. In Section 3 we obtain a planar packing for certain classes of unicyclic graphs.

In the following sections we shall need the following important results:

Theorem 2. (Jordan Curve Theorem [5]) *Let c be a Jordan curve in the plane. Then the complement of the image of c consists of two distinct connected components. One of these components (interior(c)) is bounded and the other (exterior(c)) is unbounded.*

Theorem 3. (Kuratowski Theorem [4]) *A graph is planar iff it does not contain any subgraph homeomorphic to K_5 or $K_{3,3}$.*

2. Planar packing of cycles

It is known [2] that any cycle of order n is packable into K_n except the case $n = 3$ and $n = 4$. We prove the following theorem:

Theorem 4. *Let C_n be a packable cycle of order n. There exists a planar packing of C_n into a complete graph K_n iff $n \neq 5$ and $n \neq 7$.*

Proof. First we show that there does not exist a planar packing of C_n for $n = 5$ or $n = 7$. Observe that for every packing σ of C_5 a graph $C_5 \oplus \sigma(C_5)$ is isomorphic to a complete graph K_5 and by Kuratowski Theorem it is not a planar graph.

Due to Jordan Curve Theorem, by drawing any cycle in a plane we divide this plane into two parts: one inside our cycle and the other outside the cycle. Let σ be a packing of C_7. Consider a plane graph C_7 and a plane embedding of $\sigma(C_7)$. Notice that at least four edges of $\sigma(C_7)$ are mapped either to the interior of the plane graph C_7 or to the exterior. Without loss of generality we can assume that four edges of a plane graph $\sigma(C_7)$ are in the interior (see Fig. 1). Fig. 1 shows the only way, up to isomorphism, to draw them

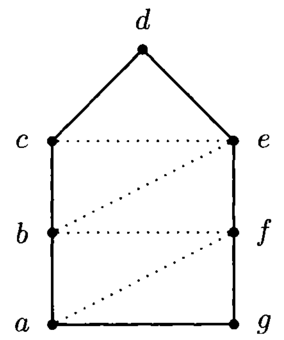

Fig. 1. A plane embedding of C_7.

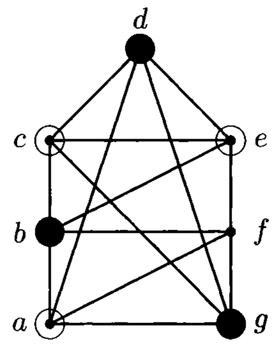

Fig. 2. A graph homeomorphic to $K_{3,3}$.

without intersections. Hence, the edges ad and cg occur in $\sigma(C_7)$. Thus, $C_7 \oplus \sigma(C_7)$ contains a subgraph homeomorphic to $K_{3,3}$ (see Fig. 2) and by Kuratowski Theorem it is not a planar graph.

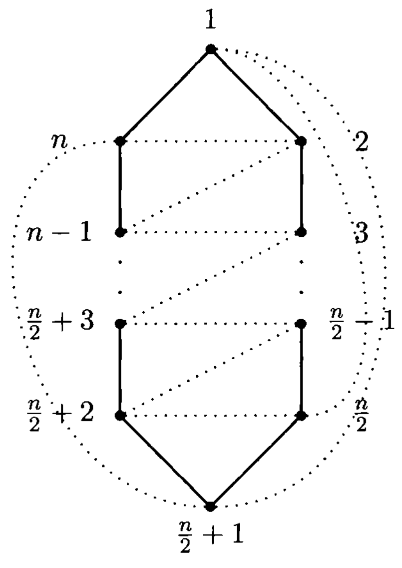

Fig. 3. A plane embedding of $C_n \oplus \sigma(C_n)$ for n even.

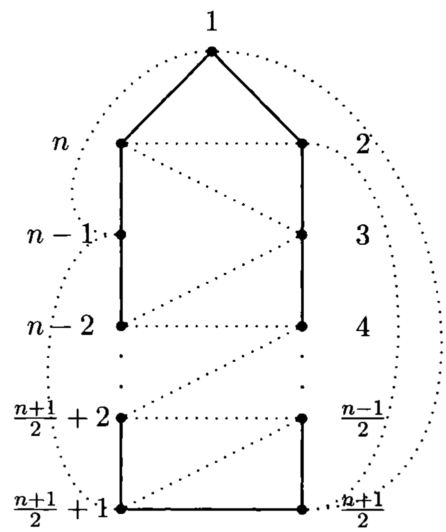

Fig. 4. A plane embedding of $C_n \oplus \sigma(C_n)$ for n odd.

Assume now that C_n is a packable cycle and $n \neq 5$ and $n \neq 7$. Let us consider a subgraph of K_n isomorphic to an edge-disjoint union of two copies of this cycle. There exists a plane embedding of such graph (see Fig. 3 and Fig. 4). Thus, there exists a planar packing of C_n and the proof is finished. ∎

3. Planar packings of unicyclic graphs

We start with the following lemma.

Lemma 1. *Let T be a tree with size k rooted at $x \in V(T)$. Then there is a plane embedding of T with a labeling of images of vertices by $x_0, \ldots, x_k$ such that:*

1. *x_0 is the image of x,*
2. *$x_0, \ldots, x_k$ are placed along a cycle segment in the natural order,*
3. *all images of edges are drawn either on the cycle segment or as chords of the segment,*
4. *for any $i \in \{1, \ldots, k-1\}$ and $j \in \{1, \ldots, k-1\}$ if x_{i-1} and x_{j-1} are joined by an image of an edge, then x_i and x_j are not joined.*

If moreover T is not a star rooted at the central vertex, then the points x_0 and x_1 are not joined.

Proof. The basic idea is that we draw a tree in such a way that its vertices are consecutive points of a cycle segment and its edges are either arcs of a circle if ends of edges are consecutive points, or chords if ends of edges are not consecutive points.

Observe, that for $j \notin \{i-1, i+1\}$ part (4) of above Lemma is obvious, since if both $x_i x_j$ and $x_{i+1} x_{j+1}$ are present, then the drawing cannot be a plane embedding.

If T is a star rooted at the central vertex, then the claim is obvious. In the opposite case the proof is by induction on k. For $k = 2$ we draw a path of size two placing the points x_0, x_1, x_2 along a cycle segment in the natural order and drawing edges $x_0 x_2$, $x_1 x_2$ as arcs of the circle. Suppose the claim is true for every tree with size $k-1$. Let T be a tree with size k rooted at $x \in V(T)$ different from a star rooted at the central vertex. Let y be a vertex of T such that $y \neq x$ and $\deg(y) = 1$. Let us consider a tree $T' = T - \{y\}$ rooted at x. Suppose first that T' is a star and x is it central vertex. Then we can place $x_0, \ldots, x_k$ along a cycle segment such that the pairs $\{x_1, x_2\}$, $\{x_0, x_2\}, \ldots, \{x_0, x_k\}$ are joined by images of edges. Suppose now that T' is different from a star rooted at the central vertex. Hence, by induction, there exists an adequate plane embedding of T'. Let $z \in V(T)$ be such that $yz \in E(T)$. Thus, there exists exactly one $i \in \{0, \ldots, k-1\}$ such that x_i is the image of z in the plane embedding of T'. We extend this plane embedding as follows.

• If $i \geq 1$ and x_{i-1} and x_i are not joined in the plane embedding of T', then we put the image of y between x_i and x_{i+1}. Then we relabel all images of vertices x_j for $j \in \{i+1, \ldots, k-1\}$ increasing the subscript by one (i.e., $x_j \to x_{j+1}$) and we label a new point by x_{i+1}. Then the arcs $x_i x_l$ in the plane embedding of T' are still x_ix_l if $l \in \{1, \dots, i-1\}$ and become x_ix_{l+1} if $l \in \{i+1, \dots, k-1\}$ and the new arc is x_ix_{i+1}. Notice that while x_ix_{i+1} is an arc, the vertices x_{i-1} and x_i are not joined by an arc in the plane embedding of T.

• If $i \geq 1$ and x_{i-1} and x_i are joined in the plane embedding of T', then we put the image of y between x_{i-1} and x_i. Then we relabel all images of vertices x_j for $j \in \{i, \dots, k-1\}$ increasing the subscript by one and we label a new point by x_i. Then the arcs x_ix_l in the plane embedding of T' become $x_{i+1}x_l$ if $l \in \{1, \dots, i-1\}$ and become $x_{i+1}x_{l+1}$ if $l \in \{i+1, \dots, k-1\}$ and the new arc is x_ix_{i+1}. Again, x_ix_{i+1} is an arc in the plane embedding of T, while the vertices x_{i-1} and x_i are not joined by an arc.

• If $i = 0$ we put the image of y between x_0 and x_{k-1} and label it by x_k. Then the arcs x_ix_l in the plane embedding of T' are still x_ix_l for any $l \in \{1, \dots, k-1\}$ and the new arc is x_0x_k.

Because no edges were intersecting in T', then no edges intersect in T either. Moreover, since for T' the conditions 1., 2., 3. and 4. hold, they also hold for T and if T is different from a star rooted at the central vertex, then the points x_0 and x_1 are not joined by an image of an edge. Thus, the proof is finished. ∎

Let G be an unicyclic graph with order n and with the cycle C such that $V(C) = \{c_1, \dots, c_m\}$ and $E(C) = \{c_1c_m, c_ic_{i+1}; i \in \{1, \dots, m-1\}\}$. For any $i \in \{1, \dots, m\}$ we denote by T_i a maximal connected subgraph of G such that it has exactly one common vertex c_i with a cycle C. Observe, that T_i is a tree (in particular, the vertices of degree zero are considered as trivial trees). We define $c_{m+1} = c_1$ and $T_{m+1} = T_1$.

Now, we prove the following theorem:

Theorem 5. *If one of the following conditions hold:*

1. $\deg(c_i) \geq 3$ *for every* $i \in \{1, \dots, m\}$,
2. *for any* $i \in \{1, \dots, m\}$ *if* $\deg(c_i) = 2$, *then* $\deg(c_{i+1}) \geq 4$ *and* T_{i+1} *is different from a star with the central vertex* c_{i+1},

then there exists a planar cyclic packing of G *into a complete graph* K_n.

Proof. We show that there exists a plane embedding of G such that images of its vertices, labeled by $x_1, \dots, x_n$, are placed along a cycle segment in the natural order, all edges are drawn either on the cycle segment or as chords of the segment. Moreover, for any $i, j \in \{1, \dots, n\}$ and $x_{n+1} = x_1$ if x_i and x_j are joined by an image of an edge, then x_{i+1}, x_{j+1} are not joined. Then $\sigma = (x_1 \dots x_n)$ is a cyclic packing of this plane graph G. Identifying a cycle segment with the equator of a sphere and drawing edges of G on the northern hemisphere and edges of $\sigma(G)$ on the southern hemisphere one can see that σ is a planar cyclic packing.

For any $i \in \{1, \ldots, m\}$ let T_i be a tree rooted at c_i. Let k_i denote the number of edges of T_i. We draw the cycle C placing its vertices along a cycle segment and labeling them by $x_1, \ldots, x_m$. Let G_1 be a subgraph of G induced by $V(C) \cup V(T_1)$. We draw G_1 increasing labels of x_j by k_1 for all $j > 1$ (i.e. $x_j \to x_{j+k_1}$), placing the image of the root of T_1 at x_1 and then, using Lemma 1, we put images of remaining vertices of T_1 between x_1 and x_{2+k_1} and we label them by $x_2, \ldots, x_{k_1+1}$. Thus, we obtain a plane embedding of G_1.

For any $l \in \{2, \ldots, m\}$ let G_l be a subgraph of G induced by $V(G_{l-1}) \cup V(T_l)$. We extend a plane embedding of G_{l-1} to a plane embedding of G_l increasing labels of x_j by k_l for all $j > l + k_1 + \ldots + k_{l-1}$, placing the image of the root of T_l at $x_{l+k_1+\ldots+k_{l-1}}$ and, using Lemma 1, we put images of its remaining vertices between $x_{l+k_1+\ldots+k_{l-1}}$ and $x_{l+1+k_1+\ldots+k_l}$ and we label them by $x_{l+k_1+\ldots+k_{l-1}+1}, \ldots, x_{l+k_1+\ldots k_{l-1}+k_l}$.

Notice, that for $G = G_m$ we obtain a plane embedding such that images of vertices of G, labeled by $x_1, \ldots, x_n$, are placed along a cycle segment in the natural order and all edges are drawn either on the cycle segment or as chords of the segment. We denote vertices in G by $\{y_1, \ldots, y_n\}$ in such a way that $u \in V(G)$ is denoted by y_i if x_i is the image of u. Then we part the set of vertices of G into m disjoint subsets $V_1,\ldots,V_m$ such that $V_l = \{y_{l+k_1+\ldots+k_{l-1}}, \ldots, y_{l+k_1+\ldots+k_l}\}$ for any $l \in \{1, \ldots, m\}$ and $k_0 = 0$. Observe that if $uv \in E(G)$, then there exists $l \in \{1, \ldots, m\}$ such that $u, v \in V_l$ or (defining $y_{n+1} = y_1$) u and v are two distinct vertices from the set $\{y_{l+k_1+\ldots+k_{l-1}}, y_{l+1+k_1+\ldots+k_l}\}$. Notice, that there are no two consecutive edges $y_{l-1}y_l$ and $y_l y_{l+1}$. When $\deg_G(y_{l-1}) = 2$ and the edge $y_{l-1}y_l \in V(G_l)$, then by our assumption T_l is not a star with the central vertex y_l and by Lemma 1 can be placed such that $y_l y_{l+1}$ is not an edge in G_l. When $\deg_G(y_{l-1}) \geq 3$, then we have inserted at least one vertex between x_{l-1} and x_l before relabeling x_l to x_{l+k_l} and the edge $y_{l+k_l-1}y_{l+k_l}$ is not in G_l. Hence, it does not matter whether the edge $y_{l+k_l}y_{l+k_l+1}$ is in G and T_l can be any tree. Therefore, by Lemma 1, for any $i, j \in \{1, \ldots, n\}$ if $y_i y_j \in E(G)$, then $y_{i+1}y_{j+1} \notin E(G)$. Then $\sigma := (y_1 \ldots y_n)$ is a planar cyclic packing of G and the proof is finished. ■

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