DEMONSTRATIO MATHEMATICA
Vol. XLII No 3 2009

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# TOPOLOGIES AND SMOOTH MAPS ON INITIAL AND FINAL OBJECTS IN THE CATEGORY OF FRÖLICHER SPACES

**Abstract.** In this paper, we show that when the Frölicher smooth structure is induced on a subset or a quotient set, there are three natural topologies underlying the resulting object. We study these topologies and compare them in each case. It is known that the topology generated by strucure functions is the weakest one in which all functions and curves on the space are continuous. We show that on a subspace, it is rather the trace topology which has this property, while the three topologies are coincident on the quotient space. We construct a base for the Frölicher topology and using either a base or a subbase in the sense of A. Frölicher [9], we characterise the morphisms of this category.

## 1. Introduction

The topology of a Frölicher space $(M, \mathcal{C}_M, \mathcal{F}_M)$ was defined in [9] as the initial topology generated by structure functions $f \in \mathbb{R}^M$, with $\{f^{-1}(0,1)\}_{f \in \mathcal{F}_M}$ as a subbase. It was studied by A. Cap [4] for the purpose of $K$-theory, and investigated by B. Dugmore [7] and P. Cherenack [6] as well. In [9], Frölicher and Kriegl proved that the category of smooth spaces (now called Frölicher spaces) is complete and co-complete, so that it has both initial and final objects. Later on, Cherenack showed that this category is topological over $\mathcal{SETS}$. In a further work, Dugmore constructed a base for the topology of these spaces and focused his study on homotopy theory. He used the function $\phi : (0, +\infty) \to (0,1)$ defined by $\phi(t) = e^{-\frac{1}{t}}$ and $\phi : (-\infty, 0] \to \{0\}$, which is not a bijection onto the whole $\mathbb{R}$. In this work we rather speak of topologies on a Frölicher space, as they are many indeed and we will have to compare them. Using the $C^\infty$ diffeomorphism $\phi : (0, +\infty) = (0,1) \cup [1, +\infty) \to \mathbb{R}$ given by $\phi(t) = -t + \frac{1}{t}$, the inverse of which maps each structure function onto a positive one, we shall obtain the same base $\{f^{-1}(0, +\infty)\}_{f \in \mathcal{F}_M}$ for the initial topology from

2000 *Mathematics Subject Classification*: 54A10, 54F65.
*Key words and phrases*: Frölicher space.

which open (-closed) sets, open (-closed) maps, and smooth maps between Frölicher spaces can be easily characterised. From the topologies underlying a Frölicher space, it will be clear that most usual properties in the category **Top** hold true in the category $\mathcal{FRL}$. As a main result of this work, we shall use our base in order to compare the topologies underlying a Frölicher subspace as initial object in the one hand, and quotient space as a final object on the other hand. Then, we show that both the canonical inclusion and projection are open maps.

Throughout this text, an $\mathcal{FRL}$-object or simply an object $M$ will mean a Frölicher space $(M, \mathcal{C}_M, \mathcal{F}_M)$, and an $\mathcal{FRL}$-morphism or a morphism will mean a smooth map in the sense of Frölicher. We shall verify that morphisms (structure functions and curves as well) are continuous in both topologies of curves and functions, and will denote by $C^\infty(M,N) := \{\varphi : M \to N \mid \varphi \text{ is } \mathbb{F}\text{-smooth}\}$ the set of all morphisms between objects $M$ and $N$.

Recall that an $\mathcal{FRL}$-morphism between two $M$ and $N$ is that map $\varphi$ such that $\mathcal{F}_N \circ \varphi \subseteq \mathcal{F}_M$ or, equivalently, $\varphi \circ \mathcal{C}_M \subseteq \mathcal{C}_N$. It follows that $\theta \circ \varphi \in C^\infty(M,P)$ whenever $\varphi \in C^\infty(M,N)$ and $\theta \in C^\infty(N,P)$. Consequently, a map $\varphi : M \to N$ between $\mathcal{FRL}$-objetcs is a morphism if, and only if $\theta \circ \varphi$ is an $\mathcal{FRL}$-morphism, where $\theta : N \to P$ is an $\mathcal{FRL}$-morphism.

## 2. Topologies on a Frölicher space

**Definition 2.1.** The topology induced by all the structure functions of an object $M$ is the collection $\tau_{\mathcal{F}_M} = \{\mathcal{U} \subset M | \mathcal{U} = \bigcup_{f \in \mathcal{F}_M} f^{-1}(V)\}$, where $V$ lies in the standard topology of $\mathbb{R}$. The topology induced by all the structure curves is $\tau_{\mathcal{C}_M} = \{\mathcal{U} \subset M | c^{-1}(\mathcal{U}) \in \tau_{\mathbb{R}}\}$, where $c \in \mathcal{C}_M$.

**Lemma 2.1.** $\tau_{\mathcal{F}_M} \subset \tau_{\mathcal{C}_M}$.

**Proof.** Let $\mathcal{U} \in \tau_{\mathcal{F}_M}$. That is, $\mathcal{U} = \bigcup_{f \in \mathcal{F}_M} f^{-1}(V)$ where $V$ is open in $\mathbb{R}$. For an arbitrary $c \in \mathcal{C}_M$, $c^{-1}(\mathcal{U}) = c^{-1}(\bigcup_{f \in \mathcal{F}_M} f^{-1}(V)) = \bigcup_{f \in \mathcal{F}_M} (f \circ c)^{-1}(V)$ $\in \tau_{\mathcal{C}_M}$.

But $V \in \tau_{\mathbb{R}}$ and $f \circ c$ is $C^\infty$. Hence $c^{-1}(\mathcal{U})$ is an open set in $\mathbb{R}$ as arbitrary union of elements of $\tau_{\mathbb{R}}$. Thus $\mathcal{U} \in \tau_{\mathcal{C}_M}$. ∎

An $\mathcal{FRL}$-object $M$ where $\tau_{\mathcal{F}_M} = \tau_{\mathcal{C}_M}$ is called a balanced space in [4], where a compact Hausdorff balanced $\mathbb{F}$-space is called a base space. Note that $M$ is Hausdorff if $\tau_{\mathcal{F}_M}$ and $\tau_{\mathcal{C}_M}$ are both Hausdorff.

From the lemma above, the topology of a Frölicher space $M$ shall be its weakest topology $\tau_{\mathcal{F}_M}$ induced by structure functions, unless otherwise specified.

**Lemma 2.2.** *Let $M$ be an $\mathbb{F}$-space and $\mathbb{R}$ endowed with the canonical $\mathbb{F}$-structure. Then $\phi : (0, +\infty) \to \mathbb{R}, \quad t \mapsto \phi(t) = -t + \frac{1}{t}$ and its inverse $\phi^{-1}$*

*are bijective and smooth functions in the usual sense. For any $g \in \mathcal{F}_M$ there exists a unique $f : M \to (0, +\infty)$ such that $f \in \mathcal{F}_M$ and $g = \phi \circ f$ and $f = \phi^{-1} \circ g$.*

**Proof.** 1. Considering the partition below for both the domain and codomain of the function $\phi$: $(0, +\infty) = (0, 1) \cup [1, +\infty)$ and $\mathbb{R} = (0, +\infty) \cup (-\infty, 0]$ it is easy to see that $\phi$ is $C^\infty$, monotonic decreasing and a bijection and also its inverse is $C^\infty$. So, $\phi$ is a diffeomorphism.

2. Let $g \in \mathcal{F}_M$ and $\phi$ as defined above. Then there is a unique $f : M \to (0, +\infty)$ such that $f = \phi^{-1} \circ g$ and $g = \phi \circ f$. Hence $f \in \mathcal{F}_M$. Furthermore, $\phi(0, 1) = (0, +\infty)$ and $\phi^{-1}(0, +\infty) = (0, 1)$. Hence, $g^{-1}(0, +\infty) = g^{-1}(\phi(0, 1)) = f^{-1}(\phi^{-1}(\phi(0, 1))) = f^{-1}(0, 1)$, which yields the required bijection $\mathcal{F}_M \to \mathcal{F}_M$ in such a way that $g \mapsto f = \phi^{-1} \circ g$ and $\{g^{-1}(0, +\infty)\}_{g \in \mathcal{F}_M} \to \{f^{-1}(0, 1)\}_{f \in \mathcal{F}_M}$, with $g^{-1}(0, +\infty) = f^{-1}(0, 1)$. The obtained collection is a base for the initial topology $\tau_{\mathcal{F}_M}$. ∎

## 2.1. Examples

**Example 2.1.1.** Let $(\mathbb{R}^n, \mathcal{C}, \mathcal{F})$ be the canonical $\mathbb{F}$-space. That is, $\mathcal{C} = C^\infty(\mathbb{R}, \mathbb{R}^n)$ and $\mathcal{F} = C^\infty(\mathbb{R}^n, \mathbb{R})$. The topology $\tau_{\mathcal{C}_{\mathbb{R}^n}}$ coincides with the Euclidean topology. Also $\mathbb{R}^n$ is a differentiable manifold. So, $\tau_{\mathcal{F}_M} = \tau_{\mathcal{C}_M}\varphi$ (see [6]). Thus, $(\mathbb{R}^n, \mathcal{C}, \mathcal{F})$ is a base space.

**Example 2.1.2.** Consider the Frölicher space $(\mathbb{Q}, \mathcal{C}_\mathbb{Q}, \mathcal{F}_\mathbb{Q})$, with the canonical structure generated by its inclusion in $\mathbb{R}$ which is an $\mathbb{F}$-smooth map. It is easy to see that $\mathcal{C}_\mathbb{Q} = \{c : \mathbb{R} \to \mathbb{Q} \mid c \text{ is constant } \}$ and $\mathcal{F}_\mathbb{Q} = \mathbb{R}^\mathbb{Q}$. In effect, as observed by P. Iglesias (see [10]), we have what follows.

$$\begin{aligned}
\mathcal{F}_o &= \{\iota : \mathbb{Q} \hookrightarrow \mathbb{R} \mid \iota = id_{\mathbb{R}|\mathbb{Q}}\} = \{\iota\}, \\
\Gamma\mathcal{F}_o &= \{c : \mathbb{R} \to \mathbb{Q} \mid f \circ c \in C^\infty(\mathbb{R}) \text{ for all } f \in \mathcal{F}_o\} \\
&= \{c : \mathbb{R} \to \mathbb{Q} \mid \iota \circ c = id_\mathbb{Q} \circ c = c \in C^\infty(\mathbb{R})\} \\
&= \{c : \mathbb{R} \to \mathbb{Q} \mid c \in C^\infty(\mathbb{R}) \text{ and } c(\mathbb{R}) \subset \mathbb{Q}\} \\
&= \mathbb{Q}^\mathbb{R} \cap C^\infty(\mathbb{R}, \mathbb{R}).
\end{aligned}$$

Thus $c \in \mathcal{C}_\mathbb{Q}$ means that $c \in C^\infty(\mathbb{R}, \mathbb{R})$ i.e. $c$ is continuous in the usual sense. Suppose $r, r' \in \mathbb{R}$ with $r < r'$ and $c(r) \neq c(r')$. Assume $c(r) < c(r')$, thus it follows, from the Intermediate Values Theorem that, for each $s \in [c(r), c(r')] \subset \mathbb{R}$, there exists $t \in [r, r']$ such that $s = c(t)$. That is, $c$ takes all real values (rational and irrational) between $c(r)$ and $c(r')$. This yields a contradiction since the range of c consists with only rational numbers. Therefore, $c(r) = c(r')$, for all $r, r' \in \mathbb{R}$ with $r \neq r'$. Thus, $c$ is a constant function. Furthermore,

$$\begin{aligned}\mathcal{F}_{\mathbb{Q}} &= \Phi\Gamma\mathcal{F}_o = \Phi\mathcal{C}_{\mathbb{Q}} \\ &= \{f : \mathbb{Q} \to \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}), \text{ for all } c \in \mathcal{C}_{\mathbb{Q}}\} \\ &= \{f : \mathbb{Q} \to \mathbb{R} \mid f \circ c_k \in C^\infty(\mathbb{R}), \text{ s.t. } c_k(t) = k, t \in \mathbb{R}, k \in \mathbb{Q}\} \\ &= \{f : \mathbb{Q} \to \mathbb{R} \mid f_k \in C^\infty(\mathbb{R}), \text{ s.t. } f_k(t) = f(k), t \in \mathbb{R}, k \in \mathbb{Q}\}.\end{aligned}$$

We say that $\mathcal{F}_{\mathbb{Q}} \subset \mathbb{R}^{\mathbb{Q}}$ such that $f \circ c_k \in C^\infty(\mathbb{R})$ is the set of all real-valued functions with source $\mathbb{Q}$. For, let $f \in \mathbb{R}^{\mathbb{Q}}$ and $k \in \mathbb{Q}$. Then $f(k)$ determines a constant function

$$f_k : \mathbb{R} \to \mathbb{R}, \quad f_k(t) = (f \circ c_k)(t) = f(c_k(t)) = f(k).$$

From the required condition $f \circ c_k \in C^\infty(\mathbb{R})$, it follows that $f \in \mathcal{F}_{\mathbb{Q}}$ iff $\mathcal{F}_{\mathbb{Q}} = \mathbb{R}^{\mathbb{Q}}$.

Now, let $c \in \mathcal{C}_{\mathbb{Q}}$. Then $c^{-1}(\mathbb{Q}) = \mathbb{R}$, $c^{-1}(\emptyset) = \emptyset$ and for any $S \in \mathcal{P}(\mathbb{Q})$, such that $\emptyset \subsetneqq S \subsetneqq \mathbb{Q}$,

$$c^{-1}(S) = \begin{cases} \emptyset & \text{for all } t \in \mathbb{R}, c(t) = a, \ a \notin S \\ \mathbb{R} & \text{for all } t \in \mathbb{R}, c(t) = a, \ a \in S. \end{cases}$$

Since

$$c(c^{-1}(S)) = S \cap c(\ ) = S \cap \{a\}, \text{ where } c(\mathbb{R}) = \{a\}.$$

It follows that

- $c(c^{-1}(S)) = \emptyset$ whenever $a = c(t) \notin S$ i.e. $c(t) = a \in \mathbb{Q} - S$
- $c(c^{-1}(S)) = \{a\}$ whenever $a = c(t) \in S$.

Hence, $c^{-1}(a) = c^{-1}(S) = \mathbb{R},\ a \in S$ or $c^{-1}(a) = c^{-1}(\mathbb{Q} - S) = \mathbb{R}$, where $a \in \mathbb{Q} - S$. Therefore, $\mathbb{R} = c^{-1}(\mathbb{Q}) - c^{-1}(S) = \mathbb{R} - c^{-1}(S)$. Thus, $c^{-1}(S) = \emptyset$. We conclude that $\tau_{\mathcal{C}_{\mathbb{Q}}} = \mathcal{P}(\mathbb{Q})$, that is the discrete topology. Now recall the inclusion $\tau_{\mathcal{F}_{\mathbb{Q}}} \subset \tau_{\mathcal{C}_{\mathbb{Q}}} = \mathcal{P}(\mathbb{Q})$, where $\mathcal{F}_{\mathbb{Q}} = \mathbb{R}^{\mathbb{Q}}$. It follows that for each $x$ in $\mathbb{Q}$, there exists a unique structure curve $c_x$ such that

$$c_x(t) = x \text{ and } (f \circ c_x)(t) = f(c_x(t)) = f(x)$$

for all $t \in \mathbb{R}$ and $f \in \mathcal{F}_{\mathbb{Q}}$. Thus $f \circ c_x$ is also constant. For any $S$ where $\emptyset \subsetneqq S \subsetneqq \mathbb{Q}$, there exists $f \in \mathcal{F}_{\mathbb{Q}}$ such that $f$ is constant on $S$ and taking its value in $(0, \infty)$, but $f$ applies $\mathbb{Q} - S$ in $(-\infty, 0]$. Thus

$$S = f^{-1}(0, +\infty) \in \tau_{\mathcal{F}_{\mathbb{Q}}}.$$

So, each subset of $\mathbb{Q}$ is open for $\tau_{\mathcal{F}_{\mathbb{Q}}}$, since $\emptyset$ and $\mathbb{Q}$ are open for any Frölicher canonically generated topology on $\mathbb{Q}$. Hence $\tau_{\mathcal{F}_{\mathbb{Q}}} = \mathcal{P}(\mathbb{Q}) = \tau_{\mathcal{C}_{\mathbb{Q}}}$ i.e. $\mathbb{Q}$ is a balanced space.

**EXAMPLE 2.1.3.** Let $(\mathbb{R}, \mathcal{C}_{\mathbb{R}}, \mathcal{F}_{\mathbb{R}})$ be the Frölicher space where the generating set is the set of constant curves. By a similar reasoning as in the example above, $\mathcal{F}_{\mathbb{R}} = \mathbb{R}^{\mathbb{R}}$ and structure curves are constant. The topologies $\tau_{\mathcal{F}_M} = \tau_{\mathcal{C}_M}$ are discrete. We have a balanced space.

Now, it is easy to show the following.

**LEMMA 2.3.** *Let $\varphi : M \to N$ be a $\mathcal{SETS}$-map between $\mathcal{FRL}$-objects. The following are equivalent:*

1. *$\varphi$ is a morphism.*
2. *The inverse image by $\varphi$ of each closed set in $N$ is closed in $M$.*
3. *If $g^{-1}(0,\infty)$ is a basic open in $N$ for $\tau_{\mathcal{F}_N}$ then $\varphi^{-1}(g^{-1}(0,\infty))$ is a basic open in $M$ for $\tau_{\mathcal{F}_M}$.*
4. *For each $p \in M$ and each open neighborhood $W_{\varphi(p)}$ of $\varphi(p)$ in $N$, there exists an open neighborhood $V_p$ of $p$ in $M$ such that $\varphi(V_p) \subset W_{\varphi(p)}$.*
5. *The inverse image by $\varphi$ of each open set in $N$ is open set in $M$ relative to topologies $\tau_{\mathcal{F}_N}$ and $\tau_{\mathcal{F}_M}$.*

**COROLLARY 2.1.** *Let $\varphi : M \to N$ be a morphism. Then*

1. *The family $\{\varphi^{-1}(g^{-1}(0,\infty)) \mid g \in \mathcal{F}_N\}$ is a base for the topology $\tau_{\mathcal{F}_N \circ \varphi} \subset \tau_{\mathcal{F}_M}$.*
2. *If $\varphi$ is an $\mathcal{FRL}$-diffeomorphism, then $\varphi$ induces an isomorphism of rings $\mathcal{F}_N \to \mathcal{F}_M$ such that $g \mapsto g \circ \varphi = \varphi^*(g)$. It turns out that $\{\varphi^{-1}(g^{-1}(0,\infty)) \mid g \in \mathcal{F}_N\}$ and $\{f^{-1}(0,\infty) \mid f \in \mathcal{F}_M\}$ are equipollent. Thus $\tau_{\mathcal{F}_N \circ \varphi} = \tau_{\mathcal{F}_M}$.*

**COROLLARY 2.2.** *If $\varphi$ is an $\mathcal{FRL}$-morphism then $\varphi$ is a* **Top**-*morphism for both $\tau_{\mathcal{C}_M}$ and $\tau_{\mathcal{F}_M}$.*

**Proof.** Let $\mathcal{U} \in \tau_{\mathcal{C}_M}$ i.e. for every $c \in \mathcal{C}_M$, $c^{-1}(\mathcal{U})$ is open in $\mathbb{R}$. $\varphi$ smooth means that $\varphi^{-1}(g^{-1}(0,1))$ is a subbasis open for $\tau_{\mathcal{F}_M}$, iff for every $c \in \mathcal{C}_M$, $\varphi \circ c = d$ where $d \in \mathcal{C}_N$.

Assume $\mathcal{U} \subset N$ such that $\mathcal{U} \in \tau_{\mathcal{C}_N}$. It follows that $d^{-1}(\mathcal{U}) = c^{-1}(\varphi^{-1}(\mathcal{U}))$ is open in $\mathbb{R}$ where

$$\tau_{\mathcal{C}_N} = \{\mathcal{U} \mid d^{-1}(\mathcal{U}) \text{ open in } \mathbb{R}, d \in \mathcal{C}_N, \mathcal{U} \subset N\}.$$

Thus $\varphi^{-1}(\mathcal{U})$ is open in $M$ for

$$\tau_{\mathcal{C}_M} = \{V \mid c^{-1}(V) \text{ open in } \mathbb{R}, c \in \mathcal{C}_M, V \subset M\}.$$

Hence $\varphi$ is continuous for $\tau_{\mathcal{C}_M}$.

Also, assume $\mathcal{U} \subset N$ such that $\mathcal{U} \in \tau_{\mathcal{F}_N}$. It follows that

$$\mathcal{U} = \bigcup_{i \in I} \Big[ \bigcap_{j=1}^{n} g_j^{-1}(0,1) \Big]_i \quad \text{where } g_j \in \mathcal{F}_N.$$

Now we can show that $\varphi^{-1}(\mathcal{U})$ is open in $M$ for $\tau_{\mathcal{F}_M}$.

$$\varphi^{-1}(\mathcal{U}) = \varphi^{-1}\Big(\bigcup_{i\in I}\bigcap_{j=1}^{n} g_j^{-1}(0,1)\Big) = \bigcup_{i\in I}\bigcap_{j=1}^{n} \varphi^{-1}g_j^{-1}(0,1)$$
$$= \bigcup_{i\in I}\bigcap_{j=1}^{n}(g_j\circ\varphi)^{-1}(0,1) = \bigcup_{i\in I}\bigcap_{j=1}^{n} f_j^{-1}(0,1),$$

which is open in $M$ for $\tau_{\mathcal{F}_M}$. Here $\varphi$ is continuous, but for $\tau_{\mathcal{F}_M}$. ∎

**Lemma 2.4.** *Let $\varphi : M \to N$ be a map between underlying sets of $\mathcal{FRL}$-objects. If $(\mathcal{U}_i)_{i\in I}$ is a $\tau_{\mathcal{C}_M}$-open covering of $M$ such that for any $i$, the restriction of $\varphi$ to $\mathcal{U}_i$ is morphism then $\varphi$ is a morphism.*

**Proof.** See [4]. ∎

**Remark 2.1.** The last assertion was proved in [4] using functions of compact support. There is no need of the compact support assumption in the present setting. Moreover, let $\varphi$ be a **Top**-morphism, that is $\varphi^{-1}(\mathcal{U})$ is open in $M$ if $\mathcal{U}$ is open in $N$. For all $g \in \mathcal{F}_N$, $\mathcal{U} = \bigcup_{g\in\mathcal{F}_N} g^{-1}(0,\infty)$, thus

$$\varphi^{-1}(\mathcal{U}) = \bigcup_{g\in\mathcal{F}_N} \varphi^{-1}g^{-1}(0,\infty) = \bigcup_{g\in\mathcal{F}_N} (g\circ\varphi)^{-1}(0,\infty)$$

is open in $M$. But we are not sure whether $f \in \mathcal{F}_M$. So, as in real analysis, if a map $\varphi$ between Frölicher spaces is continuous, then $\varphi$ is not necessarily smooth.

Open and closed $\mathcal{FRL}$-morphisms have same behaviour as in the usual setting. Note that if $f \in \mathcal{F}_M$ for an $\mathcal{FRL}$-object $M$, then $f^{-1}(0)$ is closed in $\tau_{\mathcal{F}_M}$ if and only if $f^{-1}(\{t|\ t \neq 0\})$ is $\tau_{\mathcal{F}_M}$-open.

**Example 2.1.4.** Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a polynomial. The graph of $\varphi$ is smooth. Thus $\varphi$ is smooth. Moreover, $\varphi[a,b]$ is equal to $\pi_2(G(\varphi))$, where $\pi_2$ is the second canonical projection. Hence $\varphi[a,b]$ is closed in $\mathbb{R}$. Therefore, $\varphi$ is a closed map.

**Example 2.1.5.** Let

$$S = \{(x,y)\,|\,xy = 1\} = \{(x,y)\,|\,y = \frac{1}{x} \text{ and } x \neq 0\}.$$

It is clear that

$$\pi_1(S) = \mathbb{R} - \{0\},$$

which is open in $\mathbb{R}$ in the one hand, and the map $\varphi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ such that $\varphi(x,y) = xy$ is $\mathbb{F}$-smooth on the other hand. For, let $f \in \mathcal{F}_{\mathbb{R}}$ and $c \in \mathcal{C}_{\mathbb{R}\times\mathbb{R}}$. We have $c = (c_1, c_2)$; $c_i \in \mathcal{C}_{\mathbb{R}}$ and

$$f\circ\varphi\circ c = f\circ\varphi(c_1,c_2) = f\circ(c_1\cdot c_2).$$

But $f \circ (c_1 \cdot c_2) = f \circ c_1 \cdot f \circ c_2$ is $C^\infty(\mathbb{R}, \mathbb{R})$ since each factor is $C^\infty(\mathbb{R}, \mathbb{R})$. That is, $f$ is $\mathbb{F}$-smooth and $c_1 \cdot c_2 \in \mathcal{C}_\mathbb{R}$. It follows that for $c \in \mathcal{C}_{\mathbb{R}^2}$, $\varphi$ is $\mathbb{F}$-smooth with respect to the canonical structures on $\mathbb{R}^2$ and $\mathbb{R}$ since

$$(\varphi \circ c)(t) = \varphi(c_1(t), c_2(t)) = c_1(t) \cdot c_2(t) = (c_1 \cdot c_2)(t),$$

thus $\varphi \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$. The inverse image of this closed set by $\varphi$ is a closed set. In particular $\{1\}$ is closed in $\mathbb{R}$, then

$$\varphi^{-1}\{1\} = \{(x,y) \in \mathbb{R}^2 \mid \varphi(x,y) = 1\} = \{(x,y) \in \mathbb{R}^2 \mid xy = 1\} = S$$

is closed. Finally, the image of a closed set $S$ by $\pi_1$ is an open set $\mathbb{R} - \{0\}$, that is $\pi_1$ is not a closed map.

## 3. Topologies on an $\mathbb{F}$-subspace

Let $M$ be an $\mathcal{FRL}$-object and $S$ a subset in its underlying set. Since the category $\mathcal{FRL}$ is topological over $\mathcal{SETS}$, complete and co-complete, $S$ can be made into a subobject (see [9]) so as to carry two $\mathbb{F}$-topologies. That is $\tau_{\mathcal{F}_S}$ and $\tau_{\mathcal{C}_S}$ induced respectively by structure functions and structure curves on $S$, in which all smooth functions and smooth curves are continuous. The collection $\{g^{-1}(0,\infty) \mid g \in \mathcal{F}_S\}$ is a base for $\tau_{\mathcal{F}_S}$. Moreover, $S$ has the relative topology as a **Top**-subobject, that is $\tau_{\mathcal{F}_M}(S) = \{S \cap \mathcal{U} \mid \mathcal{U} \in \tau_{\mathcal{F}_M}\}$. We shall now discuss the three topologies on an $\mathbb{F}$-subspace.

**Lemma 3.1.** *Let $M$ be an $\mathcal{FRL}$-object, $S$ a subset of its underlying set and $f \in \mathcal{F}_M$. Then*

1. *$S \cap f^{-1}(0, +\infty)$ is $\tau_{\mathcal{F}_M}(S)$-basic open in $S$.*
2. *$\iota_S$ is continuous in $\tau_{\mathcal{F}_M}(S)$.*

**Proof.** 1. In the topology $\tau_{\mathcal{F}_M}$, a set $V$ is open if $V = \bigcup_{f \in \mathcal{F}_M}[f^{-1}(0,\infty)]$. Thus

$$S \cap V = S \cap \Big( \bigcup_{f \in \mathcal{F}_M} [f^{-1}(0,\infty)] \Big) = \bigcup_{f \in \mathcal{F}_M} [S \cap f^{-1}(0,\infty)].$$

It remains to show that the family $\{S \cap f^{-1}(0,\infty) \mid f \in \mathcal{F}_M\}$ is closed under finite intersection. Let $\{f_i(0,\infty) \mid 1 \le i \le n\}$ be a finite collection of $\tau_{\mathcal{F}_M}$-basic open sets. Since $\{f^{-1}(0,\infty) \mid f \in \mathcal{F}_M\}$ is closed under finite intersection, $\bigcap_{i=1}^n f_i^{-1}(0,\infty) = g^{-1}(0,\infty)$ with $g \in \mathcal{F}_M$. Since $S \cap f_i^{-1}(0,\infty)$ lies in the collection $\{S \cap f^{-1}(0,\infty) \mid f \in \mathcal{F}_M\}$, then $\bigcap_{i=1}^{n}(S \cap f_i^{-1}(0,\infty)) =$ $S \cap (\bigcap_{i=1}^{n} f_i^{-1}(0,\infty)) = S \cap g^{-1}(0,\infty)$ also lies in $\{S \cap f^{-1}(0,\infty) | f \in \mathcal{F}_M\}$. In the sequel $\{S \cap f^{-1}(0,\infty) \mid f \in \mathcal{F}_M\}$ is closed under finite intersection.

Hence it is a base for $\tau_{\mathcal{F}_M}(S)$. That is, $S \cap f^{-1}(0,\infty)$ is a $\tau_{\mathcal{F}_M}(S)$-basic open set in $S$.

2. For $\mathcal{U} \in \tau_{\mathcal{F}_M}$, we have $\iota_S^{-1}(\mathcal{U}) = S \cap \mathcal{U}$. So $\iota_S^{-1}(\mathcal{U}) \in \tau_{\mathcal{F}_M}(S)$. Thus, $\iota_S$ is continuous. Moreover, $\iota_S^{-1}(f^{-1}(0,\infty)) = S \cap f^{-1}(0,\infty)$ is a $\tau_{\mathcal{F}_M}(S)$-basic open set. ∎

The following lemma states the transitivity principle known from general topology.

**Lemma 3.2.** *Let $P$ and $N$ be $\mathcal{FRL}$-subobjects of an $\mathcal{FRL}$-object $M$ such that $P \subset N \subset M$ holds on the underlying sets. If $P$ and $N$ are endowed with the trace topologies $\tau_{\mathcal{F}_N}(P)$ and $\tau_{\mathcal{F}_M}(N)$ respectively, then $P$ is also endowed with the trace topology $\tau_{\mathcal{F}_M}(P)$.*

**Proof.** Let $W \in \tau_{\mathcal{F}_N}(P)$. Then $W = P \cap V$, where $V \in \tau_{\mathcal{F}_M}(N)$, that is $V = N \cap \mathcal{U}$ with $\mathcal{U} \in \tau_{\mathcal{F}_M}$. Therefore $W = P \cap V = P \cap (N \cap \mathcal{U}) = (P \cap N) \cap \mathcal{U} = P \cap \mathcal{U}$. Hence $W \in \tau_{\mathcal{F}_M}(P)$. ∎

Now we can characterise open and closed sets in a subspace of a Frölicher space.

**Lemma 3.3.** *Let $M$ be an $\mathcal{FRL}$-object and $\tau_{\mathcal{F}_M}(S)$ be the trace topology on its subobject $S$. Let $\mathcal{U} \subset S$ be a $\tau_{\mathcal{F}_M}(S)$-open set. Then $S$ is a $\tau_{\mathcal{F}_M}$-open set if, and only if $\mathcal{U}$ is a $\tau_{\mathcal{F}_M}$-open set.*

**Proof.**

Let $\mathcal{U}$ be a $\tau_{\mathcal{F}_M}(S)$-open and assume that $S$ is a $\tau_{\mathcal{F}_M}$-open. Hence

$$\mathcal{U} = \bigcup_{i \in I} (S \cap f_i^{-1}(0,\infty)) = S \cap \Big( \bigcup_{i \in I} (f_i^{-1}(0,\infty)) \Big) \tag{1}$$

and

$$S = \bigcup_{j \in J} (g_j^{-1}(0,\infty) \tag{2}$$

where $f_i, g_j \in \mathcal{F}_M$. From equations 1 and 2, we have

$$\begin{aligned} \mathcal{U} &= \Big[ \bigcup_{j \in J} (g_j^{-1}(0,\infty)) \Big] \cap \Big[ \bigcup_{i \in I} (f_i^{-1}(0,\infty)) \Big] \\ &= \bigcup_{(j,i) \in J \times I} [(g_j^{-1}(0,\infty)) \cap (f_i^{-1}(0,\infty))] = \bigcup_{k \in K} h_k^{-1}(0,\infty) \end{aligned}$$

with $K = J \times I$, and

$$(h_k^{-1}(0,\infty)) = (g_j^{-1}(0,\infty)) \cap (f_i^{-1}(0,\infty))$$

since $\{f^{-1}(0,\infty) \mid f \in \mathcal{F}_M\}$ is stable under taking of finite intersections. Hence $\mathcal{U} \in \tau_{\mathcal{F}_M}$. That is, $\mathcal{U}$ is a $\tau_{\mathcal{F}_M}$-open set in $M$.

Conversely, let $\mathcal{U}$ be both a $\tau_{\mathcal{F}_M}(S)$-open set and a $\tau_{\mathcal{F}_M}$-open set. We need to show that $S$ is open in $\tau_{\mathcal{F}_M}$. From assumption,

$$\mathcal{U} = \bigcup_{j\in J}(g_j^{-1}(0,\infty)) \subset S.$$

Then, there is $g_j \in \mathcal{F}_M$ such that $g_j^{-1}(0,\infty) \subset S$. That is, $S$ contains a $\tau_{\mathcal{F}_M}$-basic open set. So $S$ is a $\tau_{\mathcal{F}_M}$-open set in $M$. ∎

One can state Lemma 3.3 above for closed sets and prove it in a similar way.

**Corollary 3.1.** *If $\mathcal{U}$ is open (closed) for $\tau_{\mathcal{F}_M}(S)$ and $S$ is open (closed) for $\tau_{\mathcal{F}_M}$ then $\mathcal{U}$ is open (closed) for $\tau_{\mathcal{F}_M}$.*

**Lemma 3.4.** *Let $M$ be an $\mathcal{FRL}$-object, $S$ a subobject of $M$ and $\tau_{\mathcal{F}_M}$ the $\mathbb{F}$-topology on $M$ making the inclusion map $\iota_S : S \hookrightarrow M$ continuous. Then $\iota_S$ is an open (closed) map if, and only if $S$ is an open (closed) set in $M$ irrespective to the given topology $\tau_{\mathcal{F}_M}$.*

**Proof.** 1. The open case.

Let $\iota_S$ be an open map. We have

$$\iota_S^{-1}(f^{-1}(0,\infty)) = (f\circ\iota_S)^{-1}(0,\infty) = (f|S)^{-1}(0,\infty).$$

But $\iota_S$ is smooth (so continuous), so $(f|S)^{-1}(0,\infty)$ is a basic open set since $f|S$ is a generator of the smooth structure on $S$, for $f \in \mathcal{F}_M$. Thus $\iota_S(f|S)^{-1}(0,\infty))$ is an open set in $\tau_{\mathcal{F}_M}$ by assumption. It follows that

$$\iota_S^{-1}(M) = \iota_S^{-1}\Big(\bigcup_{i\in I} f_i^{-1}(0,\infty)\Big) = \bigcup_{i\in I}(f_i|S)^{-1}(0,\infty) \tag{3}$$

and also

$$\iota_S^{-1}(M) = S\cap M = S. \tag{4}$$

It follows from equations 3 and 4 above that $S = \bigcup_{i\in I}(f_i|S)^{-1}(0,\infty)$. Hence $S \in \tau_{\mathcal{F}_M} \subset \tau_{\mathcal{C}_M}$.

Let $S$ be open set in $M$ for $\tau_{\mathcal{F}_M}$. From Lemma 3.3, any $\tau_{\mathcal{F}_M}(S)$-open $\mathcal{U}$ is also $\tau_{\mathcal{F}_M}$-open. That is,

$$\mathcal{U} = \bigcup_{t\in T}\Big[S\cap\Big(\bigcup_{i\in I} f_{it}^{-1}(0,\infty)\Big)\Big] \text{ with } S = \bigcup_{j\in J} h_j^{-1}(0,\infty),\ h_j \in \mathcal{F}_M \text{ for all j.}$$

Therefore,

$$\begin{aligned}\mathcal{U} &= \bigcup_{t\in T}\Big[\Big(\bigcup_{j\in J} h_j^{-1}(0,\infty)\Big)\cap\Big(\bigcup_{i\in I} f_{it}^{-1}(0,\infty)\Big)\Big]\\ &= \bigcup_{t\in T}\Big[\bigcup_{(j,i,t)\in J\times I\times T}(h_j^{-1}(0,\infty)\cap\bigcup_{i\in I} f_{it}^{-1}(0,\infty))\Big]\\ &= \bigcup_{t\in T}\Big[\bigcup_{(j,i,t)\in J\times I\times T} g_{jit}^{-1}(0,\infty)\Big] = \bigcup_{k\in K} g_k^{-1}(0,\infty),\end{aligned}$$

for $g_k \in \mathcal{F}_M$. It follows that $\mathcal{U} = \iota_S(\mathcal{U})$ is open in $\tau_{\mathcal{F}_M}$ for every $\tau_{\mathcal{F}_M}(S)$-open $\mathcal{U}$. Therefore, $\iota_S$ is an open map.

2. Obviously, the closed case can be proved in a similar way. ∎

**Proposition 3.1.** *Let $S$ be a subspace of an $\mathcal{FRL}$-object $M$. Then $\tau_{\mathcal{F}_M}(S) \subset \tau_{\mathcal{F}_S} \subset \tau_{\mathcal{C}_S}$. That is, the trace topology $\tau_{\mathcal{F}_M}(S)$ is the smallest topology on $S$ for which the inclusion map $\iota_S$ is continuous.*

**Proof.** 1. Let $\mathcal{U} \in \tau_{\mathcal{F}_M}(S)$. That is,

$$\mathcal{U} = S \cap V = \bigcup_{i\in I}(S\cap f_i^{-1}(0,\infty)), \text{ with } V = \bigcup_i f_i^{-1}(0,\infty).$$

It follows that

$$\mathcal{U} = \bigcup_{i\in I}(\iota_S^{-1}(f_i^{-1}(0,\infty))) = \bigcup_{i\in I}((f_i\circ\iota_S)^{-1}(0,\infty)) = \bigcup_{i\in I}(g_i^{-1}(0,\infty)) \in \tau_{\mathcal{F}_S},$$

where $g_i = f_i|S$. So the required inclusions hold.

2. Let $V \in \tau_{\mathcal{F}_M}(S)$ and $\tau$ is any topology on $S$, where $\iota_S$ is continuous. Then $V = S\cap\mathcal{U}$, with $\mathcal{U}\in\tau_{\mathcal{F}_M}$ and $\iota_S^{-1}(\mathcal{U}) \in \tau$ since $\iota_S$ is continuous for $\tau$. So, $\iota_S^{-1}(\mathcal{U}) = S\cap\mathcal{U} = V$. Hence $V\in\tau$ and $\tau_{\mathcal{F}_M}(S)\subset\tau$ is the smallest topology on $S$ for which $\iota_S$ is continuous. ∎

**Proposition 3.2.** *Let $M$ be an $\mathcal{FRL}$-object and $S$ a subset of its underlying set. The following hold: if $S \in \tau_{\mathcal{F}_M}$, then $\tau_{\mathcal{F}_S} = \tau_{\mathcal{F}_M}(S)$. If $S\in\tau_{\mathcal{C}_M}$, then $\tau_{\mathcal{C}_S} = \tau_{\mathcal{C}_M}(S)$.*

**Proof.** 1. Assume $\mathcal{U}\in\tau_{\mathcal{F}_S}$, that is $\mathcal{U} = \bigcup_{i\in I}(f_i|S)^{-1}(0,\infty))$, where $f_i \in \mathcal{F}_M$ and $f_i|S$ is a generator of the structure $(\mathcal{C}_S,\mathcal{F}_S)$. It follows that

$$\begin{aligned}\mathcal{U} &= \iota_S^{-1}\iota_S(\mathcal{U}) = \iota_S^{-1}\Big[\bigcup_{i\in I}\iota_S(f_i|S)^{-1}(0,\infty))\Big] = \iota_S^{-1}\Big[\bigcup_{i\in I} f_i^{-1}(0,\infty)\Big]\\ &= \bigcup_{i\in I}[\iota_S^{-1}(f_i^{-1}(0,\infty))] = \bigcup_{i\in I}[S\cap f_i^{-1}(0,\infty)] \in \tau_{\mathcal{F}_M}(S),\end{aligned}$$

using the fact that $S$ is open and $\iota_S$ is an open map. That is, $\tau_{\mathcal{F}_S}\subset\tau_{\mathcal{F}_M}$. The reverse inclusion $\tau_{\mathcal{F}_M}(S)\subset\tau_{\mathcal{F}_S}$ was proved in Proposition 3.1 above.

2. Assume $\mathcal{U} \in \tau_{\mathcal{C}_S}$, that is $d^{-1}(\mathcal{U}) \in \tau_{\mathcal{C}_\mathbb{R}}$, with $d \in \mathcal{C}_S$. But $S \in \tau_{\mathcal{C}_M}$, hence for some $c \in \mathcal{C}_M$, $c^{-1}(S) \in \tau_\mathbb{R} = \tau_\mathbb{R}$. (We used the fact that $\tau_{\mathcal{C}_\mathbb{R}} = \tau_\mathbb{R}$). Let $d \in \mathcal{C}_S$. It follows that

$$d^{-1}(\mathcal{U}) = d^{-1}(\iota_S^{-1}(\iota_S(\mathcal{U})) = (\iota_S \circ d)^{-1}(\iota_S(\mathcal{U})) = c^{-1}(\mathcal{U}),$$

where $c \in \mathcal{C}_M$. Since $\iota_S$ is smooth, $d^{-1}(\mathcal{U}) = c^{-1}(\mathcal{U}) \in \tau_\mathbb{R}$. Now $\mathcal{U} \in \tau_{\mathcal{C}_M}$, that is $\mathcal{U} \subset S \subset M$. It follows that

$$\mathcal{U} = \iota_S^{-1}(\iota_S(\mathcal{U})) = S \cap \iota_S(\mathcal{U}) = S \cap \mathcal{U} \in \tau_{\mathcal{C}_M}(S)$$

since $\mathcal{U} \in \tau_{\mathcal{C}_M}$. Therefore $\tau_{\mathcal{C}_M}(S) \supset \tau_{\mathcal{C}_S}$. Hence, $\tau_{\mathcal{C}_S} = \tau_{\mathcal{C}_M}(S)$. ∎

## 4. $\mathbb{F}$-quotient space and associated topologies

In this section, an $\mathbb{F}$-quotient space is regarded as a final object whose structure is obtained by the process of lifting from the category $\mathcal{SETS}$ to the category $\mathcal{FRL}$. That the quotient structure exists in the category $\mathcal{FRL}$ was proved in [9].

In what follows, we are given an equivalence relation $\sim$ on the underlying set of an $\mathcal{FRL}$-object $M$ such that the quotient $\tilde{M} := M/\sim$ in $\mathcal{SETS}$ is given the final Frölicher structure generated by the canonical map $\pi_\sim : M \to \tilde{M}$. Recall the universality condition as follows. For an arbitrary object $N \in \mathcal{SETS}$ and a map $f : M \to N$, one obtains an equivalence relation $\sim_f$ in the underlying set $M$ by defining $(x, y) \in \sim_f$ if and only if $f(x) = f(y)$, for $x, y \in M$, the equivalence classes of which are the fibers of $f$. They are $f^{-1}(s)$, where $s \in im(f)$.

Taking $f = \pi_\sim$, it is clear that every equivalence relation $\sim$ arises in this way. The map $f$ is said to be consistent with $\sim$ if $x \sim y$ implies $f(x) = f(y)$, i.e. $f$ is constant on each equivalence class modulo $\sim$ and there exists a unique one-to-one map $\tilde{g} : \tilde{M} \to N$ such that $\tilde{g} \circ \pi_\sim = f$. That is, $f(x) = \tilde{g}(\pi_\sim(x)) = \tilde{g}([x])$. This associated map $\tilde{g}$ is one-to-one due to the fact that $\sim_f$ is the kernel equivalence of $f$, that is, the consistency of $\sim$ with the smooth map $f \in \mathcal{F}_M$. In this case, $[x] \neq [y]$ implies $f(x) \neq f(y)$ and thus $\tilde{g}([x]) \neq \tilde{g}([y])$ or alternatively $\tilde{g}([x]) = \tilde{g}([y])$ reads $f(x) = f(y)$. Thus $f^{-1}(f(x)) = f^{-1}(f(y))$ yields

$$\{s \in M \mid f(s) = f(x)\} = \{t \in M \mid f(t) = f(y)\},$$

that is, $[x] = [y]$. Without the consistency of $\sim$ with $f \in \mathcal{F}_M$, the inclusions $\pi \circ \mathcal{C}_M \subset \mathcal{C}_{\tilde{M}}$ and $\mathcal{F}_{\tilde{M}} \circ \pi \subset \mathcal{F}_M$ have to be strict. $[x] \in \tilde{M}$ if and only if $\pi^{-1}([x]) = \{y \in M \mid f(y) = f(x)\}$ if and only if $f(\pi^{-1}([x])) = f(x) = \tilde{g}([x])$, if and only if $f \circ \pi^{-1} = \tilde{g}$.

Let $Hom_\sim(M, N)$ denote the subalgebra of $Hom(M, N)$ consisting of functions which are constant on equivalence classes of $\sim$. Hence there is an algebra isomorphism among $Hom(M, N)$ and $Hom(\tilde{M}, N)$, in particular among $Hom_{\sim}(M,\mathbb{R})$ and $Hom(\tilde{M},\mathbb{R})$ for $N=\mathbb{R}$ in our case. And from this property of $\tilde{g}$, the algebra $Hom(\tilde{M},\mathbb{R})$ separates points of $\tilde{M}$, which are the equivalence classes of $\sim$ . This is the reason why we choose to work with the kernel equivalence. More general equivalence relations can be considered, but the conclusions of these investigations are beyond this work.

Now we need to describe the Frölicher structure on the quotient. Let $\mathcal{C}_{oM}$ be a set of curves generating an $\mathbb{F}$-structure on $M$. Then $\mathcal{C}_o=\{\pi\circ c \mid c\in \mathcal{C}_{oM}\}$ will generate an $\mathbb{F}$-structure on $\tilde{M}$ as follows.

$$\begin{aligned}\mathcal{F}_{\tilde{M}} &= \Phi\mathcal{C}_o\\ &= \{\tilde{g}:\tilde{M}\to\mathbb{R} \mid \tilde{g}\circ(\pi\circ c)\in C^{\infty}(\mathbb{R}),\quad \text{for all } \pi\circ c\in\mathcal{C}_o\}\\ &= \{\tilde{g}:\tilde{M}\to\mathbb{R} \mid \tilde{g}\circ(\pi\circ c)\in C^{\infty}(\mathbb{R}),\quad \text{for all } c\in\mathcal{C}_{oM}\}\\ &= \{\tilde{g}:\tilde{M}\to\mathbb{R} \mid (\tilde{g}\circ\pi)\circ c\in C^{\infty}(\mathbb{R}),\quad \text{for all } c\in\mathcal{C}_{oM}\}\\ &= \{\tilde{g}:M\to\mathbb{R},\ \tilde{g}\circ\pi\in\mathcal{F}_M\}.\end{aligned}$$

The structure curves are given by

$$\begin{aligned}\mathcal{C}_{\tilde{M}} &= \Gamma\Phi\mathcal{C}_o=\Gamma\mathcal{F}_{\tilde{M}}\\ &= \{\tilde{c}:\mathbb{R}\to\tilde{M} \mid \tilde{g}\circ\tilde{c}\in C^{\infty}(\mathbb{R}),\quad \text{for all } \tilde{g}\in\mathcal{F}_{\tilde{M}}\}.\end{aligned}$$

Since $\tilde{g}\circ\pi\in\mathcal{F}_M$, we have $\tilde{g}\circ\pi\circ c\in C^{\infty}(\mathbb{R})$ for all $c\in\mathcal{C}_M$. That is, $C_{\tilde{M}}=\{\pi\circ c,\ c\in\mathcal{C}_M\}$. Also $\mathcal{C}_{0M}\subseteq\mathcal{C}_M$ and $\mathcal{C}_0\subseteq\tilde{\mathcal{C}}$ (see [9], [3]), then from Lemma 2.3, $\tilde{c}=\pi\circ c$ shows that the canonical map is an $\mathcal{FRL}$-morphism. Its smoothness reads $\mathcal{F}_{\tilde{M}}\circ\pi\subset\mathcal{F}_M$ if, and only if $\pi\circ\mathcal{C}_M\subset\mathcal{C}_{\tilde{M}}$.

**Definition 4.1.** The $\mathbb{F}$-space $(\tilde{M},\mathcal{C}_{\tilde{M}},\mathcal{F}_{\tilde{M}})$ is called an $\mathbb{F}$-quotient space of the $\mathbb{F}$-space $M$ by the equivalence relation $\sim$. The pair $(\mathcal{C}_{\tilde{M}},\mathcal{F}_{\tilde{M}})$ is the final $\mathbb{F}$-quotient structure (quotient structure for short) making $\pi$ into a smooth map.

**Definition 4.2.** Let $M$ be an $\mathcal{FRL}$-object and $\sim_f$ a kernel equivalence on $M$. The topology generated on the quotient space $\tilde{M}=M/\sim_f$ by structure functions is $\tau_{\mathcal{F}_{\tilde{M}}}=\{\mathcal{U}\subseteq\tilde{M}|f^{-1}(V)=\mathcal{U},\ V\in\tau_{\mathcal{F}_{\mathbb{R}}},f\in\mathcal{F}_{\tilde{M}}\}$ with subbase $\mathcal{S}=\{f^{-1}(0,1)\mid f\in\mathcal{F}_{\tilde{M}}\}$ and base given by $\mathcal{B}=\{f^{-1}(0,+\infty)\mid f\in\mathcal{F}_{\tilde{M}}\}$. The topology generated by structure curves on the quotient space is given by $\tau_{\mathcal{C}_{\tilde{M}}}=\{O\subseteq\tilde{M}|c^{-1}(O)\in\tau_{\mathcal{F}_{\mathbb{R}}}\}$, where $c$ is a structure curve on $\tilde{M}$. Both $\tau_{\mathcal{F}_{\tilde{M}}}$ as well as $\tau_{\mathcal{C}_{\tilde{M}}}$ are called $\mathbb{F}$-topologies on $\tilde{M}$ or $\mathbb{F}$-quotient topologies.

Recall that the quotient topology (or standard quotient topology or identification topology) on $\tilde{M}$ is the one which is generated by the canonical map $\pi:M\to\tilde{M}=M/\sim$. It is defined by $\tau_{\sim}=\{V\subseteq\tilde{M}:\pi^{-1}(V)\in\tau_{\mathcal{F}_M}\}$ and known to be the strongest one in which $\pi$ is continuous. In this section we need to compare three topologies on $\tilde{M}$, two of which arise from the Frölicher quotient structure. Recall that:

1. $G \subset \tilde{M}$ is a $\tau_\sim$-closed set in $\tilde{M}$ if, and only if $\pi^{-1}(G) = F$ is a $\tau_{\mathcal{F}_M}$-closed set in M. For $\pi^{-1}(G) = \pi^{-1}(\tilde{M} - V) = \pi^{-1}(\tilde{M}) - \pi^{-1}(V) = M - \mathcal{U} = F$, where $V \in \tau_\sim$, and $\mathcal{U} \in \tau_{\mathcal{F}_M}$.
2. The identification topology is Hausdorff. For, let $\tilde{g} \in \mathcal{F}_{\tilde{M}}$ and let $[x] \neq [y]$. Hence $\tilde{g}([x]) \neq \tilde{g}([y])$ since $\tilde{g}$ is injective. Thus $\tilde{g}$ separates points in $\tilde{M}$.
3. The identification topology is the largest (finest) topology in $\tilde{M}$ for which $\pi$ is continuous. So $\tau_{\mathcal{F}_{\tilde{M}}} \subset \tau_\sim$. For, let $\tau$ be another topology making $\pi$ a continuous map on $\tilde{M}$. Let $V \in \tau$. It follows from the continuity of $\pi$, that $\pi^{-1}(V) \in \tau_{\mathcal{F}_M}$ that is $V \in \tau_\sim$. Hence $\tau \subset \tau_\sim$. In particular, $\tau_{\mathcal{F}_{\tilde{M}}} \subset \tau_\sim$.
4. Let $\pi : M \to \tilde{M}$ be the canonical projection. Then $\pi$ is open (closed) map with respect to $\tau_{\mathcal{F}_M}$ and $\tau_\sim$. In effect, let $\mathcal{U}$ be a $\tau_{\mathcal{F}_M}$-open set in $M$, i.e. $\mathcal{U} = \bigcup_{j\in J} f_j^{-1}(0,+\infty)$. Then
$$\pi(\mathcal{U}) = \pi\Big(\bigcup_{j\in J}((f_j^{-1}(0,+\infty)))\Big) = \bigcup_{j\in J}(\pi(f_j^{-1} \circ (0,+\infty)).$$
We need to show that $\pi(\mathcal{U})$ is open in $\tilde{M}$. That is, $\pi^{-1}(\pi(\mathcal{U})$ must lie in $\mathcal{F}_M$. But
$$\pi^{-1}(\pi(\mathcal{U})) = \pi^{-1}\Big(\bigcup_j \pi(f_j^{-1}(0,\infty))\Big) = \bigcup_j \pi^{-1}(\pi(f_j^{-1}(0,\infty))$$
contains $\bigcup_j f^{-1}(0,\infty)$. It follows that $\pi^{-1}(\pi(\mathcal{U}))$ is open in $\tau_{\mathcal{F}_M}$. Then so is $\pi(\mathcal{U})$. Thus, $\pi$ is an open map. The proof is similar using a closed set.

**Lemma 4.1.** *Let $\pi : M \to \tilde{M}$ be the canonical projection. Let $\tilde{g} \in \mathcal{F}_{\tilde{M}}$ such that $\tilde{g} \circ \pi = f$, $f \in \mathcal{F}_M$. Then $\tilde{g}$ is open (closed) map with respect to $\tau_\sim$ and $\tau_{\mathbb{R}}$ if, and only if $f(\mathcal{U})$ is open (closed) set for each open (closed) set $\mathcal{U} = \pi^{-1}\pi\mathcal{U}$. Let us say that $\mathcal{U}$ is $\pi$-satured.*

**Proof.** "$\Rightarrow$" Assume that $\tilde{g}$ is an open map with respect to $\tau_\sim$ and $\tau_{\mathbb{R}}$. That is $\tilde{g}(V) \in \tau_{\mathbb{R}}$ for any $V \in \tau_\sim$. Hence $\pi^{-1}(V) = \mathcal{U}$ is a $\tau_{\mathcal{F}_M}$-open set in $M$ by definition of $\tau_\sim$. Applying $\pi$ to both sides yields naturally $V = \pi(\mathcal{U})$ by surjectivity of $\pi$. Thus $\mathcal{U} = \pi^{-1}(V) = \pi^{-1}\pi\mathcal{U}$. It follows that $\tilde{g}(V) = \tilde{g}\pi(\mathcal{U}) = f(\mathcal{U})$ is an open set in $\tau_{\mathcal{F}_{\mathbb{R}}}$ such that $\mathcal{U} = \pi^{-1}\pi\mathcal{U}$ and $f \in \mathcal{F}_M$.

"$\Leftarrow$" Assume $f(\mathcal{U})$ be a $\tau_{\mathcal{F}_{\mathbb{R}}}$-open set with $\mathcal{U} = \pi^{-1}\pi\mathcal{U}$. That is,

$$f(\mathcal{U}) = f(\pi^{-1}\pi\mathcal{U}) = (f\pi^{-1})(\pi\mathcal{U}) = \tilde{g}(\pi\mathcal{U}). \tag{5}$$

Let $V \in \tau_\sim$. By the definition of $\tau_\sim$ and the surjectivity of $\pi$, it follows that

(6) $$\pi^{-1}(V) = \mathcal{U}, \quad V \in \tau_{\mathcal{F}_M} \text{if, and only if } V = \pi(\mathcal{U}).$$

Therefore, from equations 5 and 6, $f(\mathcal{U}) = \tilde{g}(V)$ is a $\tau_{\mathbb{R}}$-open set, with $V$ any $\tau_\sim$-open set in $\tilde{M}$. Hence $\tilde{g}$ is an open map.

It is no difficult to prove the closeness of $\pi$. ∎

**Corollary 4.1.** *Let $\tau_{\mathcal{F}_{\tilde{M}}}$ and $\tau_{\mathcal{F}_M}$ be given on $M$. Then $\mathcal{B} = \{\pi\mathcal{U} \mid \mathcal{U} \in \tau_{\mathcal{F}_M}\}$ is a base for $\tau_\sim$ and $\mathcal{B} = \{\pi(f^{-1}(0, +\infty)) \mid f \in \mathcal{F}_M\}$ is a base for $\tau_{\mathcal{F}_{\tilde{M}}}$.*

**Proof.** Let $V \in \tau_\sim$. That is, $V = \pi\mathcal{U}$ with $\mathcal{U} \in \tau_{\mathcal{F}_M}$ by definition of $\tau_\sim$ and Lemma 4.1. Thus $\mathcal{B} = \tau_\sim$ is the trivial base. From the universality condition, $\pi(f^{-1}(0, +\infty)) = \tilde{g}^{-1}(0, +\infty)$. Thus $\mathcal{B}$ is the standard base of the $\mathbb{F}$-space $\tilde{M}$. ∎

**Proposition 4.1.** *Given the three topologies defined on $\tilde{M}$. Then $\tau_{\mathcal{F}_{\tilde{M}}} = \tau_{\mathcal{C}_{\tilde{M}}} = \tau\sim$.*

**Proof.** In the above section, we proved that $\tau_{\mathcal{F}_{\tilde{M}}} \subset \tau_{\mathcal{C}_{\tilde{M}}} \subset \tau\sim$. We need to show that one can reverse these inclusions. Let $V \in \tau_\sim$. From assumption, $\pi^{-1}(V)$ lies in $\tau_{\mathcal{F}_M}$, the weakest topology on $M$ in which $\pi$ is continuous. Hence, $\pi^{-1}(V) = \bigcup_{f \in \mathcal{F}_M} f^{-1}(0, \infty)$. But $\pi$ is surjective, so $\pi(\pi^{-1}(V)) = V = \bigcup_{f \in \mathcal{F}_M} \pi f^{-1}(0, \infty)$. From the universality condition on $\mathbb{F}$-quotient, there exists a unique map $\tilde{g} \in \mathcal{F}_{\tilde{M}}$ such that $f = \tilde{g} \circ \pi$. So,

$$f^{-1}(0, \infty) = (\tilde{g}\pi)^{-1}(0, \infty) = \pi^{-1}\tilde{g}^{-1}(0, \infty)$$

and

$$\pi f^{-1}(0, \infty) = \pi(\pi^{-1}(\tilde{g}^{-1}(0, \infty))) = \tilde{g}^{-1}(0, \infty)$$

again since $\pi$ is surjective. This ends the proof. ∎

**Remark 4.1.** Note that because the three topologies coincide, they can indiscriminately be denoted by one of the three symbols $\tau\sim$ or $\tau_{\mathcal{F}_{\tilde{M}}}$ or $\tau_{\mathcal{C}_{\tilde{M}}}$.

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*Received March 11, 2008; revised version February 11, 2009.*