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ENTIRE GRAPHS UNDER A GENERAL FLOW

Abstract. An initial entire graph with bounded second fundamental form in R^{n+1} over some hyperplane is evolving under a general flow defined in the paper. For an additionally suitable condition in the main theorem, we obtain gradient and curvature estimates, leading to long-time existence of the flow, and convergence to an entire graph in the limit.

1. Introduction

Consider n-dimensional hypersurfaces M_t, defined by a one parameter family of smooth immersions $X_t : M^n \to R^{n+1}$, with $M_t = X_t(M^n)$. The hypersurfaces M_t are said to move by mean curvature, if $X_t = X(\cdot, t)$ satisfies

$$(1) \qquad \frac{d}{dt}X(p,t) = -H(p,t)v(p,t), \quad p \in M^n,\ t > 0.$$

By $v(p,t)$ we denote a choice of unit normal of M_t at $X(p,t)$, and by $H(p,t)$ the mean curvature with respect to this normal. The surface area $|M_t|$ of the hypersurface is known to decrease under the flow. So the evolution can be used for obtaining minimal surface in the limit, if it converges.

Here we are interested in the evolution of entire graphs M_t over some hyperplane. In particular, we consider the evolution equation

$$(2) \qquad \begin{cases} \frac{d}{dt}X(p,t) = -H(p,t)\vec{v}(p,t) + cX(p,t), & p \in M^n,\ t > 0 \\ X(\cdot, 0) = X_0, \end{cases}$$

where H is the mean curvature of $M_t = X_t(M^n)$ and c is bounded non-negative constant. As initial hypersurface we choose a locally Lipschitz continuous entire graph over some hyperplane. The vector $\vec{v}(p,t)$ is the outer unit normal.

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The smooth solutions M_t of (2) are still entire graphs over R^n (see section 3). In this case, the hypersurfaces can be expected to converge to a surface which is an entire graph over R^n in the limit.

The mean curvature flow has been studied by many mathematicians and obtained some good results. Readers can get the basic notations and methods from Huisken's classical works, such as [1], [2] and so on.

The main theorem we prove is

THEOREM. *Let $X_0 : R^n \to R^{n+1}$ be a locally Lipschitz continuous entire graph over R^n. The Cauchy problem (2) has a smooth solution $X(\cdot,t) : R^n \to R^{n+1}$ for all time $t \in [0,+\infty)$. Moreover, each $X(\cdot,t)$ is also an entire graph over R^n.*

The paper is organized as follows:

In Section 2 we give some definitions and the evolution equations of the flow. Gradient estimates (see section 3) and curvature estimates (see section 4) lead to long-time existence (see section 5).

The methods we use here are those introduced by Ecker-Huisken [3] for the mean curvature flow, and also used for instance in [4].

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2. The evolution equations

Let M^n be an n-dimensional smooth manifold and let

$$X(\cdot,t) : M^n \to R^{n+1}$$

be a one-parameter family of smooth hypersurface immersions in R^{n+1}.

In a local coordinate system $\{x^i\}$, $1 \leqslant i \leqslant n$, the metric and the second fundamental form of M_t can be computed as follows

$$g_{ij}(p,t) = \left(\frac{\partial X(p,t)}{\partial x^i}, \frac{\partial X(p,t)}{\partial x^j}\right), \quad h_{ij}(p,t) = -\left(\vec{v}, \frac{\partial^2 X(p,t)}{\partial x^i \partial x^j}\right).$$

The Gauss-Weingarten relations can be given as follows at the same time

$$\frac{\partial^2 X(p,t)}{\partial x^i \partial x^j} = \Gamma^k_{ij}\frac{\partial X}{\partial x^i} - h_{ij}\vec{v},$$

$$\frac{\partial \vec{v}}{\partial x^j} = h_{jl}g^{lm}\frac{\partial X}{\partial x^m}.$$

If $X(\cdot,t)$ is locally given as a graph over some hyperplane in R^{n+1} such that $(\vec{v},\vec{w}) > 0$ for some fixed vector $\vec{w} \in R^n, |\vec{w}| = 1$, we will consider the gradient function v defined by

$$v = (\vec{v},\vec{w})^{-1}.$$

Because the second fundamental form of initial entire graph is bounded, one can get the short time existence of (2) according to the standard theory of parabolic equation. The gradient on M_t and Beltrami-Laplace operator on M_t are denoted by ∇ and $\triangle$ respectively. We first have the following equations.

Lemma 2.1. *If $X(\cdot, t)$ satisfies (2), we have*

(i) $\frac{\partial}{\partial t} g_{ij} = -2Hh_{ij} + 2cg_{ij}$,

(ii) $\frac{\partial}{\partial t}\vec{v} = \nabla^i H \cdot \frac{\partial X}{\partial x^i}$,

(iii) $\frac{\partial}{\partial t} h_{ij} = \triangle h_{ij} - 2Hh_{il}g^{lm}h_{mj} + |A|^2 h_{ij} + ch_{ij}$,

(iv) $\frac{\partial H}{\partial t} = \triangle H + |A|^2 H - cH$,

(v) $\frac{\partial}{\partial t}|A|^2 = \triangle |A|^2 - 2|\nabla A|^2 + 2|A|^4 - 2c|A|^2$.

Proof. In the following computations we will use the definition of metric, the normal and the second fundamental form, and we will also use the Gauss-Weingarten relations.

(i)
$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= \frac{\partial}{\partial t}\left(\frac{\partial X}{\partial x^i}, \frac{\partial X}{\partial x^j}\right) \\
&= \left(\frac{\partial}{\partial x^i}(-H\vec{v} + cX), \frac{\partial X}{\partial x^j}\right) + \left(\frac{\partial X}{\partial x^i}, \frac{\partial}{\partial x^j}(-H\vec{v} + cX)\right) \\
&= -H\left(\frac{\partial \vec{v}}{\partial x^i}, \frac{\partial X}{\partial x^j}\right) - H\left(\frac{\partial \vec{v}}{\partial x^j}, \frac{\partial X}{\partial x^i}\right) + 2cg_{ij} \\
&= 2H\left(\vec{v}, \frac{\partial^2 X}{\partial x^i \partial x^j}\right) + 2cg_{ij} \\
&= -2Hh_{ij} + 2cg_{ij}
\end{aligned}$$

(ii)
$$\begin{aligned}
\frac{\partial \vec{v}}{\partial t} &= \left(\frac{\partial \vec{v}}{\partial t}, \frac{\partial X}{\partial x^i}\right)\frac{\partial X}{\partial x^j} g^{ij} \\
&= -\left(\vec{v}, \frac{\partial}{\partial t}\frac{\partial X}{\partial x^i}\right)\frac{\partial X}{\partial x^j} g^{ij} \\
&= -\left(\vec{v}, \frac{\partial}{\partial x^i}(-H\vec{v} + cX)\right) g^{ij}\frac{\partial X}{\partial x^j} \\
&= \frac{\partial H}{\partial x^i} g^{ij} \frac{\partial X}{\partial x^j} = \nabla^i H \cdot \frac{\partial X}{\partial x^i} = \nabla H
\end{aligned}$$

(iii) $$\frac{\partial h_{ij}}{\partial t} = -\frac{\partial}{\partial t}\left(\frac{\partial^2 X}{\partial x^i \partial x^j}, \vec{v}\right)$$
$$= -\left(\frac{\partial^2}{\partial x^i \partial x^j}(-H\vec{v} + cX), \vec{v}\right) - \left(\frac{\partial^2 X}{\partial x^i \partial x^j}, \frac{\partial H}{\partial x^l} g^{lm} \frac{\partial X}{\partial x^m}\right)$$
$$= \left(\frac{\partial^2}{\partial x^i \partial x^j}(H\vec{v}), \vec{v}\right) + ch_{ij} - \left(\frac{\partial^2 X}{\partial x^i \partial x^j}, \frac{\partial H}{\partial x^l} g^{lm} \frac{\partial X}{\partial x^m}\right)$$
$$= \left(\frac{\partial}{\partial x^i}(\frac{\partial H}{\partial x^j}\vec{v} + H h_{jl} g^{lm} \frac{\partial X}{\partial x^m}), \vec{v}\right)$$
$$- \left(\Gamma^k_{ij}\frac{\partial X}{\partial x^k} - h_{ij}\vec{v}, \frac{\partial H}{\partial x^l} g^{lm} \frac{\partial X}{\partial x^m}\right) + ch_{ij}$$
$$= \frac{\partial^2 H}{\partial x^i \partial x^j} + H\left(\frac{\partial}{\partial x^i}(h_{jl} g^{lm} \frac{\partial X}{\partial x^m}), \vec{v}\right) - \Gamma^k_{ij}\frac{\partial H}{\partial x^l} g^{lm} g_{km} + ch_{ij}$$
$$= \frac{\partial^2 H}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial H}{\partial x^k} + H\left(h_{jl} g^{lm} \frac{\partial^2 X}{\partial x^i \partial x^m}, \vec{v}\right) + ch_{ij}$$
$$= \nabla_i \nabla_j H - H h_{jl} g^{lm} h_{im} + ch_{ij}.$$

Lemma 2.2.

$$(1) \quad \triangle h_{ij} = \nabla_i \nabla_j H + H h_{il} g^{lm} h_{mj} - |A|^2 h_{ij}$$
$$(2) \quad \frac{1}{2}\triangle |A|^2 = \langle h_{ij}, \nabla_i \nabla_j H\rangle + |\nabla A|^2 + Z$$

where $Z = Htr(A^3) - |A|^4$, $tr(A^3) = g^{ij} g^{kl} g^{mn} h_{ik} h_{lm} h_{nj}$.

Proof. Lemma 2.2 is the same as Lemma 2.3 in [4].

Let us come back to the proof of (iii). Substitute (1) in Lemma 2.2 into the above computation, we get
$$\frac{\partial}{\partial t} h_{ij} = \triangle h_{ij} - 2H h_{il} g^{lm} h_{mj} + |A|^2 h_{ij} + ch_{ij}.$$

(iv) $$\frac{\partial H}{\partial t} = \frac{\partial}{\partial t}(g^{ij} h_{ij})$$
$$= -g^{il}\frac{\partial g_{lm}}{\partial t} g^{mj} h_{ij} + g^{ij}\frac{\partial h_{ij}}{\partial t}$$
$$= 2H h^{ij} h_{ij} - 2c g^{ij} h_{ij} + g^{ij}(\triangle h_{ij} - 2H h^m_i h_{mj} + |A|^2 h_{ij} + ch_{ij})$$
$$= \triangle H + |A|^2 H - cH.$$

(v) $$\frac{\partial}{\partial t}|A|^2 = \frac{\partial}{\partial t}(g^{ik} g^{jl} h_{ij} h_{kl})$$
$$= 2\left(\frac{\partial}{\partial t} g^{ik}\right) g^{jl} h_{ij} h_{kl} + 2 g^{ik} g^{jl}\left(\frac{\partial}{\partial t} h_{ij}\right) h_{kl}$$

$$
\begin{aligned}
&= 4g^{im}(Hh_{mn} - cg_{mn})g^{nk}g^{jl}h_{ij}h_{kl} \\
&\quad + 2g^{ik}g^{jl}(\Delta h_{ij} - 2Hh_{il}g^{lm}h_{mj} + |A|^2 h_{ij} + ch_{ij})h_{kl} \\
&= -4cg^{ik}g^{jl}h_{ij}h_{kl} + 2h^{ij}\Delta h_{ij} + 2|A|^4 + 2c|A|^2 \\
&= 2\langle h_{ij}, \Delta h_{ij}\rangle + 2|A|^4 - 2c|A|^2.
\end{aligned}
$$

However

$$\Delta|A|^2 = g^{kl}\nabla_k\nabla_l\langle h_{ij}, h_{ij}\rangle = 2g^{kl}\nabla_k\langle h_{ij}, \nabla_l h_{ij}\rangle = 2|\nabla A|^2 + 2\langle h_{ij}, \Delta h_{ij}\rangle.$$

Thus

$$\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + 2|A|^4 - 2c|A|^2.$$

Lemma 2.3. *If $X(\cdot, t)$ satisfies equation (2), we have*

(1) $\frac{\partial}{\partial t}|X|^2 = \Delta|X|^2 - 2n + c|X|^2$,

(2) $\frac{\partial}{\partial t}u = \Delta u + cu$, *where* $u = (X, \vec{w})$,

(3) $\frac{\partial}{\partial t}v = \Delta v - |A|^2 v - 2v^{-1}|\nabla v|^2$.

Proof. By direct computations, we have

$$
\begin{aligned}
(1)\quad \frac{\partial}{\partial t}|X|^2 &= 2\left(X, \frac{\partial}{\partial t}X\right) = 2(X, \Delta X + cX) \\
&= \Delta|X|^2 - 2(\nabla_i X, \nabla^i X) + c|X|^2 = \Delta|X|^2 - 2n + c|X|^2,
\end{aligned}
$$

$$
\begin{aligned}
(2)\quad \frac{\partial}{\partial t}u &= \left(\frac{\partial}{\partial t}X, \vec{w}\right) = (\Delta X + cX, \vec{w}) \\
&= \Delta(X, \vec{w}) + c(X, \vec{w}) = \Delta u + cu,
\end{aligned}
$$

$$(3)\quad \frac{\partial}{\partial t}v = -v^2\left(\frac{\partial}{\partial t}\vec{v}, \vec{w}\right) = -v^2(\nabla H, \vec{w}).$$

On the other hand,

$$
\begin{aligned}
\Delta v &= g^{ij}\nabla_i\nabla_j((\vec{v}, \vec{w})^{-1}) = g^{ij}\nabla_i(-v^2(\nabla_j\vec{v}, \vec{w})) \\
&= g^{ij}\nabla_i\left(-v^2(h_{jl}g^{lm}\frac{\partial X}{\partial x^m}, \vec{w})\right) \\
&= 2v^3\left(h_{ik}g^{kn}\frac{\partial X}{\partial x^n}, \vec{w}\right)\left(h_{jl}g^{lm}\frac{\partial X}{\partial x^m}, \vec{w}\right) - v^2(g^{ij}\nabla_i h_{jl}\cdot g^{lm}\frac{\partial X}{\partial x^m}, \vec{w}) \\
&\quad - v^2\left(h_{jl}g^{lm}g^{ij}\nabla_i\frac{\partial X}{\partial x^m}, \vec{w}\right) \\
&= 2v^{-1}g^{ij}\nabla_i v\nabla_j v - v^2(\nabla H, \vec{w}) + v^2(h_{jl}g^{lm}g^{ij}h_{im}\vec{v}, \vec{w}) \\
&= v^2(\nabla H, \vec{w}) + v|A|^2 + 2v^{-1}|\nabla v|^2.
\end{aligned}
$$

Then we get the evolution equation for v.

3. Gradient estimates

As an immediate consequence of Lemma 2.3 (3), we know from the maximum principle that if $M^n = R^n$ and $X(\cdot, 0)$ is an entire graph with uniform bounded gradient v, then the solution $X(\cdot, t)$ remains to be entire graphs and its gradient also uniformly bounded by the same constant. The localized version of this gradient estimate is the following proposition.

Let $R > 0$ and $x_0 \in R^{n+1}$ be arbitrary, we define

$$\varphi(X, t) = R^2 - |X - x_0|^2 - 2nt$$

and denote φ_+ to be the positive part of φ.

Proposition 3.1.

$$v(X, t)\varphi_+ \leqslant \sup_{X(\cdot,0)} v\varphi_+$$

as long as $v(X, t)$ is defined everywhere on the support of φ_+.

Proof. Without loss of generality, we may assume $x_0 = 0$. For $R > 0$ we define

$$\eta(r) = (R^2 - r)^2.$$

Note that η satisfies

$$\eta^{-1}(\eta')^2 = 4 \quad \text{and} \quad \eta'' = 2.$$

If $r = |X|^2 + 2nt$, we derive from Lemma 1.3 (1) that

$$\begin{aligned} \frac{\partial}{\partial t}\eta &= 2(r - R^2)\frac{\partial r}{\partial t} \\ &= 2(r - R^2)\frac{\partial}{\partial t}(|X|^2 + 2nt) \\ &= 2(r - R^2)\triangle|X|^2 + 2c(r - R^2)|X|^2. \end{aligned}$$

On the other hand,

$$\begin{aligned} \triangle\eta &= \nabla^i(2(r - R^2)\nabla_i r) \\ &= 2(r - R^2)\triangle r + 2\nabla^i r \cdot \nabla_i r \\ &= 2(r - R^2)\triangle|X|^2 + 2|\nabla|X|^2|^2. \end{aligned}$$

Thus we get

$$\frac{\partial}{\partial t}\eta = \triangle\eta - 2|\nabla|X|^2|^2 + 2c(r - R^2)|X|^2. \tag{3}$$

Combining (3) with Lemma 2.3 (3) we obtain

$$\begin{aligned} \frac{\partial}{\partial t}(v^2\eta) &= 2v\frac{\partial v}{\partial t} \cdot \eta + v^2\frac{\partial \eta}{\partial t} \\ &= 2v(\triangle v - |A|^2 v - 2v^{-1}|\nabla v|^2)\eta \\ &\quad + v^2(\triangle\eta - 2|\nabla|X|^2|^2 + 2c(r - R^2)|X|^2) \end{aligned}$$

and

$$\begin{aligned}\Delta(v^2\eta) &= \Delta(v^2)\eta + v^2\Delta\eta + 2\nabla v^2 \cdot \nabla\eta \\ &= (2v\Delta v + 2|\nabla v|^2)\eta + v^2\Delta\eta + 2\nabla v^2 \cdot \nabla\eta.\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial}{\partial t}(v^2\eta) = \Delta(v^2\eta) - 6|\nabla v|^2\eta - 2\nabla v^2 \cdot \nabla\eta \\ - 2v^2|A|^2\eta - 2v^2|\nabla|X|^2|^2 + 2cv^2(r - R^2)|X|^2.\end{aligned}$$

Observe that

$$-2\nabla v^2 \cdot \nabla\eta = -6v\nabla v\nabla\eta + \eta^{-1}\nabla\eta\nabla v^2\eta - \eta^{-1}|\nabla\eta|^2 v^2,$$

we have

$$\frac{\partial}{\partial t}(v^2\eta) \leqslant \Delta(v^2\eta) - 2|A|^2(v^2\eta) + \eta^{-1}\nabla\eta\nabla v^2\eta + 2cv^2(r - R^2)|X|^2. \tag{4}$$

If we replace $R^2 - r$ by φ_+, this computation remains valid on the support of φ_+ as long as v is defined. The inequality (4) will change into

$$\begin{aligned}\frac{\partial}{\partial t}(v^2\varphi_+^2) &\leqslant \Delta(v^2\varphi_+^2) - 2|A|^2(v^2\varphi_+^2) + \varphi_+^{-2}\nabla\varphi_+^2\nabla(v^2\varphi_+^2) - 2cv^2\varphi_+|X|^2 \\ &\leqslant \Delta(v^2\varphi_+^2) + \varphi_+^{-2}\nabla\varphi_+^2\nabla(v^2\varphi_+^2).\end{aligned}$$

That is

$$\frac{\partial}{\partial t}(v^2\varphi_+^2) - \Delta(v^2\varphi_+^2) \leqslant \varphi_+^{-2}\nabla\varphi_+^2\nabla(v^2\varphi_+^2).$$

By the maximum principle we can get the result.

COROLLARY 3.2. *Let $X(\cdot,t)$ be a solution of (2) on $[0,w)$. Suppose the initial hypersurface $X(\cdot,0)$ is an entire graph over R^n. Then $X(\cdot,t)$ remains to be an entire graph over R^n for any $t \in [0,w)$.*

4. Curvature estimate

In the followings we shall prove that as long as $X(\cdot,t)$ can be written as a graph with bounded gradient, the curvature remains bounded as well. We begin with a global version of curvature estimate. Recall from Proposition 3.1 and Lemma 2.3 that

$$\frac{\partial|A|^2}{\partial t} \leqslant \Delta|A|^2 - 2|\nabla|A||^2 + 2|A|^4$$

and

$$\frac{\partial v^2}{\partial t} = \Delta v^2 - 2|A|^2v^2 - 6|\nabla v|^2.$$

Then we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)|A|^2v^2 \leqslant -2\nabla|A|^2\nabla v^2 - 2|\nabla|A||^2v^2 - 6|\nabla v|^2|A|^2,$$

where

$$\begin{aligned} -2\nabla|A|^2\nabla v^2 &= \nabla|A|^2\nabla v^2 - 4v|A|\nabla|A|\nabla v \\ &= -v^{-2}\nabla v^2\nabla|A|^2v^2 + v^{-2}|\nabla v^2|^2|A|^2 - 4v|A|\nabla|A|\nabla v. \end{aligned}$$

This implies

$$\left(\frac{\partial}{\partial t} - \triangle\right)|A|^2v^2 \leqslant -v^{-2}\nabla v^2\nabla|A|^2v^2.$$

Thus we obtain the curvature estimate

$$\sup_{X(\cdot,t)} |A|^2v^2 \leqslant \sup_{X(\cdot,0)} |A|^2v^2.$$

To deduce the localized version of this curvature estimate, we use the notations and methods introduced in [4].

Let $R > 0$ and $x_0 \in R^{n+1}$ be such that $\{X \in X(\cdot,t)|2nt+|X-x_0|^2 \leqslant R^2\}$ can be written as graph over some hyperplane for $t \in [0,T]$.

Denote

$$K(x_0,t,R^2) = \{X \in X(\cdot,t)|2nt + |X - x_0|^2 \leqslant R^2\}.$$

Then we have the following proposition

Proposition 4.1. *For any $0 \leqslant \theta \leqslant 1$ we have the estimate*

$$\sup_{K(x_0,t,\theta R^2)} |A|^2 \leqslant D(n)(1-\theta)^{-2}t^{-1} \sup_{0\leqslant s\leqslant t} \sup_{K(x_0,s,R^2)} v^2,$$

where $D(n)$ is a constant which is depend on n.

Proof. We proceed as in [4] (proof of Proposition 7.5) and calculate the evolution inequality of product $g\eta$, where $g = |A|^2\varphi v^2$ and $\varphi(v^2) = \frac{v^2}{1-kv^2}$, $k > 0$. The only difference is the evolution equation of η, which is affected by an additional term $2c(r - R^2)|X|^2$ (see (3) above), we end up with the inequality

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \triangle\right)g\eta \leqslant{}& -2kg^2\eta - \frac{2k}{(1-kv^2)^2}|\nabla v|^2 g\eta - 2\varphi v^{-3}\nabla v\nabla g\cdot\eta - 2\nabla g\nabla\eta \\ & - 2|\nabla|X|^2|^2 g + 2gc|X|^2(r - R^2). \end{aligned}$$

Since c, g are non-negative and $r - R^2 \leqslant 0$ in $K(x_0,t,R^2)$, that means the last term $2gc|X|^2(r - R^2)$ is non-positive. Then the inequality above will change into

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \triangle\right)g\eta \leqslant{}& -2kg^2\eta - \frac{2k}{(1-kv^2)^2}|\nabla v|^2 g\eta \\ & - 2\varphi v^{-3}\nabla v\nabla g\cdot\eta - 2\nabla g\nabla\eta - 2|\nabla|X|^2|^2 g, \end{aligned}$$

which is the same as the corresponding inequality in [4]. In the forthcoming, the remainder is totally the same with the proof of Proposition 7.5 in [4].

We know from the proof of Proposition 4.2 in [4] that the derivatives of the curvature in the case of mean curvature flow satisfy the following equation

$$\frac{\partial}{\partial t}|\nabla^m A|^2 = \triangle|\nabla^m A|^2 - 2|\nabla^{m+1}A|^2 + \sum_{i+j+k=m} \nabla^m A * \nabla^i A * \nabla^j A * \nabla^k A,$$

where $S * T$ denotes linear combination of traces of S and T. By the similar argument, we can get the derivatives of the curvature under the flow (2) satisfy the following equation

LEMMA 4.2. *Under the flow* (2) *we have*

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^m A|^2 = {} & \triangle|\nabla^m A|^2 - 2|\nabla^{m+1}A|^2 + \sum_{i+j+k=m} \nabla^m A * \nabla^i A * \nabla^j A * \nabla^k A \\ & - c\nabla^m A * P(\nabla^m A, \nabla^{m-1}A, \dots, \nabla A),\end{aligned}$$

where $P(\nabla^m A, \nabla^{m-1}A, \dots, \nabla A)$ is a polynomial which denotes linear combination of $\nabla^m A, \nabla^{m-1}A, \dots, \nabla A$ and the operator $*$ has the same meaning as above.

By the similar argument it is not hard to extend the exterior estimate in Proposition 4.1 to all derivatives of A.

PROPOSITION 4.3. *For any $m \geqslant 0$, $0 \leqslant \theta < 1$ and $t \in [0,T]$ we have the estimate*

$$\sup_{K(x_0,t,\theta R^2)} |\nabla^m A|^2 \leqslant D_m\left(\frac{1}{R^2} + \frac{1}{t}\right)^{m+1},$$

where $D_m = D_m(\theta, n, m, c, \sup_{0\leqslant s\leqslant t} \sup_{K(x_0,s,R^2)} v)$.

5. Long time existence

THEOREM 5.1. *Let $X_0 : R^n \to R^{n+1}$ be a locally Lipschitz continuous entire graph over R^n. The Cauchy problem* (2) *has a smooth solution $X(\cdot,t) : R^n \to R^{n+1}$ for all time $t \in [0,+\infty)$. Moreover, each $X(\cdot,t)$ is also an entire graph over R^n.*

Proof. According to the general theory of linear parabolic equation, the equation (2) has a unique smooth solution on some short time interval. Our gradient and curvature estimates then ensure that this solution extends to all $t > 0$. Readers can also see the detailed proof of Theorem 7.1 in [4].

REMARK. We consider the following equation

$$\begin{cases} \frac{d}{dt}X(p,t) = -H(p,t)\vec{v}(p,t) + cX(p,t), & p \in M^n,\ t > 0 \\ X(\cdot,0) = X_0, \end{cases}$$

where the initial hypersurface X_0 with boundary satisfies

(i) X_0 is an entire graph over some hyperplane Π if we can't consider the boundary,

(ii) X_0 intersects Π orthogonally at the free boundary,

in addition,

(iii) the boundary $X_t(\partial M)$ is contained in Π.

Then the Cauchy problem (2) will change into the Neumann boundary problem. In this case, one can use the same method to deal with the problem, then you can get a similar conclusion.

By the way, if the nonnegative constant c in evolution equation (2) changes into a bounded continuous function $c(t)$, we could also get the same conclusion by the totally same argument in the paper. Obviously, that will be an extension to our consequence. In order to embody the continuity of the authors' thought about the problem and the surprising identity between the proofs of two cases, so we just consider the case of nonnegative constant in the main body of this paper. Specially, if $c = 0$ in this paper, then the flow will change into the mean curvature flow which has done by Ecker-Huisken in [3].

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