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GENERALIZED PAPPUS' THEOREM

Abstract. The paper contains a generalization to the n-dimensional projective space over a commutative field of a famous theorem of Pappus.

The well-known theorem of Pappus, as one of the most important theorems of the projective geometry, was a subject of many investigations. We may mention here for instance the works [1], [2], [3] and [4], where this theorem was generalized to the n-dimensional projective space P^n (projective space over an arbitrary commutative field). In particular, the generalization from [1] concerns two sets of points $A = \{a_0, \ldots, a_n\}$ and $B = \{b_0, \ldots, b_n\}$ on two hyperplanes H_1 and H_2, respectively. The theorem says that the dimension of the join of subspaces (points in general) $S_0, \ldots, S_n$ is not greater than $n-1$ ($S_j = \bigcap_{i=0, j\neq j}^{n} S_{ij}$, where $S_{ij} = J(b_i, A \setminus \{a_i, a_j\})$, $i \neq j$ (the symbol $J(P_1, \ldots, P_m)$ denotes the join of subspaces $P_1, \ldots, P_m$). Points $a_0, \ldots, a_n$ as well as $b_0, \ldots, b_n$ are assumed to be in a general position i.e. no n of them are in an $(n-2)$-dimensional subspace. Obviously, when $n = 2$, it is the usual plane Pappus' theorem. In this work we present a more general theorem than that from [1]. Throughout the paper we investigate two sets of points $A = \{a_0, \ldots, a_n\}$ and $B = \{b_0, \ldots, b_n\}$ such that $\dim J(A) = n-1$, $\dim J(B) = k$, $1 \leq k \leq n-1$, and points $a_0, \ldots, a_n$ as well as $b_0, \ldots, b_n$ are in a general position (no $k+1$ points of $b_0, \ldots, b_n$ are in a $(k-1)$-dimensional subspace). Then we shall show that $\dim J(S_0, \ldots, S_n) \leq k$. First we prove

Lemma 1. *If points $b_0, \ldots, b_m$ $(0 \leq m \leq k-1)$ are in $H_1 = J(A)$, then* $\dim J(S_0, \ldots, S_n) \leq \max(1, m)$.

Proof. First suppose $b_0 \in H_1$ and $b_i \notin H_1$ for $i = 1, \ldots n$. Then [1] $S_j = a_0$ (when $b_0 \notin J(A \setminus \{a_0, a_j\})$ or $S_j = \emptyset$ (when $b_0 \in J(A \setminus \{a_0, a_j\})$ for all $j = 1, \ldots, n$. Since $\dim S_0 = 0$, $\dim J(S_0, \ldots, S_n) = 1$.

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Let now $m \geq 1$. For $j \leq m$ $S_j = A_j \cap B_j$, where $A_j = \bigcap_{i=0, i\neq j}^n S_{ij}$, $B_j = \bigcap_{i=m+1}^n S_{ij}$. Notice that $\{a_0, \dots a_m\} \setminus \{a_j\} \subseteq B_j$ and $\dim B_j = m$. Hence $B_j \cap H_1 = J(\{a_0, \dots, a_m\} \setminus \{a_j\})$. Since $A_j \subseteq H_1$, $S_j \subseteq J(a_0, \dots a_m)$. On the other hand, for $j \geq m+1$ $S_j = C_j \cap D_j$, where $C_j = \bigcap_{i=0}^m S_{ij}$, $D_j = \bigcap_{i=m+1, i\neq j}^n S_{ij}$. We have $\{a_0, \dots, a_m\} \subseteq D_j$, $\dim D_j = m+1$ and, consequently, $\dim(D_j \cap H_1) = m$. It means that $D_j \cap H_1 = J(a_0, \dots, a_m)$. As in the previous case $C_j \subseteq H_1$, hence $S_j \subseteq J(a_0, \dots, a_m)$. Thus we see that $J(S_0, \dots, S_n) \subseteq J(a_0, \dots, a_m)$. This ends the proof. ∎

Lemma 2. *Let $A = \{a_0, \dots, a_n\}$ and $B = \{b_0, \dots, b_n\}$ be two sets of points on two hyperplanes H_1 and H_2, respectively. Points $a_0, \dots, a_n$ are assumed to be in a general position. If some of points b_i coincide, then* $\dim J(S_0, \dots, S_n) \leq n-1$.

Proof. In view of Lemma 1 we may assume that $b_i \notin H_1$ for $i = 0, \dots, n$. Hence [1] $\dim S_j = 0$, all j (i.e. S_j are points). Suppose e.g. $b_0 = b_1$. We have $S_{01} = J(b_0, a_2, \dots, a_n) = S_{10} = J(b_1, a_2, \dots, a_n)$. Hence $S_0, S_1 \in S_{01}$. Observe that for $j \geq 2$, $S_j \in J(b_0, \{a_2, \dots a_n\} \setminus \{a_j\}) \subset S_{01}$. ∎

Let now $A = \{a_0, \dots, a_n\}$ and $B = \{b_0, \dots, b_n\}$ be two sets of points in a general position such that $b_i \notin J(A)$ for all i, and $\dim J(A) = n-1$, $\dim J(B) = k$, $1 \leq k \leq n-2$. There are, among points a_i, at least $n-k+1$ not belonging to $J(B) = H_2$. We choose $n-k-1$, say $a_0, \dots, a_{n-k-2}$, from them in such a way that $\dim J(a_0, \dots, a_{n-k-2}, H_2) = n-1$.

Lemma 3. *There are, in $\overline{H}_2 = J(a_0, \dots, a_{n-k-2}, B)$, points $c_{k+1}, \dots, c_n$ such that points $\overline{b}_0, \dots \overline{b}_n$ are in a general position, where $\overline{b}_i = b_i$ for $i = 0, \dots, k$, $\overline{b}_i = c_i$ for $i = k+1, \dots, n$, and $\overline{S}_j = S_j$ for $j = n-k-1, \dots, n$ ($\overline{S}_j = \bigcap_{i=0, i\neq j}^n \overline{S}_{ij}$, $\overline{S}_{ij} = J(\overline{b}_i, A \setminus \{a_i, a_j\})$).*

Proof. We choose a point $c_{k+1+i} \neq b_{k+1+i}, a_i$ on a line $l_i = J(a_i, b_{k+1+i})$, $i = 0, \dots, n-k-2$. Obviously, $c_{k+1} \notin H_2$, $c_{k+2} \notin H_3 = J(H_2, c_{k+1}), \dots$, $c_{n-1} \notin H_{n-k-2} = J(H_{n-k-3}, c_{n-2})$. Thus we have n linearly independent points $b_0, \dots, b_k, c_{k+1}, \dots, c_{n-1}$ which are vertices of an $(n-1)$-dimensional simplex S contained in $\overline{H}_2$. Consider the $(n-k-1)$-dimensional subspace G determined by points $a_0, \dots, a_{n-k-2}, b_n$. G cuts the faces of S in subspaces G_i $i = 1, \dots, n$. Finally we choose a point c_n in G in such a way that $c_n \notin G_i$, all i. We have still to show that $\overline{S}_j = S_j$ for $j = n-k-1, \dots, n$. Observe that

$$S_j = \bigcap_{i=0}^{k} S_{ij} \cap \bigcap_{i=k+1, i\neq j}^{n} S_{ij}, \quad \overline{S}_j = \bigcap_{i=0}^{k} S_{ij} \cap \bigcap_{i=k+1, i\neq j}^{n} \overline{S}_{ij} \text{ for } j = n-k-1, \dots, n.$$

Nevertheless, when $k+1 \le i \le n-1$,

$$S_{ij} = J(a_0, \dots, a_{n-k-2}, b_i, \{a_{n-k-1}, \dots, a_n\} \setminus \{a_i, a_j\})$$
$$= J(a_0, \dots, a_{n-k-2}, c_i, \{a_{n-k-1}, \dots, a_n\} \setminus \{a_i, a_j\}) = \overline{S}_{ij},$$

since $c_i \in J(a_{i-k-1}, b_i)$ and points c_i, a_{i-k-1}, b_i are all distinct. If $i = n$, then

$$S_{nj} = J(a_0, \dots, a_k, \{a_{k+1}, \dots, a_{n-1}\} \setminus \{a_j\}, b_n)$$
$$= J(a_0, \dots, a_k, \{a_{k+1}, \dots, a_{n-1}\} \setminus \{a_j\}, c_n) = \overline{S}_{nj},$$

since $c_n \in J(a_0, \dots, a_{n-k-2}, b_n) = G$ and $b_n \in J(a_0, \dots, a_{n-k-2}, c_n) = G$. This completes the proof. ∎

Lemma 4. *As previously, we consider two sets of points, in a general position, $A = \{a_0, \dots, a_n\}$ and $B = \{b_0, \dots, b_n\}$ such that $\dim H_1 = \dim H_2 = n-1$, where $H_1 = J(A)$, $H_2 = J(B)$. If $a_0, \dots, a_{k-1} \in H_2$ and $b_i \notin H_1$, all i, then $\dim J(S_k, \dots, S_n) \le n-k-1$.*

Proof. Obviously, without loss of generality, we may assume that $a_j \notin H_2$ for $j \ge k$. Suppose we choose in P^n an allowable coordinate system in such a way that the j-th coordinate of point a_i equals to δ_i^j, the equation of H_2 is $\sum_{i=k}^n x_i$ and $a_n = (1, 1, \dots, 1.0)$, where $i = 0, \dots, n-1$, $j = 0, \dots, n$, and δ_i^j is the Kronecker δ. By b_{ij} we denote the j-th coordinate of point b_i, $i = 0, \dots, n$, $j = 0, \dots, n-1$. Then b_{in} will equal to $-\sum_{j=k}^{n-1} b_{ij}$. Notice that $\sum_{j=k}^{n-1} b_{ij} \ne 0$ for $i = 0, \dots, n$. Let us denote the sums $\sum_{j=k}^{n-1} b_{ij}$, $\sum_{j=k}^{n-1} b_{nj}$ by M_i and M, respectively. One can check easily that the hyperplane S_{ni} has the equation $x_i M_i + x_n b_{ii} = 0$, $i = 0, \dots, n-1$. Consequently, the i-th coordinate s_{ni} of point S_n is b_{ii}/M_i, $i = 0, \dots, n-1$, while the n-th coordinate of this point equals to -1. Similarly, we check that hyperplane S_{ji} has the equation $(x_j - x_i)M_i + x_n(b_{ij} - b_{ii}) = 0$ $j = k, \dots, n-1$, $i = 0, \dots, n-1$, and the equation of S_{jn} is $x_j M + x_n b_{nj} = 0$, $j = 0, \dots, n-1$. Hence the i-th coordinate s_{ji} of point S_j is

$$\frac{b_{ij} - b_{ii}}{M_i} - \frac{b_{nj}}{M}, \; i = 0, \dots, n-1, \; i \ne j, \; s_{jj} = -\frac{b_{nj}}{M}, \; s_{jn} = 1, \; j = k, \dots, n-1.$$

Observe that

$$(n-k)s_{ni} + \sum_{j=k}^{n-1} s_{ji} = 0 \quad \text{for } i = 0, \dots, n.$$

It means that points $S_k, \dots, S_n$ are linearly dependent i.e. $\dim J(S_k, \dots, S_n) \le n-k-1$.

Let now $A = \{a_0, \dots, a_n\}$, $B = \{b_0, \dots, b_n\}$ be two sets of points like those described in the introduction.

THEOREM. *If* $\dim J(A) = n-1$ *and* $\dim J(B) = k$, $1 \le k \le n-1$ *and* $J(B) \not\subset J(A)$, *then* $\dim J(S_0, \ldots, S_n) \le k$.

Proof. Of course, we may consider $k \le n-2$. In view of Lemma 1, we may assume that $b_i \notin H_1$ for $i = 0, \ldots, n$. In fact, from $b_0, \ldots, b_m \in H_1$ it follows that $\dim J(S_0, \ldots, S_n) \le \max(1, m)$, but $m \le k-1$. Thus [1] the subspaces $S_0, \ldots, S_n$ are points. Suppose $\dim J(S_0, \ldots, S_n) > k$. Hence there exist $k+2$ points, among $S_0, \ldots, S_n$, say $S_{n-k-1}, \ldots, S_n$, such that $\dim J(S_{n-k-1}, \ldots, S_n) = k+1$. Take into account points $a_0, \ldots, a_{n-k-2}$. Denote $J(B, a_0, \ldots, a_{n-k-2})$ by H_2. If $\dim H_2 = n-1$, then by Lemma 3, there are points $c_{k+1}, \ldots, c_n$ in H_2 such that the respective points $\overline{S}_j$ are equal to S_j for $j = n-k-1, \ldots, n$. According to Lemma 4 , $\dim J(S_{n-k-1}, \ldots, S_n) \le k$, a contradiction. If $\dim H_2 < n-1$, we add points $a_{n-k-1}, \ldots, a_m$ to the points $a_0, \ldots, a_{n-k-2}$ in such a way that $\dim J(B, a_0, \ldots, a_m) = n-1$ and $\dim J(B, a_0 \ldots, a_{m-1}) < n-1$. There is, among points $a_0, \ldots, a_m$, a subset of $n-k-1$ points, say $a_{i_1}, \ldots, a_{i_{n-k-1}}$ such that $\dim J(a_{i_1}, \ldots, a_{i_{n-k-1}}, B) = n-1$. Hence, Lemma 3, there exist points $c_{k+1}, \ldots, c_n$ such that the respective points $\overline{S}_j$ are equal to S_j for $j \notin \{i_1, \ldots, i_{n-k-1}\}$. In particular, it has place when $j = m+1, \ldots, n$. According to Lemma 4, $\dim J(S_{m+1}, \ldots, S_n) \le n-m-2$. It implies $\dim J(S_m, \ldots, S_n) \le n-m-1$, $\dim J(S_{m-1}, \ldots, S_n) \le n-m$ and so on. Finally, we obtain $\dim J(S_{n-k-1}, \ldots, S_n) \le k$ which contradicts with the supposition $\dim J(S_0, \ldots, S_{k+1}) = k+1$. This ends the proof.

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