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**A QUADRATIC TRANSFORMATION OF THE  
THREE-DIMENSIONAL SPACE OF ALL CONICS  
THROUGH TWO POINTS OF A PROJECTIVE PLANE**

**Abstract.** Using the Steiner's method of projective generation of conics and its dual we define two projective mappings of a double contact pencil of conics into itself and we prove that one is the inverse of the other. We show that these projective mappings are induced by quadratic transformations of the three-dimensional projective space of all conics through two distinct points of a projective plane.

**1. Introduction**

In 1832 Jacob Steiner proved that conics may be generated using two projectively related pencils of lines. Let  $\mathcal{P}_A$  and  $\mathcal{P}_B$  be the pencils of lines with vertices two distinct points  $A, B$  of a projective plane over a field  $K$ . Let  $\Phi$  be a projectivity between  $\mathcal{P}_A$  and  $\mathcal{P}_B$  that does not map the line  $A \vee B$  onto itself. In [5] (see also [2], [3] or [4]) it is proved that the set of points of intesection of corresponding lines under  $\Phi$  is a non-singular conic  $\gamma$  containing  $A$  and  $B$ . The projectivity  $\Phi$  maps the tangent line to  $\gamma$  at  $A$  onto the line  $A \vee B$  and the line  $A \vee B$  onto the tangent line to  $\gamma$  at  $B$ . Also due to Steiner is the following dual construction. Let  $a$  and  $b$  be two distinct lines and let  $\Psi$  be a projectivity between  $a$  and  $b$  that does not map the point  $a \cap b$  onto itself. By the principle of duality it follows that the lines joining corresponding points under  $\Psi$  are tangent lines to a non-singular conic  $\gamma'$  with the property that  $a$  and  $b$  are tangent lines to  $\gamma'$ .

A conic is a self-dual concept only if the characteristic of  $K$  is different from two. In fact if the characteristic is two all tangent lines to a conic contain a special point called *nucleus*. Moreover two projectively related pencils of lines generate a conic, but two projectively related lines fail to generate tangent lines to a conic.

Let  $R$  and  $S$  be two distinct points and let  $t_R$  and  $t_S$  be two lines containing  $R$  and  $S$  (respectively), both different from the line  $R \vee S$ . The points  $R$ ,  $S$  and the lines  $t_R$ ,  $t_S$  define a double contact pencil of conics  $\mathcal{F}_{R,S;t_R,t_S}$ , formed by all the non-singular conics through  $R$  and  $S$  which have  $t_R$ ,  $t_S$  as tangent lines at  $R$  and  $S$  (respectively) and by the two singular conics  $t_R \cup t_S$  and  $R \vee S$  (counted twice). The points  $R$  and  $S$  are called the *base* of the pencil and  $t_R$ ,  $t_S$  are called the *tangent lines* of the pencil at  $R$  and  $S$ .

Let  $K$  be a field of characteristic different from two. A conic of the projective plane  $PG(2, K)$  over  $K$  may be represented by a point of the projective space  $PG(5, K)$ , and vice versa. Let  $\gamma$  be a conic of  $PG(2, K)$  and let  $A = (a_{ij})_{i,j=1,2,3}$  be a  $3 \times 3$  symmetric matrix over  $K$  that represents  $\gamma$ . The six elements  $(a_{ij})_{i \leq j}$  completely determine  $A$  and the conic  $\gamma$  may be represented by the point  $((a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33}))$  of  $PG(5, K)$ , as the elements  $a_{ij}$  are defined up to scalar multiplication. Finally we recall that a pencil of conics of  $PG(2, K)$  is represented by a line of  $PG(5, K)$ .

A *quadratic transformation* of a 3-dimensional projective space  $PG(3, K)$  into itself is a birational mapping  $f : PG(3, K) \rightarrow PG(3, K)$ , such that  $f$  and  $f^{-1}$  are given by homogeneous polynomials of degree two. The points of a plane of  $PG(3, K)$  are mapped, under  $f$ , onto the points of a quadric. A quadratic transformation  $f$  of  $PG(3, K)$  and its inverse  $f^{-1}$  are represented by the equations

$$\begin{aligned} x'_i &= \varphi_i(x_1, x_2, x_3, x_4), \\ x_i &= f_i(x'_1, x'_2, x'_3, x'_4), \quad i = 1, 2, 3, 4 \end{aligned}$$

where  $f_i$ ,  $i = 1, 2, 3, 4$ , is a homogeneous polynomial of degree two. Take  $S = f^*(O(1))$ . There exist three different types of quadratic transformations of a three-dimensional projective space (see [1]):

- Type I. The linear system  $\mathcal{S}$  is formed by all quadrics containing a conic  $\gamma$  and a point  $P_1$ . Lines of  $PG(3, K)$  are mapped onto conics through  $P_1$  intersecting  $\gamma$  in two points.
- Type II. The linear system  $\mathcal{S}$  is formed by all quadrics containing a line  $\ell$  and three points  $P_1, P_2, P_3$ . Lines of  $PG(3, K)$  are mapped onto twisted cubics with  $\ell$  as a chord and containing  $P_1, P_2, P_3$ .
- Type III. The linear system  $\mathcal{S}$  is formed by all quadrics containing four points  $P_1, P_2, P_3, P_4$  with the same tangent plane at  $P_1$ . Lines of  $PG(3, K)$  are mapped onto quartics containing  $P_2, P_3, P_4$  and with  $P_1$  as a double point.

## 2. A projectivity of the double contact pencil of conics

Let  $PG(2, K)$  be a projective plane over a field  $K$  of characteristic different from two. Let  $\Gamma$  be a non-singular conic of  $PG(2, K)$  with induced

polarity  $\perp$ . Let  $A$  and  $B$  be two distinct points on  $\Gamma$  and let  $\mathcal{P}_A, \mathcal{P}_B$  be the pencils of lines with vertices  $A$  and  $B$  (respectively). The map

$$\Phi' : \ell \in \mathcal{P}_A \longrightarrow B \vee \ell^\perp \in \mathcal{P}_B,$$

is a projectivity between  $\mathcal{P}_A$  and  $\mathcal{P}_B$  mapping the line  $A \vee B$  onto the tangent line to  $\Gamma$  at  $B$ . Therefore the set of points of intersection of corresponding lines under  $\Phi'$  is a non-singular conic  $\Gamma'$  through  $A$  and  $B$ .

**PROPOSITION 2.1.** *The conics  $\Gamma$  and  $\Gamma'$  intersect exactly on the points  $A$  and  $B$ .*

**Proof.** Suppose, on the contrary, that there exists a point  $C$  that belongs to  $(\Gamma \cap \Gamma') \setminus \{A, B\}$ . Let  $\ell = A \vee C$ , since  $C \in \Gamma'$  we have  $C = \Phi'(\ell) \cap \ell = (B \vee \ell^\perp) \cap \ell$ . Hence the points  $B, C, \ell^\perp$  are collinear, a contradiction because  $\ell^\perp \vee C$  is the tangent line to  $\Gamma$  at  $C$ . ■

**PROPOSITION 2.2.** *The conics  $\Gamma$  and  $\Gamma'$  have the same tangent lines at  $A$  and at  $B$ .*

**Proof.** Let  $t_A$  and  $t_B$  be the tangent lines to  $\Gamma$  at  $A$  and  $B$  (respectively) and let  $t'_A$  and  $t'_B$  be the tangent lines to  $\Gamma'$  at  $A$  and  $B$  (respectively). Since  $\Phi'(t_A) = B \vee t_A^\perp = B \vee A$ , it follows that  $t_A = \Phi'^{-1}(B \vee A) = t'_A$ . Moreover  $t'_B = \Phi'(A \vee B) = t_B$ . ■

Let  $\mathcal{F} = \mathcal{F}_{A,B;t_A,t_B}$  be a double contact pencil of conics of  $PG(2, K)$  with  $A, B$  as base and  $t_A, t_B$  as tangent lines at  $A$  and  $B$ . Using the previous results we may define a map  $\sigma$  of the pencil  $\mathcal{F}$  into itself in the following way: to every non-singular conic  $\Gamma$  of  $\mathcal{F}$ ,  $\sigma$  corresponds the conic  $\Gamma'$ . The two singular conics of the pencil  $\mathcal{F}$  are assumed to be both fixed by  $\sigma$ .

**PROPOSITION 2.3.** *The map  $\sigma$  is a projectivity of the line  $\mathcal{F}$  into itself.*

**Proof.** Any two double contact pencils of conics of  $PG(2, K)$  are projectively equivalent, so we may assume  $A = \langle(1, 0, 0)\rangle$ ,  $B = \langle(0, 1, 0)\rangle$ ,  $t_A$  with equation  $x_2 = 0$ ,  $t_B$  with equation  $x_1 = 0$  and  $\mathcal{F}$  with equation  $\alpha x_3^2 + \beta x_1 x_2 = 0$ . Let  $\Gamma$  be a non-singular conic of  $\mathcal{F}$  with equation  $\lambda x_3^2 + \mu x_1 x_2 = 0$  ( $\lambda \mu \neq 0$ ).

Let  $\ell$  be a line through  $A$  with equation  $ax_2 + bx_3 = 0$ . The pole  $\ell^\perp$  of  $\ell$ , with respect to  $\Gamma$ , is the point  $\langle(2a\mu^{-1}, 0, b\lambda^{-1})\rangle$  and the line  $B \vee \ell^\perp$  has an equation of the form  $b\lambda^{-1}x_1 - 2a\mu^{-1}x_3 = 0$ . Thus  $\Phi'(\ell) \cap \ell = (B \vee \ell^\perp) \cap \ell$  is the point  $\langle(2a^2\mu^{-1}, -b^2\lambda^{-1}, ab\lambda^{-1})\rangle$ , and this gives a parametric representation of the conic  $\Gamma'$ . It follows that  $\Gamma'$  may be represented by the equation  $2\lambda x_3^2 + \mu x_1 x_2 = 0$ . Hence  $\sigma$  maps the non-singular conic of  $\mathcal{F}$  with equation  $\lambda x_3^2 + \mu x_1 x_2 = 0$  onto the non-singular conic with equation  $2\lambda x_3^2 + \mu x_1 x_2 = 0$ . The map  $\sigma$  is then a projectivity of the line  $\mathcal{F}$  into itself. ■

Observe that the set of all conics of  $PG(2, K)$  containing two distinct points  $A$  and  $B$  form a linear system of dimension three, so it is represented by a three-dimensional subspace  $S^3_{A,B}$  of  $PG(5, K)$ .

**PROPOSITION 2.4.** *The set of all singular conics of  $PG(2, K)$  through two distinct points  $A$  and  $B$  is represented in  $PG(5, K)$  by a cubic surface of  $S^3_{A,B}$ , formed by the union of a hyperbolic quadric  $Q$  with the tangent plane  $\pi$  to  $Q$  at the point which represents the singular conic  $A \vee B$ .*

**Proof.** As before we may assume  $A = \langle(1, 0, 0)\rangle$  and  $B = \langle(0, 1, 0)\rangle$ . A conic of  $PG(2, K)$  through  $A$  and  $B$  has an equation of the form  $2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 + a_{33}x_3^2 = 0$  and it is a singular conic if, and only if,  $a_{12}(2a_{13}a_{23} - a_{12}a_{33}) = 0$ . Hence the singular conics containing  $A$  and  $B$  are represented by the union of the plane  $\pi$  with equations  $a_{11} = a_{22} = a_{12} = 0$ , which represent all the singular conics of  $PG(2, K)$  containing the line  $A \vee B$ , with the hyperbolic quadric  $Q$  with equations  $a_{11} = a_{22} = 2a_{13}a_{23} - a_{12}a_{33} = 0$ . The plane  $\pi$  and the quadric  $Q$  are both contained in the three-dimensional subspace  $S^3_{A,B}$ , which is represented by the equations  $a_{11} = a_{22} = 0$ . Finally, a straightforward calculation shows that  $\pi$  is the tangent plane to  $Q$  at the point which represents the conic  $A \vee B$ . ■

As before let  $\Gamma$  be a non-singular conic of  $PG(2, K)$  with induced polarity  $\perp$ . Let  $A$  and  $B$  be two distinct points on  $\Gamma$  and let  $t_A$  and  $t_B$  be the tangent lines to  $\Gamma$  at  $A$  and  $B$  (respectively). The map

$$\Psi'' : P \in t_A \longrightarrow t_B \cap P^\perp \in t_B,$$

is a projectivity between the lines  $t_A$  and  $t_B$  mapping the point  $t_A \cap t_B$  onto the point  $B$ . Hence the lines joining corresponding points under  $\Psi''$  are tangent lines to a non-singular conic  $\Gamma''$  of  $PG(2, K)$ . By dualizing the previous arguments the following results holds.

**PROPOSITION 2.5.** *The conics  $\Gamma$  and  $\Gamma''$  intersect exactly on the points  $A$  and  $B$ . ■*

**PROPOSITION 2.6.** *The conics  $\Gamma$  and  $\Gamma''$  have the same tangent lines at  $A$  and at  $B$ . ■*

Let  $\mathcal{F} = \mathcal{F}_{A,B;t_A,t_B}$  be a double contact pencil of conics of  $PG(2, K)$  with  $A, B$  as base and  $t_A, t_B$  as tangent lines at  $A$  and  $B$ . We may define a map  $\omega$  of the pencil  $\mathcal{F}$  into itself in the following way: to every non-singular conic  $\Gamma$  of  $\mathcal{F}$ ,  $\omega$  corresponds the non-singular conic  $\Gamma''$ . The two singular conics of  $\mathcal{F}$  are assumed to be both fixed by  $\omega$ .

**PROPOSITION 2.7.** *The map  $\omega$  is a projectivity of the line  $\mathcal{F}$  into itself.*

**Proof.** As in the proof of Proposition 2.3 we may assume  $A = \langle(1, 0, 0)\rangle$ ,  $B = \langle(0, 1, 0)\rangle$ ,  $t_A$  with equation  $x_2 = 0$ ,  $t_B$  with equation  $x_1 = 0$  and  $\mathcal{F}$

with equation  $\alpha x_3^2 + \beta x_1 x_2 = 0$ . Let  $\Gamma$  be a non-singular conic of  $\mathcal{F}$  with equation  $\lambda x_3^2 + \mu x_1 x_2 = 0$ , with  $\lambda\mu \neq 0$ . Let  $P = \langle(a, 0, b)\rangle$  be a point of  $t_A$ . The polar  $P^\perp$  of  $P$ , with respect to  $\Gamma$ , may be represented by the equation  $\mu a x_2 + 2\lambda b x_2 - \mu b x_3 = 0$  and  $t_b \cap P^\perp$  is the point  $\langle(0, 2\lambda b, -\mu a)\rangle$ . Hence the line  $P \vee (t_b \cap P^\perp)$  has an equation of the form  $-2\lambda b^2 x_1 + \mu a^2 x_2 + 2\lambda a b x_3 = 0$ . Thus the projective coordinates of this line satisfy the equation

$$(*) \quad \mu X_3^2 + 2\lambda X_1 X_2 = 0.$$

The conic  $\Gamma''$  is the locus of  $PG(2, K)$  whose tangent lines have projective coordinates satisfying  $(*)$ , hence this locus has an equation of the form  $\lambda x_3^2 + 2\mu x_1 x_2 = 0$ . Therefore  $\omega$  maps the non-singular conic of  $\mathcal{F}$  with equation  $\lambda x_3^2 + \mu x_1 x_2 = 0$ , onto the non-singular conic with equation  $\lambda x_3^2 + 2\mu x_1 x_2 = 0$ . This proves that  $\sigma$  is a projectivity of the line  $\mathcal{F}$  into itself. ■

**PROPOSITION 2.8.** *The projectivities  $\sigma$  and  $\omega$  are inverses of each other.*

**Proof.** The property easily follows by the analytic representation of  $\sigma$  and  $\omega$ . ■

### 3. The quadratic transformation

The correspondence  $\Gamma \rightarrow \Gamma'$  defines a map  $\Sigma$  of the set  $S_{A,B}^3 \setminus (Q \cup \pi)$  into itself. Observe that the point  $\Gamma'$  lie on the line joining  $\Gamma$  with the point  $\Gamma_{AB}$  representing the conic  $A \vee B$ . The map  $\Sigma$  induces, as proved in Proposition 2.3, on every line through  $\Gamma_{AB}$ , not contained in  $\pi$ , a projectivity with two fixed points. Let  $\Gamma$  be a non-singular conic through  $A$  and  $B$ , and let  $t_A(\Gamma)$ ,  $t_B(\Gamma)$  be the tangent lines to  $\Gamma$  at  $A$  and  $B$  (respectively), observe that the line of  $PG(5, K)$  containing  $\Gamma$  and  $\Gamma_{AB}$  represent the double contact pencil of conics with  $A, B$  as base and  $t_A(\Gamma)$ ,  $t_B(\Gamma)$  as tangent lines at  $A$  and at  $B$  (respectively).

Similarly, the correspondence  $\Gamma \rightarrow \Gamma''$  defines a map  $\Omega$  of the set  $S_{A,B}^3 \setminus (Q \cup \pi)$  into itself. Observe that a point  $\Gamma$  is mapped under  $\Omega$  onto the point  $\Gamma''$ , which lie on the line joining  $\Gamma$  with the point  $\Gamma_{AB}$ . The map  $\Omega$  induces on every line through  $\Gamma_{AB}$ , not contained in  $\pi$ , a projectivity with two fixed points. It is clear that  $\Sigma$  and  $\Omega$  are inverses of each other.

**PROPOSITION 3.1.** *The map  $\Sigma$  (as well as  $\Omega$ ) is induced by a quadratic transformation  $\bar{\Sigma}$  of type I of  $S_{AB}^3$  such that the exceptional divisor of  $\bar{\Sigma}$  is a singular cubic union of a smooth quadric with the plane which is tangent to this quadric.*

**Proof.** Assume  $A = \langle(1, 0, 0)\rangle$  and  $B = \langle(0, 1, 0)\rangle$ . Let  $\Gamma$  be a non-singular conic through  $A$  and  $B$  with equation  $2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 + a_{33}x_3^2 = 0$ . Let  $\ell$  be a line through  $A$  with equation  $b_2x_2 + b_3x_3 = 0$ . The

pole  $\ell^\perp$  of  $\ell$ , with respect to  $\Gamma$ , is the point

$$\langle ((a_{13}a_{23} - a_{12}a_{33})b_2 + a_{12}a_{23}b_3, -a_{13}^2b_2 + a_{12}a_{13}b_3, a_{12}a_{13}b_2 - a_{12}^2b_3) \rangle,$$

and the line  $B \vee \ell^\perp$  can be represented by the equation

$$(a_{12}a_{13}b_2 - a_{12}^2b_3)x_1 - ((a_{13}a_{23} - a_{12}a_{33})b_2 + a_{12}a_{23}b_3)x_3 = 0.$$

Thus  $(B \vee \ell^\perp) \cap \ell$  is the point

$$\langle ((a_{13}a_{23} - a_{12}a_{33})b_2^2 + a_{12}a_{23}b_2b_3, a_{12}^2b_3^2 - a_{12}a_{13}b_2b_3, \\ -a_{12}^2b_2b_3 + a_{12}a_{13}b_2^2) \rangle,$$

and this gives a parametric representation of  $\Gamma'$ . It follows that  $\Gamma'$  has an equation of the form

$$-a_{12}^2x_1x_2 - a_{12}a_{13}x_1x_3 - a_{12}a_{23}x_2x_3 + (a_{13}a_{23} - a_{12}a_{33})x_3^2 = 0.$$

Since the coefficients of  $x_i x_j$  are homogeneous polynomials in  $a_{hk}$  of degree two, it follows that  $\Omega$  is induced by a quadratic transformation  $\bar{\Omega}$  of  $S_{AB}^3$ . Let  $x_1 = a_{12}$ ,  $x_2 = a_{13}$ ,  $x_3 = a_{23}$ ,  $x_4 = a_{33}$  be projective coordinates of  $S_{AB}^3$ , it follows that  $\Sigma$  is induced by the algebraic transformation  $\bar{\Sigma}$  of  $S_{AB}^3$  given by

$$x'_1 = -x_1^2,$$

$$x'_2 = -x_1x_2,$$

$$x'_3 = -x_1x_3,$$

$$x'_4 = x_2x_3 - x_1x_4.$$

Let  $\delta$  be a plane of  $S_{AB}^3$ . If  $\delta$  contains the point  $\Gamma_{AB}$  then it is clear that  $\bar{\Sigma}$  maps  $\delta$  onto itself, that is onto a quadric of  $S_{AB}^3$ . If  $\Gamma_{AB} \notin \delta$  then  $\delta$  can be represented by the equation

$$x_4 = a_1x_1 + a_2x_2 + a_3x_3,$$

and a straightforward calculation shows that  $\delta$  is mapped under  $\bar{\Sigma}$  onto the quadric with equation

$$a_1x_1'^2 + a_2x_1'x_2' + a_3x_1'x_3' - x_1'x_4' - x_2'x_3' = 0.$$

It follows that  $\bar{\Sigma}$  is a quadratic transformation of  $S_{AB}^3$  into itself. Let  $\ell$  be a line of  $S_{AB}^3$ . If  $\ell$  contains the point  $\Gamma_{AB}$  then it is clear that  $\ell$  is mapped under  $\bar{\Sigma}$  onto itself, that is onto a conic. If  $\Gamma_{AB} \notin \ell$  then  $\ell$  is the intersection of the plane  $\pi_1 = \ell \vee \Gamma_{AB}$  with a plane  $\pi_2$  through  $\ell$  not containing  $\Gamma_{AB}$ . Thus the line  $\ell$  is mapped under  $\bar{\Sigma}$  onto the conic obtained as intersection of the plane  $\bar{\Sigma}(\pi_1)$  with the quadric  $\bar{\Sigma}(\pi_2)$ . Hence  $\bar{\Sigma}$ , as well as  $\bar{\Omega}$ , is a quadratic transformation of type I. ■

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*Received September 11, 2008; revised version November 10, 2008.*

