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VOLTERRA COMPOSITION OPERATORS BETWEEN WEIGHTED BERGMAN-NEVANLINNA AND BLOCH-TYPE SPACES

Abstract. Let g and φ be holomorphic maps on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Define Volterra composition operators $J_{g,\varphi}$ and $I_{g,\varphi}$ induced by g and φ as

$$J_{g,\varphi}f(z) = \int_0^z (f \circ \varphi)(\zeta)\,(g \circ \varphi)'(\zeta)\,d\zeta \quad \text{and} \quad I_{g,\varphi}f(z) = \int_0^z (f \circ \varphi)'(\zeta)\,(g \circ \varphi)(\zeta)\,d\zeta$$

for $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$, the space of holomorphic functions on $\mathbb{D}$. In this paper, we characterize boundedness and compactness of these operators acting between weighted Bergman-Nevanlinna spaces $\mathcal{A}^\beta_\mathcal{N}$ and Bloch-type spaces. In fact, we prove that $J_{g,\varphi} : \mathcal{A}^\beta_\mathcal{N} \to \mathcal{B}^\alpha$ (or $\mathcal{B}^\alpha_0$) and $I_{g,\varphi} : \mathcal{A}^\beta_\mathcal{N} \to \mathcal{B}^\alpha$ (or $\mathcal{B}^\alpha_0$) are compact if and only if they are bounded.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. Throughout this paper, we denote by $H(\mathbb{D})$, the space of holomorphic functions on $\mathbb{D}$ and by $S(\mathbb{D})$ the class of holomorphic self-maps of $\mathbb{D}$. Let $dA(z) = \frac{1}{\pi}dxdy = \frac{1}{\pi}rdrd\theta$ be the normalized area measure on $\mathbb{D}$. For each $\beta \in (-1, \infty)$, we set $d\nu_\beta(z) = (\beta+1)(1-|z|^2)^\beta dA(z)$, $z \in \mathbb{D}$. Then $d\nu_\beta$ is a probability measure on $\mathbb{D}$. For $0 < p < \infty$ the weighted Bergman space $\mathcal{A}^p_\beta$ is defined as

$$\mathcal{A}^p_\beta = \Big\{ f \in H(\mathbb{D}) : ||f||_{\mathcal{A}^p_\beta} = \Big(\int_\mathbb{D} |f(z)|^p d\nu_\beta(z) \Big)^{1/p} < \infty \Big\}.$$

Note that $||f||_{\mathcal{A}^p_\beta}$ is a Banach space only if $1 \le p < \infty$. When $0 < p < 1$, $\mathcal{A}^p_\beta$ is a an F-space with respect to the translation invariant metric defined by $d^\beta_p(f,g) = ||f-g||_{\mathcal{A}^p_\beta}$. The weighted Bergman-Nevanlinna space $\mathcal{A}^\beta_\mathcal{N}$ defined

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as

$$\mathcal{A}_{\mathcal{N}}^{\beta} = \left\{ f \in H(\mathbb{D}) : \int_{\mathbb{D}} \log^{+} |f(z)| d\nu_{\beta}(z) < \infty \right\},$$

where

$$\log^{+} x = \begin{cases} \log x & \text{if} \quad x \geq 1 \\ 0 & \text{if} \quad x < 1. \end{cases}$$

The space $\mathcal{A}_{\mathcal{N}}^{\beta}$ appears in the limit as $p \to 0$ of the weighted Bergman space $\mathcal{A}_{\beta}^{p}$, in the sense of

$$\lim_{p \to 0} \frac{t^{p} - 1}{p} = \log t, \qquad 0 < t < \infty$$

and it contains all the Bergman spaces $\mathcal{A}_{\beta}^{p}$. Obviously, the inequality

$$\log^{+} x \leq \log(1 + x) \leq 1 + \log^{+} x, \quad x \geq 0$$

implies that $f \in \mathcal{A}_{\mathcal{N}}^{\beta}$ if and only if

$$||f||_{\mathcal{A}_{\mathcal{N}}^{\beta}} = \int_{D} \log(1 + |f(z)|) d\nu_{\beta}(z) < \infty.$$

Of course, we are abusing the term norm since $||f||_{\mathcal{A}_{\mathcal{N}}^{\beta}}$ fails to satisfy the properties of norm, but in this case $(f, g) \to ||f - g||_{\mathcal{A}_{\mathcal{N}}^{\beta}}$ defines a translation invariant metric on $\mathcal{A}_{\mathcal{N}}^{\beta}$ and this turns $\mathcal{A}_{\mathcal{N}}^{\beta}$ into a complete metric space. Also, by subharmonicity of $\log(1 + |f(z)|)$, we have

$$\log(1 + |f(z)|) \leq C_{\beta} \frac{||f||_{\mathcal{A}_{\mathcal{N}}^{\beta}}}{(1 - |z|^{2})^{\beta + 2}}, \quad z \in \mathbb{D} \tag{1.1}$$

for all $f \in \mathcal{A}_{\mathcal{N}}^{\beta}$. In particular, (1.1) tells us that if $f_n \to f$ in $\mathcal{A}_{\mathcal{N}}^{\beta}$, then $f_n \to f$ locally uniformly. Here, locally uniform convergence refers to the uniform convergence on every compact subset of $\mathbb{D}$.

Let $\alpha > 0$. A function f holomorphic in $\mathbb{D}$ is in α-Bloch space $\mathcal{B}^{\alpha}$ if

$$\sup_{z \in \mathbb{D}} (1 - |z|^{2})^{\alpha} |f'(z)| < \infty$$

and in the little α-Bloch space $\mathcal{B}_{0}^{\alpha}$ if

$$\lim_{|z| \to 1} (1 - |z|^{2})^{\alpha} |f'(z)| = 0.$$

For $f \in \mathcal{B}^{\alpha}$ define

$$||f||_{\mathcal{B}^{\alpha}} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^{2})^{\alpha} |f'(z)|.$$

With this norm $\mathcal{B}^{\alpha}$ is a Banach space and the little α-Bloch space $\mathcal{B}_{0}^{\alpha}$ is a closed subspace of the α-Bloch space. Note that $\mathcal{B}^{1} = \mathcal{B}$ is the usual Bloch space and $\mathcal{B}_{0}^{1} = \mathcal{B}_{0}$ is the usual little Bloch space.

Moreover, for $f \in \mathcal{B}^\alpha$,

$$|f(z)| \leq \begin{cases} C\,\|f\|_{\mathcal{B}^\alpha} & \text{if } 0 < \alpha < 1; \\ C \log \frac{2}{1-|z|}\,\|f\|_{\mathcal{B}^\alpha} & \text{if } \alpha = 1; \\ C \frac{\|f\|_{\mathcal{B}^\alpha}}{(1-|z|^2)^{\alpha-1}} & \text{if } \alpha > 1. \end{cases} \tag{1.2}$$

For general background on weighted Bergman spaces $\mathcal{A}^p_\beta$, weighted Bergman-Nevanlinna spaces $\mathcal{A}^\beta_\mathcal{N}$ and Bloch spaces $\mathcal{B}^\alpha$ and $\mathcal{B}^\alpha_0$ one may consult [5] and the references therein.

Let $g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Associated with $\varphi \in S(\mathbb{D})$ is the composition operator defined by

$$C_\varphi f = f \circ \varphi \qquad (f \in H(\mathbb{D})).$$

For $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$, we can define a linear operator $J_{g,\varphi}$ induced by g and φ as

$$J_{g,\varphi} f(z) = \int_0^z (f \circ \varphi)(\zeta)\,(g \circ \varphi)'(\zeta)\,d\zeta.$$

The operator $J_{g,\varphi}$ can be viewed as a generalization of the Riemann-Stieltjes operator J_g induced by g, defined by

$$J_g f(z) = \int_0^z f(\zeta) dg(\zeta) = \int_0^1 f(tz) z g'(tz) dt, \qquad z \in \mathbb{D}.$$

Ch. Pommerenke [8] initiated the study of Riemann-Stieltjes operator on H^2, where he showed that J_g is bounded on H^2 if and only if g is in $BMOA$. This was extended to other Hardy spaces H^p, $1 \leq p < \infty$, in [1] and [2] where compactness of J_g on H^p and Schatten class membership of J_g on H^2 was also completely characterized in terms of the symbol g. Similar questions on weighted Bergman spaces were considered by A. Aleman and A. G. Siskakis in [3]. We also consider another integral operator $I_{g,\varphi}$ induced by g and φ and defined as

$$I_{g,\varphi} f(z) = \int_0^z (f \circ \varphi)'(\zeta)\,(g \circ \varphi)(\zeta)\,d\zeta.$$

The operator $I_{g,\varphi}$ is the generalization of the operator I_g, recently defined by Yoneda in [13] as

$$I_g f(z) = \int_0^z f'(\zeta) g(\zeta) d\zeta, \qquad z \in \mathbb{D}.$$

Recently, several authors have studied these operators on different spaces of analytic functions. For example, one can refer to ([6], [9], [10], [12], [13]) and related references therein for the study of these operators on different spaces of analytic functions.

In fact, when $\varphi(z) = z$, we have $J_{g,\varphi} = J_g$ and $I_{g,\varphi} = I_g$, whereas when $g \equiv 1$, then $I_{g,\varphi} = C_\varphi$. Let X be an F-space of analytic functions and Y is a Banach space of analytic functions. Then one may ask, for what symbols g and φ, $J_{g,\varphi} : X \to Y$ and $I_{g,\varphi} : X \to Y$ are bounded operators, are compact operators? Furthermore, for what X and Y, $J_{g,\varphi} : X \to Y$ and $I_{g,\varphi} : X \to Y$ are compact if and only if they are bounded? If $X = \mathcal{A}^p_\beta$ and $Y = \mathcal{B}^\alpha$ or $\mathcal{B}^\alpha_0$, then by Theorem 1, Theorem 2, Theorem 3, and Theorem 4 of [6], there do exist p, α and β such that $J_{g,\varphi} : X \to Y$ and $I_{g,\varphi} : X \to Y$ are bounded but not compact.

In this paper, we note that $J_{g,\varphi} : \mathcal{A}^\beta_\mathcal{N} \to \mathcal{B}^\alpha$ (or $\mathcal{B}^\alpha_0$) and $I_{g,\varphi} : \mathcal{A}^\beta_\mathcal{N} \to \mathcal{B}^\alpha$ (or $\mathcal{B}^\alpha_0$) are compact if and only if they are bounded.

2. Boundedness and compactness of $J_{g,\varphi}$ and $I_{g,\varphi}$

In this section, we characterize the boundedness and compactness of $J_{g,\varphi} : \mathcal{A}^\beta_\mathcal{N} \to \mathcal{B}^\alpha$ (or $\mathcal{B}^\alpha_0$) and $I_{g,\varphi} : \mathcal{A}^\beta_\mathcal{N} \to \mathcal{B}^\alpha$ (or $\mathcal{B}^\alpha_0$).

A subset E of $\mathcal{A}^\beta_\mathcal{N}$ is bounded if it is bounded for the defining F-norm $\|\cdot\|_{\mathcal{A}^\beta_\mathcal{N}}$.

Given a Banach space $\mathcal{Y}$, we say that a linear map $T : \mathcal{A}^\beta_\mathcal{N} \to \mathcal{Y}$ is bounded if $T(E) \subset \mathcal{Y}$ is bounded for every bounded subset E of $\mathcal{A}^\beta_\mathcal{N}$. In addition, we say that T is compact if $T(E) \subset \mathcal{Y}$ is relatively compact for every bounded set $E \subset \mathcal{A}^\beta_\mathcal{N}$.

The following criterion for compactness is a useful tool to us and it follows from standard arguments, for example, to those outlined in Proposition 3.11 of [4]. For completeness, we include its proof.

Lemma 2.1. *Let $\alpha \in (0,\infty)$, $\beta \in (-1,\infty)$, $g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then $J_{g,\varphi}$ (or $I_{g,\varphi}$) : $\mathcal{A}^\beta_\mathcal{N} \to \mathcal{B}^\alpha$ is compact if and only if for any sequence $\{f_n\}$ in $\mathcal{A}^\beta_\mathcal{N}$ with $\sup_n \|f_n\|_{\mathcal{A}^\beta_\mathcal{N}} = M < \infty$ and which converges to zero locally uniformly on $\mathbb{D}$, we have*

$$\lim_{n\to\infty} \|J_{g,\varphi} f_n\|_{\mathcal{B}^\alpha} = 0 \ \left(or \lim_{n\to\infty} \|I_{g,\varphi} f_n\|_{\mathcal{B}^\alpha} = 0 \right).$$

Proof. We prove the result for $J_{g,\varphi}$ only. Suppose that $J_{g,\varphi} : \mathcal{A}^\beta_\mathcal{N} \to \mathcal{B}^\alpha$ is compact and $\{f_n\}$ is in $\mathcal{A}^\beta_\mathcal{N}$ with $\sup_n \|f_n\|_{\mathcal{A}^\beta_\mathcal{N}} = M < \infty$ and which converges to zero locally uniformly on $\mathbb{D}$ as $n \to \infty$. Then $\{J_{g,\varphi} f_n\}$ has a subsequence $\{J_{g,\varphi} f_{n_k}\}$ that converges to $h \in \mathcal{B}^\alpha$. Thus by (1.2) for all compact subsets $K \subset \mathbb{D}$, there is a positive constant C_K independent of f_n such that

$$|J_{g,\varphi} f_{n_k}(z) - h(z)| \le C_K \|J_{g,\varphi} f_{n_k} - h\|_{\mathcal{B}^\alpha}$$

for all $z \in \mathbb{D}$. Therefore, $\{J_{g,\varphi}f_{n_k}(z) - h(z)\}$ converges to zero uniformly on K. Notice that, there is a constant $C > 0$ such that $|(g \circ \varphi)(z)| < C$ for all $z \in K$. Also $\varphi(K)$ is compact in $\mathbb{D}$ and so we have $f_{n_k}(\varphi(z))$ converges to zero for each $z \in \mathbb{D}$. Therefore, $|J_{g,\varphi}f_{n_k}(\varphi(z))| \to 0$ uniformly on K. Thus for the arbitrariness of K, we have $h \equiv 0$. Since it is true for arbitrary subsequence of f_n, we see that $J_{g,\varphi}f_{n_k}(\varphi(z)) \to 0$ in $\mathcal{B}^\alpha$, when $n \to \infty$.

Conversely, let $\{h_k\}$ be a bounded sequence in $\mathcal{A}^\beta_{\mathcal{N}}$. Since $\sup_n \|f_n\|_{\mathcal{A}^\beta_{\mathcal{N}}} = M < \infty$, the sequence $\{h_k\}$ is uniformly bounded on compact subsets of $\mathbb{D}$ and hence a normal family by Montel's Theorem. Hence we may extract a subsequence $\{h_{k_j}\}$ which converges uniformly on compact subsets of $\mathbb{D}$ to some $h \in H(\mathbb{D})$. Moreover, $h \in \mathcal{A}^\beta_{\mathcal{N}}$ and $\|h\|_{\mathcal{A}^\beta_{\mathcal{N}}} \leq M$. Thus the sequence $\{(h_{k_j} - h)\}$ is such that $\|h_{k_j} - h\|_{\mathcal{A}^\beta_{\mathcal{N}}} \leq M$ and converges to zero on compact subsets $\mathbb{D}$. By hypothesis, we have $J_{g,\varphi}h_{k_j} \to J_{g,\varphi}h$ in $\mathcal{B}^\alpha$. Thus $J_{g,\varphi} : \mathcal{A}^\beta_{\mathcal{N}} \to \mathcal{B}^\alpha$ is compact as desired.

Lemma 2.2. [7] *Let $K \subset \mathcal{B}^\alpha_0$. Then K is compact if and only if K is closed, bounded and satisfies*

$$\lim_{|z|\to 1^-} \sup_{f\in K}(1 - |z|^2)^\alpha |f'(z)| = 0.$$

Theorem 2.3. *Let $\alpha > 0$, $\beta > -1$, $g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then the following are equivalent:*

(i) $J_{g,\varphi} : \mathcal{A}^\beta_{\mathcal{N}} \to \mathcal{B}^\alpha$ *is bounded.*
(ii) $J_{g,\varphi} : \mathcal{A}^\beta_{\mathcal{N}} \to \mathcal{B}^\alpha$ *is compact.*
(iii) *For all $c > 0$,*

$$\lim_{|\varphi(z)|\to 1} \left(1 - |z|^2\right)^\alpha |g'(\varphi(z))|\,|\varphi'(z)| \exp\left[\frac{c}{\left(1 - |\varphi(z)|^2\right)^{\beta+2}}\right] = 0$$

and

$$N = \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right)^\alpha |g'(\varphi(z))\,\varphi'(z)| < \infty.$$

Proof. It suffices to show only two implications: (i)⇒(iii) and (iii)⇒(ii).

(i)⇒(iii). Suppose (i) holds. Let $\lambda \in \mathbb{D}$ be fixed. For $w = \varphi(\lambda)$ and $c > 0$, consider the function

$$f_w(z) = \left(\frac{1 - |w|^2}{(1 - \overline{w}z)^2}\right)^{\beta+2} \exp\left\{\frac{c(1 - |w|^2)^{\beta+2}}{(1 - \overline{w}z)^{2(\beta+2)}}\right\}.$$

Using the inequalities

$$\log(1 + xy) \leq \log(1 + x) + \log(1 + y)\,; \quad x, y \geq 0$$

and

$$\log(1+x) \le 1 + \log^+ x; \quad x \ge 0,$$

we have

$$\log(1+|f_w(z)|) \le \log\left[1+\left(\frac{1-|w|^2}{(1-\overline{w}z)^2}\right)^{\beta+2}\right] + 1 + \left\{\frac{c(1-|w|^2)^{\beta+2}}{|1-\overline{w}z|^{2(\beta+2)}}\right\}$$
$$\le 1 + (1+c)\,\frac{(1-|w|^2)^{\beta+2}}{|1-\overline{w}z|^{2(\beta+2)}}$$

and so

$$\|f_w\|_{\mathcal{A}^\beta_\mathcal{N}} \le 1 + (1+c)\int_{\mathbb{D}}\left(\frac{1-|w|^2}{|1-\overline{w}z|^2}\right)^{\beta+2} dv_\beta(z) \le 2 + c.$$

Since $J_{g,\varphi} : \mathcal{A}^\beta_\mathcal{N} \to \mathcal{B}^\alpha$ is bounded, we can find a constant $M > 0$ such that

$$M \ge (1-|\lambda|^2)^\beta \left|g'(\varphi(\lambda))\varphi'(\lambda)\right| |f(\varphi(\lambda))|$$
$$= \frac{(1-|\lambda|^2)^\beta |g'(\varphi(\lambda))| |\varphi'(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\beta+2}} \exp\left[\frac{c}{(1-|\varphi(\lambda)|^2)^{\beta+2}}\right].$$

That is,

$$(1-|\lambda|^2)^\beta \left|g'(\varphi(\lambda))\varphi'(\lambda)\right| \exp\left[\frac{c}{(1-|\varphi(\lambda)|^2)^{\beta+2}}\right] \le M\left(1-|\varphi(\lambda)|^2\right)^{\beta+2}.$$

Taking limit as $|\varphi(\lambda)| \to 1$ on both sides of above inequality, we have

$$\lim_{|\varphi(\lambda)|\to 1}\left(1-|\lambda|^2\right)^\beta \left|g'(\varphi(\lambda))\varphi'(\lambda)\right| \exp\left[\frac{c}{(1-|\varphi(\lambda)|^2)^{\beta+2}}\right] = 0.$$

(iii)$\Rightarrow$(ii). Assume that (iii) is valid for all $c > 0$. Using (1.1) we have for all $f \in \mathcal{A}^\beta_\mathcal{N}$,

$$|f_n(z)| \le \exp\left\{\frac{M_0 \|f_n\|_{\mathcal{A}^\beta_\mathcal{N}}}{(1-|z|^2)^{\beta+2}}\right\}.$$

Choose a sequence $\{f_n\}$ in $\mathcal{A}^\beta_\mathcal{N}$ such that $\sup_n \|f_n\|_{\mathcal{A}^\beta_\mathcal{N}} = M < \infty$ and $f_n \to 0$ locally uniformly on $\mathbb{D}$. By Lemma 2.1, it is sufficient to show that $\|J_{g,\varphi} f_n\|_{\mathcal{B}^\alpha} \to 0$ as $n \to \infty$. For $r \in (0,1)$

$$\sup_{|\varphi(z)|\le r}\left(1-|z|^2\right)^\alpha \left|(J_{g,\varphi} f_n)'(z)\right|$$
$$= \sup_{|\varphi(z)|\le r}\left(1-|z|^2\right)^\alpha \left|g'(\varphi(z))\varphi'(z)\right| |f_n(\varphi(z))|$$
$$\le \sup_{|\varphi(z)|\le r}\left(1-|z|^2\right)^\alpha \left|g'(\varphi(z))\varphi'(z)\right| \sup_{|\varphi(z)|\le r}|f_n(\varphi(z))|$$
$$\le N \sup_{|\varphi(z)|\le r}|f_n(\varphi(z))| \to 0 \quad \text{as} \quad n \to \infty.$$

On the other hand, whenever $r \to 1$, we have

$$\sup_{|\varphi(z)|>r} (1-|z|^2)^{\alpha} \left|(J_{g,\varphi}f_n)'(z)\right|$$
$$= \sup_{|\varphi(z)|>r} (1-|z|^2)^{\alpha} \left|g'(\varphi(z))\varphi'(z)\right| \exp\left[\frac{M_0 M}{(1-|\varphi(z)|^2)^{\beta+2}}\right] \to 0.$$

Combining the above estimates we see that $\|J_{g,\varphi}f_n\|_{\mathcal{A}^{\beta}_{\mathcal{N}}} \to 0$ as $n \to \infty$. This completes the proof.

THEOREM 2.4. *Let $\alpha > 0$, $\beta > -1$, $g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then the following are equivalent:*

(i) $J_{g,\varphi} : \mathcal{A}^{\beta}_{\mathcal{N}} \to \mathcal{B}^{\alpha}_0$ *is bounded.*
(ii) $J_{g,\varphi} : \mathcal{A}^{\beta}_{\mathcal{N}} \to \mathcal{B}^{\alpha}_0$ *is compact.*
(iii) *For all $c > 0$,*

$$\lim_{|z|\to 1} (1-|z|^2)^{\alpha} \left|g'(\varphi(z))\varphi'(z)\right| \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right] = 0.$$

Proof. Once again we only need to prove two implications: (i)⇒(iii) and (iii)⇒(i).

(i)⇒(iii). By taking $f(z) = c$, a constant function in $\mathcal{A}^{\beta}_{\mathcal{N}}$, we get

$$\lim_{|z|\to 1} (1-|z|^2)^{\alpha} \left|g'(\varphi(z))\varphi'(z)\right| = 0. \tag{2.1}$$

Again using the same test functions as in Theorem 2.3, we get

$$\lim_{|\varphi(z)|\to 1} (1-|z|^2)^{\alpha} \left|g'(\varphi(z))\varphi'(z)\right| \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right] = 0. \tag{2.2}$$

By (2.2), for every $\epsilon > 0$, there exists $r_1 \in (0,1)$ such that

$$(1-|z|^2)^{\alpha} \left|g'(\varphi(z))\varphi'(z)\right| \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right] < \epsilon$$

whenever $r_1 < |\varphi(z)| < 1$. By (2.1), there exists $r_2 \in (0,1)$ such that

$$(1-|z|^2)^{\alpha} \left|g'(\varphi(z))\varphi'(z)\right| < \left(\exp\left[\frac{c}{(1-r_1^2)^{\beta+2}}\right]\right)^{-1} < \epsilon$$

for all $c > 0$, whenever $r_2 < |z| < 1$.

Thus, whenever $r_2 < |z| < 1$ and $r_1 < |\varphi(z)| < 1$, we have

$$(1-|z|^2)^{\alpha} \left|g'(\varphi(z))\varphi'(z)\right| \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right] < \epsilon. \tag{2.3}$$

If $r_2 < |z| < 1$ and $|\varphi(z)| \le r_1$, then we have

$$(2.4)\quad (1-|z|^2)^\alpha \left|g'(\varphi(z))\varphi'(z)\right| \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right] \le (1-|z|^2)^\alpha \left|g'(\varphi(z))\varphi'(z)\right| \exp\left[\frac{c}{(1-r_1^2)^{\beta+2}}\right] < \epsilon.$$

Combining (2.3) and (2.4), we get (iii).

(iii)⇒(i) Assume that (iii) holds. It follows from Lemma 2.2 that $J_{g,\varphi} : \mathcal{A}^\beta_\mathcal{N} \to \mathcal{B}^\alpha$ is compact if and only if

$$(2.5)\quad \lim_{|z|\to 1} \sup_{\|f\|_{\mathcal{A}^\beta_\mathcal{N}} \le 1} (1-|z|^2)^\alpha \left|(J_{g,\varphi} f)'(z)\right| = 0.$$

By (1.1), we have

$$(1-|z|^2)^\alpha \left|g'(\varphi(z))\varphi'(z)\right| |f(\varphi(\lambda))| \le (1-|z|^2)^\alpha \left|g'(\varphi(z))\varphi'(z)\right| \exp\left[\frac{M_0 \|f\|_{\mathcal{A}^\beta_\mathcal{N}}}{(1-|\varphi(z)|^2)^{\beta+2}}\right].$$

By (iii), the above inequality implies (2.5). Thus $J_{g,\varphi} : \mathcal{A}^\beta_\mathcal{N} \to \mathcal{B}^\alpha_0$ is compact.

Corollary 2.5. *Let $\alpha > 0$, $\beta > -1$ and $g \in H(\mathbb{D})$. Then the following are equivalent:*

(i) $J_g : \mathcal{A}^\beta_\mathcal{N} \to \mathcal{B}^\alpha$ *is bounded.*
(ii) $J_g : \mathcal{A}^\beta_\mathcal{N} \to \mathcal{B}^\alpha$ *is compact.*
(iii) $J_g : \mathcal{A}^\beta_\mathcal{N} \to \mathcal{B}^\alpha_0$ *is bounded.*
(iv) $J_g : \mathcal{A}^\beta_\mathcal{N} \to \mathcal{B}^\alpha_0$ *is compact.*
(v) g *is constant.*

Proof. The equivalence of first four conditions follows by Theorem 2.3 and Theorem 2.4. That $(i)-(iv)$ are equivalent to (v) follows by the maximum modulus principle.

Theorem 2.6. *Let $\alpha > 0$, $\beta > -1, g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then the following are equivalent:*

(i) $I_{g,\varphi} : \mathcal{A}^\beta_\mathcal{N} \to \mathcal{B}^\alpha$ *is bounded.*
(ii) $I_{g,\varphi} : \mathcal{A}^\beta_\mathcal{N} \to \mathcal{B}^\alpha$ *is compact.*
(iii) *For all $c > 0$,*

$$(2.6)\quad \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |g(\varphi(z))| |\varphi'(z)|}{(1-|\varphi(z)|^2)} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right] = 0$$

and

$$M = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha \left|g(\varphi(z))\varphi'(z)\right| < \infty.$$

Proof. Once again we only need to prove two implications: (i)⇒(iii) and (iii)⇒(ii).

(i)⇒(iii). Suppose (i) holds. By taking $f(z) = z$ in $\mathcal{A}^\beta_\mathcal{N}$, we get $\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |g(\varphi(z))\varphi'(z)| < \infty$. Fix $z_0 \in \mathbb{D}$. For $c > 0$ and $w = \varphi(z_0)$, consider the function

$$f_w(z) = \exp\left\{\frac{c(1-|w|^2)^{\beta+2}}{(1-\overline{w}z)^{2(\beta+2)}}\right\}.$$

Using the obvious inequality, $\log(1+x) \le 1 + \log^+ x$ for $x \ge 0$, we have

$$\|f_w\|_{\mathcal{A}^\beta_\mathcal{N}} \le 1 + c\int_\mathbb{D} \frac{(1-|w|^2)^{\beta+2}}{|1-\overline{w}z|^{2(\beta+2)}} d\nu_\beta(z) \le 1 + c.$$

Since $I_{g,\varphi}$ maps $\mathcal{A}^\beta_\mathcal{N}$ boundedly into $\mathcal{B}^\alpha$ and

$$f_w'(z) = \frac{2(\beta+2)c\overline{w}(1-|w|^2)^{\beta+2}}{(1-\overline{w}z)^{2(\beta+2)+1}}\exp\left\{\frac{c(1-|w|^2)^{\beta+2}}{(1-\overline{w}z)^{2(\beta+2)}}\right\},$$

there is a constant $N > 0$ depending only on c and β such that

$$\begin{aligned} N &\ge (1-|z_0|^2)^\alpha \left|g(\varphi(z_0))\varphi'(z_0)\right| \left|f_w'(\varphi(z_0))\right| \\ &= (1-|z_0|^2)^\alpha \frac{2(\beta+2)c|\varphi(z_0)||\psi(z_0)|}{(1-|\varphi(z_0)|^2)^{\beta+3}} \exp\left\{\frac{c\left(1-|w|^2\right)^{\beta+2}}{(1-|\varphi(z_0)|^2)^{2(\beta+2)}}\right\}, \end{aligned}$$

where $|\psi(z_0)| = |g(\varphi(z_0))\varphi'(z_0)|$. That is,

$$\frac{(1-|z_0|^2)^\alpha |\psi(z_0)|}{(1-|\varphi(z_0)|^2)} \exp\left[\frac{c}{(1-|\varphi(z_0)|^2)^{\beta+2}}\right] \le \frac{M(1-|\varphi(z_0)|^2)^{\beta+1}}{2(\beta+2)c|\varphi(z_0)|}.$$

Taking $\lim_{|\varphi(z_0)|\to 1}$ on both sides of above inequality, we get (2.6).

(iii)⇒(ii). Assume that (iii) is valid for all $c > 0$. Note that if $f \in \mathcal{A}^\beta_\mathcal{N}$, then by (1.1) and Cauchy's integral formula for derivatives

$$\left(1-|z|^2\right)|f'(z)| \le \frac{2}{\pi}\int\limits_{\partial\mathbb{D}}\left|f\left(z+\frac{1}{2}(1-|z|)\zeta\right)\right||d\zeta|$$
$$\le \exp\left[\frac{4^{2+\beta}M_0\|f_w\|_{\mathcal{A}^\beta_{\mathcal{N}}}}{(1-|\varphi(z)|^2)^{\beta+2}}\right].$$

Choose any sequence f_n in $\mathcal{A}^\beta_{\mathcal{N}}$ such that $\|f_n\|_{\mathcal{A}^\beta_{\mathcal{N}}} \le M$ and $f_n \longrightarrow 0$ locally uniformly on $\mathbb{D}$. By Lemma 2.1, it is sufficient to show that $I_{g,\varphi}f_n \longrightarrow 0$ as $n \longrightarrow \infty$. For $r \in (0,1)$

$$\sup_{|\varphi(z)|\le r}(1-|z|^2)^\alpha|I_{g,\varphi}f_n(z)| = \sup_{|\varphi(z)|\le r}(1-|z|^2)^\alpha|g(\varphi(z))\varphi'(z)|\,|f_n'(\varphi(z))|$$
$$\le A\sup_{|\varphi(z)|\le r}|f_n'(\varphi(z))| \longrightarrow 0$$

as $n \longrightarrow \infty$, where $A = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|g(\varphi(z))\varphi'(z)| < \infty$. On the other hand, whenever $r \longrightarrow 1$, we have

$$\sup_{|\varphi(z)|>r}(1-|z|^2)^\alpha|I_{g,\varphi}f_n(z)|$$
$$\le \sup_{|\varphi(z)|>r}\frac{(1-|z|^2)^\alpha|g(\varphi(z))\varphi'(z)|}{(1-|\varphi(z)|^2)}\exp\left[\frac{4^{2+\beta}M_0M}{(1-|\varphi(z)|^2)^{\beta+2}}\right] \longrightarrow 0.$$

Combining the above estimates, we see that $\|I_{g,\varphi}f_n\|_{\mathcal{A}^{-\alpha}} \longrightarrow 0$ as $n \longrightarrow \infty$. This completes the proof.

Theorem 2.7. *Let $\alpha > 0$, $\beta > -1$, $\psi \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then the following are equivalent:*

(i) *$I_{g,\varphi}$ maps $\mathcal{A}^\beta_{\mathcal{N}}$ boundedly into $\mathcal{B}^\alpha_0$.*
(ii) *$I_{g,\varphi}$ maps $\mathcal{A}^\beta_{\mathcal{N}}$ compactly into $\mathcal{B}^\alpha_0$.*
(iii) *For all $c > 0$,*

$$\lim_{|z|\to 1}\frac{(1-|z|^2)^\alpha|g(\varphi(z))\varphi'(z)|}{(1-|\varphi(z)|^2)}\exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right] = 0.$$

Proof. The proof follows on same lines as the proof of Theorem 2.4. So we omit the details.

Corollary 2.8. *Let $\alpha > 0$, $\beta > -1$ and $g \in H(\mathbb{D})$. Then the following are equivalent:*

(i) *$I_g : \mathcal{A}^\beta_{\mathcal{N}} \to \mathcal{B}^\alpha$ is bounded.*
(ii) *$I_g : \mathcal{A}^\beta_{\mathcal{N}} \to \mathcal{B}^\alpha$ is compact.*
(iii) *$I_g : \mathcal{A}^\beta_{\mathcal{N}} \to \mathcal{B}^\alpha_0$ is bounded.*
(iv) *$I_g : \mathcal{A}^\beta_{\mathcal{N}} \to \mathcal{B}^\alpha_0$ is compact.*
(v) *$g \equiv 0$.*

Finally, we have the following two corollaries, the first of which is a recent result of Jie Xiao [11].

COROLLARY 2.9. *Let $\alpha > 0$, $\beta > -1$ and $\varphi \in S(\mathbb{D})$. Then the following are equivalent:*

(i) $C_\varphi : \mathcal{A}_\mathcal{N}^\beta \to \mathcal{B}^\alpha$ *is bounded.*
(ii) $C_\varphi : \mathcal{A}_\mathcal{N}^\beta \to \mathcal{B}^\alpha$ *is compact.*
(iii) *For all* $c > 0$,

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |\varphi'(z)|}{(1-|\varphi(z)|^2)} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right] = 0.$$

COROLLARY 2.10. *Let $\alpha > 0$, $\beta > -1$ and $\varphi \in S(\mathbb{D})$. Then the following are equivalent:*

(i) $C_\varphi : \mathcal{A}_\mathcal{N}^\beta \to \mathcal{B}_0^\alpha$ *is bounded.*
(ii) $C_\varphi : \mathcal{A}_\mathcal{N}^\beta \to \mathcal{B}_0^\alpha$ *is compact.*
(iii) *For all* $c > 0$,

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^\alpha |\varphi'(z)|}{(1-|\varphi(z)|^2)} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right] = 0.$$

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