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A FIXED POINT THEOREM FOR MULTI-VALUED WEAKLY PICARD OPERATORS IN b-METRIC SPACE

Abstract. In this paper, we establish a fixed point theorem for multi-valued operators in a complete b-metric space using the concept of Berinde and Berinde [9] on multi-valued weak contractions for the Picard iteration in a metric space. Our main result generalizes, extends and improves some of the recent results of Berinde and Berinde [9] as well as those of Daffer and Kaneko [17] and also unifies several classical results pertainning to single and multi-valued contractive mappings in the literature.

1. Introduction

The notion of the b-metric space will be introduced in the sequel. Presently, let (X, d) be a complete metric space and $CB(X)$ denote the family of all nonempty closed and bounded subsets of X. For $A,\ B \subset X$, define the distance betweeen A and B by $D(A, B) = \inf\{d(a, b) \mid a \in A,\ b \in B\}$, the diameter of A and B by $\delta(A, B) = \sup\{d(a, b) \mid a \in A,\ b \in B\}$, and the Hausdorff–Pompeiu metric on $CB(X)$ by $H(A, B) = \max\{\sup\{d(a, B) \mid a \in A\},\ \sup\{d(b, A) \mid b \in B\}\}$. $H(A, B)$ is induced by d.

Let $P(X)$ be the family of all nonempty subsets of X and $T : X \to P(X)$ a multi-valued mapping. Then, an element $x \in X$ such that $x \in T(x)$ is called a *fixed point of* T. Denote the set of all the fixed points of T by Fix (T), that is,

$$\text{Fix }(T) = \{x \in X \mid x \in T(x)\}.$$

Markins [27] and Nadler [29] initiated the study of fixed point theorems for multi-valued operators. The celebrated Banach's fixed point theorem is extended to the following result of Nadler [29] from the single-valued maps to the multi-valued contractive maps.

THEOREM 1.1. (Nadler [29]) *Let (X, d) be a complete metric space and*

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$T : X \to CB(X)$ a set-valued α-contraction, that is, a mapping for which there exists a constant $\alpha \in (0,1)$, such that

$$(1) \qquad H(Tx, Ty) \leq \alpha d(x,y), \ \forall\ x,\ y \in X.$$

Then T has at least one fixed point.

EXAMPLE 1.2. Let $X = [0,1] \subset \mathbb{R}$ with the usual metric. Define $g(x) : X \to X$ by

$$g(x) = \begin{cases} \frac{1}{3}x + \frac{5}{6}, & x \in [0, \frac{1}{4}) \\ -\frac{1}{3}x + 1, & x \in [\frac{1}{4}, 1]. \end{cases}$$

Define $F : X \to 2^X$ by $F(x) = \{0\} \cup \{g(x)\}\ \forall\ x \in X$. Then, F is a multi-valued contraction operator and the fixed point set of $F = \left\{0, \frac{3}{4}\right\}$.

For the Banach's fixed point theorem and its various generalizations in single-valued case, we refer to Agarwal et al. [1], Banach [2], Berinde [3]–[7] and some other references in the reference section of this paper.

Apart from Markins [27] and Nadler [29], several other papers have been devoted to the treatment of multi-valued operators and these include Berinde and Berinde [9], Ciric [14], Ciric and Ume [15, 16], Daffer and Kaneko [17], Itoh [20], Kaneko [22, 23], Kubiaczyk and Ali [25], Lim [26], Mizoguchi [28] and some others in the reference section.

In Berinde and Berinde [9], the following contractive condition was employed:

DEFINITION 1.3. Let (X, d) be a metric space and $T : X \to P(X)$ a multi-valued operator. T is said to be a *multi-valued weak contraction* or a *multi-valued $(\theta,\ L)$-contraction* if and only if there exist two constants $\theta \in (0,1)$ and $L \geq 0$ such that

$$(2) \qquad H(Tx, Ty) \leq \theta d(x,y) + LD(y, Tx), \ \forall\ x,\ y \in X.$$

The following notion of b-metric space shall be employed in the sequel.

DEFINITION 1.4. (Czerwik [12, 13]) Let X be a (nonempty) set and $s \geq 1$ a real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a *b-metric* if $\forall x, y, z \in X$,

(i) $d(x,y) = 0$ iff $x = y$;
(ii) $d(x,y) = d(y,x)$;
(iii) $d(x,z) \leq s[d(x,y) + d(y,z)]$.

The pair (X, d) is called a b-metric space.

In fact, the class of b-metric spaces is effectively larger than that of metric spaces, since a b-metric is a metric when $s = 1$.

DEFINITION 1.5. (Berinde and Berinde [9]) Let (X, d) be a metric space and $T : X \to P(X)$ a multi-valued operator. T is said to be a *multi-valued* *weakly Picard (MWP) Operator* if and only if for each $x \in X$ and any $y \in T(x)$, there exists a sequence $\{x_n\}_{n=0}^{\infty}$ such that

(i) $x_0 = x,\ x_1 = y$;
(ii) $x_{n+1} \in T(x_n)$ for all $n = 0, 1, \ldots$;
(iii) the sequence $\{x_n\}_{n=0}^{\infty}$ is convergent and its limit is a fixed point of T.

Remark 1.6. A sequence $\{x_n\}_{n=0}^{\infty}$ satisfying conditions (i) and (ii) in Definition 1.4 will be called a *sequence of successive approximations* of T, starting from (x, y) or a *Picard iteration* associated to T or a *(Picard) orbit* of T at the initial point x_0.

Examples 1.7. (MWP Operators) Several examples including Examples 1.7 (a) and (b) are contained in Rus et al [39]:
(a) (Nadler [29]) Let (X, d) be a complete metric space and $T : X \to CB(X)$ a multi-valued α-contraction $(0 < \alpha < 1)$. Then T is a MWP operator.
(b) (Rus [37]) Let (X, d) be a complete metric space and $T : X \to CB(X)$ a multi-valued operator for which there exist $\alpha, \beta \in \mathbb{R}^+$, $\alpha + \beta < 1$ such that

(i) $H(Tx, Ty) \leq \alpha d(x, y) + \beta D(y, Ty)$, $\forall x \in X$ and $\forall y \in Tx$;
(ii) T is a closed multi-valued operator.

Then T is a MWP operator.
(c) (Berinde and Berinde [9]) Let (X, d) be a complete metric space and $T : X \to CB(X)$ a multi-valued operator for which there exist two constants $\theta \in (0, 1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \theta d(x, y) + LD(y, Tx), \quad \forall x, y \in X.$$

Then T is a MWP operator.
(d) (Berinde and Berinde [9]) Let (X, d) be a complete metric space and $T : X \to CB(X)$ a multi-valued operator for which there exist a constant $L \geq 0$ and a function $\alpha : [0, \infty) \to [0, 1)$ satisfying $\lim_{r \to t^+} \sup \alpha(r) < 1$, for every $t \in [0, \infty)$, such that

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y) + LD(y, Tx), \quad \forall x, y \in X.$$

Then T is a MWP operator.

A more general class of MWP operators will be presented as our main result in this paper.

In this paper, we obtain a more general result than one of the results of Berinde and Berinde [9] using the following general contractive definition:

Definition 1.8. Let (X, d) be a b-metric space and $T : X \to P(X)$ a multi-valued operator. Then, T will be called a *multi-valued* (θ_n, ϕ)*-weak contraction* if and only if there exist a sequence $\{\theta_n\}_{n=0}^{\infty} \subset (0, 1)$ and a continuous monotone increasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that

$$(\star)\quad H(Tx,Ty)\le \theta_n d(x,y)+\phi(D(y,Tx)),\qquad \forall x,y\in X,\ n=0,1,2,\ldots.$$

Remark 1.9. If in condition $(\star)$, $\theta_n=\theta$, $0<\theta<1$ and $\phi(u)=Lu$, $L\ge 0$, $\forall\ u\in\mathbb{R}^+$, then we obtain the (δ,L)-weak contraction condition in the multi-valued setting employed by Berinde and Berinde [9] defined in (2). The condition $(\star)$ is also a generalization and extension of several others in the literature.

However, we shall require the following Lemma in the sequel.

Lemma 1.10. *Let (X,d) be a metric space. Let $A,\ B\subset X$ and $q>1$. Then, for every $a\in A$, there exists $b\in B$ such that*

$$d(a,b)\le qH(A,B).$$

Lemma 1.10 is contained in Berinde and Berinde [9], Ciric [14] and Rus [35] in a metric space setting.

2. Main result

The following main result shows that any multi-valued weak contraction is a MWP operator.

Theorem 2.1. *Let (X,d) be a complete b-metric space with continuous b-metric and $T:X\to CB(X)$ multi-valued $(\theta_n,\ \phi)$-weak contraction. Suppose that $\phi:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous monotone increasing function such that $\phi(0)=0$. Then,*

(i) *Fix $(T)\neq\phi$;*

(ii) *for any $x_0\in X$, there exists an orbit $\{x_n\}_{n=0}^{\infty}$ of T at the point x_0 that converges to a fixed point x^* of T;*

(iii) *the a priori and the a posteriori error estimates are respectively given by*

$$d(x_n,x^*)\le sM_1 d(x_0,x_1),\quad s\ge 1,\ n=1,2,\cdots, \tag{5}$$

where $M_1=\sum_{j=0}^{\infty}\Pi_{k=0}^{n+j-1}h_k$; and

$$d(x_n,x^*)\le sM_2 d(x_{n-1},x_n),\quad s\ge 1,\ n=1,2,\ldots, \tag{6}$$

$M_2=\sum_{j=0}^{\infty}\Pi_{k=n-1}^{n+j-1}h_k$, *for a certain sequence $\{h_n\}_{n=0}^{\infty}\subset(0,1)$.*

Proof. Let $q>1$ and $h_n=q\theta_n\in(0,1)$, $n=0,1,2,\cdots$. Let $x_0\in X$ and $x_1\in Tx_0$. If $H(Tx_0,Tx_1)=0$, then $Tx_0=Tx_1$, that is, $x_1\in Tx_1$, which implies that Fix $(T)\neq\phi$.

Let $H(Tx_0,Tx_1)\neq 0$. Then, we have by Lemma 1.10 that there exists $x_2\in Tx_1$ such that

$$d(x_1,x_2)\le qH(Tx_0,Tx_1),$$

so that by ($\star$) we have

$$d(x_1,x_2) \leq q[\theta_0 d(x_0,x_1) + \phi(D(x_1,Tx_0))] \\ = q\theta_0 d(x_0,x_1) = h_0 d(x_0,x_1),$$

where we take $h_0 = q\theta_0 < 1$. If $H(Tx_1,Tx_2) = 0$, then $Tx_1 = Tx_2$, that is, $x_2 \in Tx_2$.

Let $H(Tx_1,Tx_2) \neq 0$. Again, by Lemma 1.10, there exists $x_3 \in Tx_2$ such that

$$(7) \qquad d(x_2,x_3) \leq qH(Tx_1,Tx_2), \\ \leq q[\theta_1 d(x_1,x_2) + \phi(D(x_2,Tx_1))] \\ = q\theta_1 d(x_1,x_2) = h_1 d(x_1,x_2) \leq h_0 h_1 d(x_0,x_1).$$

By induction, we obtain

$$(8) \qquad d(x_n,x_{n+1}) \leq \Pi_{k=0}^{n-1} h_k d(x_0,x_1).$$

Therefore, we have by (8) and the property (iii) of the Definition 1.4 that

$$(9) \quad d(x_n,x_{n+p}) \leq s[d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \cdots + d(x_{n+p-1},x_{n+p})] \\ \leq s[\Pi_{k=0}^{n-1} h_k + \Pi_{k=0}^{n} h_k + \cdots + \Pi_{k=0}^{n+p-2} h_k] d(x_0,x_1)$$

$$(10) \qquad = s\Big(\sum_{j=n-1}^{n+p-2} \Pi_{k=0}^{j} h_k\Big) d(x_0,x_1).$$

From (10), we have

$$(11) \quad d(x_n,x_{n+p}) \leq s\Big(\sum_{j=n-1}^{n+p-2} \Pi_{k=0}^{j} h_k\Big) d(x_0,x_1) \\ = s[\sum_{j=0}^{n+p-2} \Pi_{k=0}^{j} h_k - \sum_{j=0}^{n-2} \Pi_{k=0}^{j} h_k] d(x_0,x_1) \to 0 \text{ as } n \to \infty.$$

We therefore have from (11), that for any $x_0 \in X$, $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X. Since (X,d) is a complete b-metric space, then $\{x_n\}_{n=0}^{\infty}$ converges to some $x^* \in X$. That is,

$$(12) \qquad \lim_{n\to\infty} x_n = x^*.$$

Therefore, by ($\star$), we have that

$$(13) \qquad D(x^*,Tx^*) \leq s[d(x^*,x_{n+1}) + d(x_{n+1},Tx^*)] \\ \leq s[d(x^*,x_{n+1}) + H(Tx_n,Tx^*)] \\ \leq sd(x^*,x_{n+1}) + s[\theta_n d(x_n,x^*) + \phi(D(x^*,Tx_n))].$$

By using (12), the continuity of the function ϕ and the fact that $x_{n+1} \in Tx_n$, then $\phi(D(x^*, Tx_n)) \to 0$ as $n \to \infty$ and also $d(x_n, x^*) \to 0$ as $n \to \infty$. It follows from (13) that, as $n \to \infty$, $D(x^*, Tx^*) = 0$. Since Tx^* is closed, then $x^* \in Tx^*$.

To prove the a priori error estimate in (5), we have from (10) that

$$d(x_{n+p}, x_n) \leq s\Big(\sum_{j=0}^{p-1} \Pi_{k=0}^{n+j-1} h_k\Big) d(x_0, x_1),$$

from which it follows by the continuity of the b-metric that

$$d(x_n, x^*) = d(x^*, x_n) = \lim_{p\to\infty} d(x_{n+p}, x_n) \leq s\Big(\sum_{j=0}^{\infty} \Pi_{k=0}^{n+j-1} h_k\Big) d(x_0, x_1),$$

giving the result in (5).

We now prove the a posteriori estimate in (6): Let $q\theta_n = h_n \in (0,1)$, $n = 0, 1, \cdots$, we get by condition $(\star)$ and Lemma 1.10 that

$$\begin{aligned} d(x_n, x_{n+1}) &\leq qH(Tx_{n-1}, Tx_n) \leq q[\theta_{n-1} d(x_{n-1}, x_n) + \phi(D(x_n, Tx_{n-1}))] \\ &= q\theta_{n-1} d(x_{n-1}, x_n) = h_{n-1} d(x_{n-1}, x_n). \end{aligned}$$

Also, we have

$$d(x_{n+1}, x_{n+2}) \leq h_n d(x_n, x_{n+1}) \leq h_n h_{n-1} d(x_{n-1}, x_n),$$

so that in general, we obtain

$$d(x_{n+j}, x_{n+j+1}) \leq \Pi_{k=n-1}^{n+j-1} h_k d(x_{n-1}, x_n), \; j = 0, 1, \cdots. \tag{14}$$

Using (14) in (9) yields

$$\begin{aligned} d(x_n, x_{n+p}) &\leq s\Big(\sum_{j=n-1}^{n+p-2} \Pi_{k=n-1}^{j} h_k\Big) d(x_{n-1}, x_n) \\ &= s\Big(\sum_{j=0}^{p-1} \Pi_{k=n-1}^{n+j-1} h_k\Big) d(x_{n-1}, x_n). \end{aligned} \tag{15}$$

Again, by taking limits in (15) as $p \to \infty$ and using the continuity of the b-metric, we have

$$d(x_n, x^*) = d(x^*, x_n) = \lim_{p\to\infty} d(x_{n+p}, x_n) \leq s\Big(\sum_{j=0}^{\infty} \Pi_{k=n-1}^{n+j-1} h_k\Big) d(x_{n-1}, x_n),$$

giving the required a posteriori error estimate.

Remark 2.2. Theorem 2.1 is a generalization and extension of Theorem 3 of Berinde and Berinde [9]. It is also a generalization and extension of Theorem 1.1 (which is Theorem 5 of Nadler [29]). Indeed, Theorem 2.1

is a generalization and extension of a multitude of results in the literature pertainning to the single-valued and multi-valued cases. In particular, the error estimates of Theorem 2.1 indeed extend those of Berinde [8].

References

[1] R. P. Agarwal, M. Mechan, D. O'Regan, *Fixed Point Theory and Applications*, Cambridge University Press (2001).

[2] S. Banach, *Sur les operations dans les ensembles abstraits et leur applications aux equations integrales*, Fund. Math. 3 (1922), 133–181.

[3] V. Berinde, *A priori and a posteriori error estimates for a class of φ-contractions*, Bulletins for Applied & Computing Math. (1999), 183–192.

[4] V. Berinde, *Iterative Approximation of Fixed Points*, Editura Efemeride (2002).

[5] V. Berinde, *On the approximation of fixed points of weak φ-contractive operators*, Fixed Point Theory 4 (2) (2003), 131–142.

[6] V. Berinde, *On the approximation of fixed points of weak contractive mappings*, Carpathian J. Math. 19 (1) (2003), 7–22.

[7] V. Berinde, *Approximating fixed points of weak contractions using Picard iteration*, Nonlinear Anal. Forum 9 (1) (2004), 43–53.

[8] V. Berinde, *Error estimates for approximating fixed points of quasi-contractions*, General Mathematics 13 (2) (2005), 23–34.

[9] M. Berinde, V. Berinde, *On a general class of multi-valued weakly Picard mappings*, J. Math. Anal. Appl. 326 (2007), 772–782.

[10] D. W. Boyd, J. S. W. Wong, *On linear contractions*, Proc. Amer. Math. Soc. 20 (1969), 458–464.

[11] F. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. U.S.A. 54 (1965), 1041–1044.

[12] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. et Inform. Univ. Ostraviensis 1 (1993), 5–11.

[13] S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti Sem. Mat. Fis. Univ. Modena 46 (2) (1998), 263–276. MR1665883 (99j:54043).

[14] L. B. Ciric, *Fixed Point Theory, Contraction Mapping Principle*, FME Press, Beograd, 2003.

[15] L. B. Ciric, J. S. Ume, *Common fixed point theorems for multi-valued non-self mappings*, Publ. Math. Debrecen 60 (3-4) (2002), 359–371.

[16] L. B. Ciric, J. S. Ume, *On the convergence of Ishikawa iterates to a common fixed point of multi-valued mappings*, Demonstratio Math. 36 (4) (2003), 951–956.

[17] P. Z. Daffer, H. Kaneko, *Fixed points of generalized contractive multi-valued mappings*, J. Math. Anal. Appl. 192 (1995), 655–666.

[18] M. A. Geraghty, *On contractive mappings*, Proc. Amer. Math. Soc. 40 (1973), 604–608.

[19] C. Goffman, G. Pedrick, *First Course in Functional Analysis*, Prentice Hall of India, Private Limited, New Delhi-11000 (1993).

[20] S. Itoh, *Multi-valued generalized contractions and fixed point theorems*, Comment. Math. Univ. Carolin. 18 (1977), 247–258.

[21] M. C. Joshi, R. K. Bose, *Some Topics in Nonlinear Functional Analysis*, Wiley Eastern Limited (1985).

[22] H. Kaneko, *A general principle for fixed points of contractive multi-valued mappings*, Math. Japon. 31 (1986), 407–411.
[23] H. Kaneko, *Generalized contractive multi-valued mappings and their fixed points*, Math. Japon. 33 (1988), 57–64.
[24] M. A. Khamsi, W. A. Kirk, *An Introduction to Metric Spaces and Fixed Point Theory*, John Wiley & Sons, Inc. (2001).
[25] I. Kubiaczyk, N. M. Ali, *On the convergence of the Ishikawa iterates to a common fixed point for a pair of multi-valued mappings*, Acta Math. Hungar. 75 (3) (1997), 253–257.
[26] T. C. Lim, *On fixed point stability for set-valued contractive mappings with applications to generalized differential equations*, J. Math. Anal. Appl. 110 (2) (1985), 436–441.
[27] J. T. Markins, *A fixed point theorem for set-valued mappings*, Bull. Amer. Math. Soc. 74 (1968), 639–640.
[28] N. Mizoguchi, W. Takahashi, *Fixed point theorems for multi-valued mappings on complete metric spaces*, J. Math. Anal. Appl. 141 (1989), 177–188.
[29] S. B. Nadler, *Multi-valued contraction mappings*, Pacific J. Math. 30 (1969), 282–291.
[30] M. O. Olatinwo, *A generalization of some results on multi-valued weakly Picard mappings in b-metric space*, Fasc. Math. 40 (2008), 45–56.
[31] E. Picard, *Memoire sur la theorie des equations aux derivees partielles et la methode des approximations successives*, J. Math. Pures Appl. 6 (1890), 145–210.
[32] B. E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc. 226 (1977), 257–290.
[33] B. E. Rhoades, *A fixed point theorem for a multi-valued non-self mapping*, Comment. Math. Univ. Carolin. 37 (1996), 401–404.
[34] B. E. Rhoades, B. Watson, *Fixed points for set-valued mappings on metric spaces*, Math. Japon. 35 (4) (1990), 735–743.
[35] I. A. Rus, *Fixed point theorems for multi-valued mappings in complete metric spaces*, Math. Japon. 20 (1975), 21–24.
[36] I. A. Rus, *Generalized Contractions and Applications*, Cluj Univ. Press, Cluj Napoca (2001).
[37] I. A. Rus, *Basic problems of the metric fixed point theory revisited (II)*, Stud. Univ. Babeş-Bolyai 36 (1991), 81–99.
[38] I. A. Rus, A. Petrusel, G. Petrusel, *Fixed Point Theory, 1950-2000, Romanian Contributions*, House of the Book of Science, Cluj Napoca (2002).
[39] I. A. Rus, A. Petrusel, A. Sintamarian, *Data dependence of the fixed point set of some multi-valued weakly Picard operators*, Nonlinear Anal. 52 (2003), 1947–1959.
[40] S. L. Singh, C. Bhatnagar, A. M. Hashim, *Round-off stability of Picard iterative procedure for multi-valued operators*, Nonlinear Anal. Forum 10 (2005), 13–19.
[41] E. Zeidler, *Nonlinear Functional Analysis and its Applications-Fixed Point Theorems*, Springer-Verlag, New York, Inc. (1986).

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