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CHARACTERIZATION OF ASYMPTOTICAL EXPANSIONS OF COPULAS BY THE USE OF HOMOGENEOUS FUNCTIONS

Abstract. The definition, terminology and possible forms of homogeneous expansion of copulas are given. The methodology that provides homogeneous expansions with a proof of their existence is presented. Numerous examples illustrating the usage of the main theorem for valuation of the expansions are indicated.

Introduction

One of the most interesting new ideas which have entered finance in recent years is the copula. It is a function that joins a multivariate probability distribution to a collection of univariate marginal probability functions. Copulas allow us to construct models which go beyond the standard ones at the level of dependence. Both in insurance and finance, modeling of extreme events is of great importance. A book of Embrechts et al [3] is the indispensable starting point for anyone interested in contemporary applications and extensions of classical Extreme Value Theory (EVT). There are a number of texts available on EVT. There are other interesting review papers on extremes in insurance and finance: Embrechts et al. (see [5], [2]). They present one of the possible approach for modeling tail events in the multivariate case. The concept of tail dependence describes the amount of dependence in the lower-left-quadrant tail or upper-right-quadrant tail of a bivariate distribution. The paper which surveys various estimators for the tail-dependence coefficient within a parametric, semiparametric, and nonparametric framework is [7]. One of the ideas of the tail analysis is homogeneous tail expansion. The notion of copula tail expansion in the form presented in this paper was introduced by P. Jaworski (see [8], [9]). He studied the behavior of such expansions, but in contrast with this article he used one-degree

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homogeneous functions only. We extend results included in the mentioned articles to much more general case by the use of homogeneous functions of an arbitrary degree. Furthermore, we present and prove the theorem which contains conditions responsible for existence of tail expansions.

In this paper we focus on lower and upper tail k-degree expansions of a copula. We introduce the notion of such expansions and compile some basic facts about them. Then, we investigate possible forms of tail expansions, their existence and methods of computing. Theorem 2.1 and its proof provide a natural and intrinsic characterization of homogeneous expansions. The theorem states that the described expansions exist if and only if there exist some finite limit dependent on the expansion degree.

The paper is structured as follows. Section 1 introduces homogeneous expansions of copulas, some notation and terminology. It contains also a brief exposition of their basic properties. In Section 2 our main result, Theorem 2.1 is stated and proved. Moreover it contains some relevant consequences of mentioned theorem and the methodology of its proof. Section 3 contains some statistical application.

For simplicity of notation, we will restrict ourselves to the bivariate case only, however all results presented in this article holds true also for n-dimensional copulas. For more details regarding the theory of copulas and survival copulas we refer the reader to the monograph of Nelsen [11] or Joe [10].

1. Notations, definitions and properties

Throughout this paper we denote by $\mathbb{R}^+ := [0, +\infty)$ and $\mathbb{R}^2_+ := [0, +\infty]^2 \setminus \{(\infty, \infty)\}$. In this section, we introduce some basic properties of copulas and homogeneous expansions that shall be useful in the sequel.

1.1. Definition and properties of copula function

A copula is a multivariate joint distribution defined on the unit cube $[0, 1]^2$ such that every marginal distribution is uniform on the interval $[0, 1]$. Formally, a copula is a function C of 2 variables on the unit cube $[0, 1]^2$ with the following properties:

1. the range of C is the unit interval [0,1],
2. $C(\vec{u}) = 0$ whenever $\vec{u} \in [0, 1]^2$ has at least one coordinate equal to zero,
3. $C(u, 1) = C(1, u) = u$ for all $u \in [0, 1]$,
4. C is 2-increasing in the sense that for every $u_1, u_2, v_1, v_2 \in [0, 1]$, where $u_1 \leqslant u_2, v_1 \leqslant v_2$ the volume assigned by C to the box $[u_1, u_2] \times [v_1, v_2]$ is nonnegative, that is $C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geqslant 0$.

It is obvious that if C is a copula, then it is continuous and non-decreasing in each of its arguments. In addition, one can demonstrate the so-called

Fréchet inequality, which states that each copula function is bounded by the minimum and maximum one: $\max(u+v-1,0) \leqslant C(u,v) \leqslant \min(u,v)$ for all u,v in $[0,1]$, which are commonly denoted by W and M in the literature. We must say that in two dimensions, both of the Fréchet-Hoeffding bounds are copulas themselves, but as soon as the dimension increases, the Fréchet-Hoeffding lower bound is no longer n-increasing function. However, the inequality on the left-hand side cannot be improved, since for any $\vec{u}$ from the unit cube, there exists a copula C_u such that $W(\vec{u}) = C_u(\vec{u})$ (see Nelsen [11], Theorem 2.10.12).

Now let F_1 and F_2 be any two univariate distributions. It is easy to show that $F(x,y) = C(F_1(x), F_2(y))$ is a probability distribution, the margins of which are exactly F_1 and F_2. Conversely, Sklar [12] proved in 1959 that any bivariate distribution F admits such a representation and that the copula C is unique provided the margins are continuous. The theorem proposed by Sklar underlies most applications of the copula because it shows that much of the study of joint distribution functions can be reduced to the study of copulas. Furthermore, under a.s. strictly increasing transformations of X and Y, the copula C_{XY} is invariant, while the margins may be changed at will.

1.2. Definition of k-degree homogeneous expansion

We say that a copula C has a k-degree lower tail homogeneous expansion if, for arguments in the neighborhood of zero, it can be uniformly approximated by a k-degree homogeneous function. Formally, we define it as follows:

Definition 1.1. We say that a copula C has a *k-degree homogeneous lower tail expansion* ($k \geqslant 1$) if there exists a positive, homogeneous function L of degree k, that is:

$$L\colon \mathbb{R}_+^2 \to \mathbb{R} \quad \text{such that:} \quad \forall t \geqslant 0 \quad L(tu,tv) = t^k L(u,v),$$

and a function $R\colon [0,1]^2 \to \mathbb{R}$, which fulfills two conditions:

$$\begin{aligned} &1.\ |R(u,v)(u+v)^{k-1}| \leqslant M, \text{ for some } M > 0,\\ &2.\ \lim_{(u,v)\to(0,0)} R(u,v) = 0, \end{aligned} \tag{1.2.1}$$

such that the following equation holds:

$$\forall\,(u,v) \in [0,1]^2 : \ C(u,v) = L(u,v) + R(u,v)(u+v)^k. \tag{1.2.2}$$

The function L will be called *the leading part* of the expansion. When $L \equiv 0$ we shall say that the expansion is trivial.

Proposition 1.1. *Let L be the leading part of k-degree lower tail homogeneous expansion of copula C. Then its value can be calculated as a limit* *along a ray:*

$$L(u,v) = \lim_{t\to 0^+} \frac{C(tu,tv)}{t^k}. \tag{1.2.3}$$

Proof. The proof of equality (1.2.3) is based on the definition of homogeneous expansion and the equality (1.2.1). It follows that:

$$\lim_{t\to 0^+} \frac{C(tu,tv)}{t^k} = \lim_{t\to 0^+} \frac{t^k L(u,v) + t^k R(tu,tv)(u+v)^k}{t^k}$$
$$= L(u,v) + (u+v)^k \lim_{t\to 0^+} R(tu,tv) = L(u,v),$$

which completes the proof. ∎

Another important concept used in this article is that of survival copula. Given a copula C, the survival copula associated to C is defined as $\bar{C}(u,v) = u + v - 1 + C(1-u, 1-v)$. For example, it can be shown that for elliptical copulas $\bar{C} = C$. It is also true for the Frank copula. The same approach, as in Sklar's theorem, can be applied to the survival copula. Using, as before, the notation $\bar{F}$ for the joint survival function, and $\bar{F}_1$, $\bar{F}_2$ for the marginal survival functions, the survival copula is given by $\bar{C}(u,v) = \bar{F}(\bar{F}_1^{-1}(u), \bar{F}_2^{-1}(v))$.

From this relation between copula and the corresponding survival copula, one can easily obtains the definition of upper tail expansions of any copula C analogous to Definition 1.1.

Definition 1.2. We say that a copula C has a *k-degree upper tail expansion* ($k \geqslant 1$) if the corresponding survival copula $\bar{C}$ has a k-degree lower tail expansion.

1.3. Examples of the copulas leading parts

This subsection shows some of the most known families of copulas and theirs leading parts of lower tail homogeneous expansions in relation to degree of this expansion. Presented examples shows all possible situations of existing and degree of lower tail homogeneous expansions in case of two-dimensional copulas. All the copulas listed below are described in [11]. The leading parts presented below can be obtained after long but straightforward calculations.

1. Product copula: $C(u,v) = uv$

$$k \in [1,2): L(u,v) \equiv 0, \; R(u,v)(u+v)^k = C(u,v),$$
$$k = 2: L(u,v) = C(u,v), \; R(u,v)(u+v)^2 \equiv 0.$$

This is an example of a bivariate copula function, which is homogeneous of degree two. It implies that its k-degree lower tail homogeneous expansion is trivial for $k \in [1,2)$ and nontrivial for $k = 2$. For $k > 2$ such expansion does not exist, because in this case limit given by the formula (1.2.3) is infinite.

2. Lower Fréchet-Hoeffding Bound: $C(u,v) = \max(u+v-1, 0)$

$$k \geqslant 1 : L(u,v) \equiv 0, \; R(u,v)(u+v)^k = C(u,v).$$

3. Upper Fréchet-Hoeffding Bound: $C(u,v) = \min(u,v)$

$$k = 1 : L(u,v) = C(u,v), \; R(u,v)(u+v) \equiv 0.$$

This is an example of a bivariate copula function, which is homogeneous of degree one, so it has nontrivial one-degree lower tail homogeneous expansion. For $k > 1$ such expansion does not exist (from the same reason as in case of product copula).

4. The FGM family (see [11], p. 68):

$$C_\theta(u,v) = uv + \theta uv(1-u)(1-v), \; \theta \in [-1,1]$$

$$k \in [1,2) : L(u,v) \equiv 0, \; R(u,v)(u+v)^k = C_\theta(u,v),$$
$$k = 2 : L(u,v) = uv(1+\theta), \; R(u,v)(u+v)^2 = \theta uv(uv-u-v).$$

This copula has trivial k-degree lower tail homogeneous expansion for $k \in [1,2)$ and nontrivial for $k = 2$. For $k > 2$ such expansion does not exist (arguments as in example 1).

5 The Ali-Mikhail-Haq family (see [11], p. 25):

$$C_\theta(u,v) = \frac{uv}{1-\theta(1-u)(1-v)}, \; \theta \in [-1,1)$$

$$k \in [1,2) : L(u,v) \equiv 0, \; R(u,v)(u+v)^k = C_\theta(u,v),$$
$$k = 2 : L(u,v) = \frac{uv}{1-\theta}, \; R(u,v)(u+v)^2 = \frac{uv(uv-u-v)}{(1-\theta)(1-\theta(1-u)(1-v))}.$$

As before, this copula has trivial k-degree lower tail homogeneous expansion for $k \in [1,2)$ and nontrivial for $k = 2$. For $k > 2$ such expansion does not exist (arguments as in example 1).

We will end this subsection by discussing the existence of homogeneous copulas with degree of homogeneity higher than their dimension. From the third property of the definition of a copula function we can get that 2-dimensional copula which is homogeneous of degree k is of the form $C(u,v) = (\max(u,v))^{k-1}\min(u,v)$, where $k \geqslant 1$. The main part of lower tail expansion is equal to C. The function C defined above is not 2-increasing for $k > 2$. To see this, let $A = [a,1] \times [a,1]$. Then $V_C(A) = 1 - 2a + a^k$ and for $k > 2$ we can always choose such a, for which $V_C(A) < 0$. It means, that there are no copulas which are homogeneous of order higher than their dimension (see Theorem 3.4.2 in [11]).

1.4. The properties of leading part

Many properties of the leading part of the copula homogeneous expansion are closely related to copula properties, cf. [11]. For example, function L, as the limit of nondecreasing functions, is also non-decreasing. The next properties are given for the lower tail copula homogeneous expansion only however, analogous properties hold for the upper tail expansion.

Proposition 1.2. *The leading part L of copula lower tail expansion of order k has the following properties:*

i) *(Nonnegativity)* $\forall\, u,v \in [0,1]: \; L(u,v) \geqslant 0.$
ii) *(Groundedness)* $\forall u,v \in [0,1]: \; L(u,0) = 0 = L(0,v).$
iii) *(Monotonicity) $L(u,v)$ is 2-increasing and non-decreasing in each of its arguments. Moreover for $k=1$ function L is Lipschitz continuous.*
iv) *(Homogeneity)* $\forall\, u,v \in [0,1]\ \forall\, t > 0: \; L(tu,tv) = t^k L(u,v) \geqslant 0.$
v) *(Boundedness) Let M be such as in condition 2. of Definition 1.1. Then:*

$$\forall\, u,v \in [0,1]: \; L(u,v) \leqslant \min(u,v) + 2M.$$

Furthermore, if function R is non-negative or $k=1$ then the following stronger inequality is satisfied:

$$L(u,v) \leqslant \min(u,v).$$

Proof. Properties i)-iv) follow from basic properties of copulas and their proofs are left to the reader. The proof of the fifth property for $k=1$ can be found in [8]. The proof for $k>1$ is based on the similar idea and we omit the details. ∎

It is also worth to mention that in the case of one-degree homogeneous expansion its leading part is concave (see [8]). However, in the case of homogeneous expansion of higher degree it is not true. For example, the leading part of 2-degree lower tail expansion for copula from FGM family is of the form $L(u,v) = (1+\theta)uv$ (see subsection 1.3, example 4) for $\theta \in [-1,1]$. We have to check that for all $(u_1,u_2),(v_1,v_2) \in [0,1]^2$ and for all $a,b \in R_+$, $a+b=1$ the function L satisfies condition: $L(au_1+bv_1, au_2+bv_2) \geqslant aL(u_1,u_2)+bL(v_1,v_2)$. Last inequality is equivalent to $ab(u_1-v_1)(v_2-u_2) \geqslant 0$. It is easily seen that we can choose arguments in such way, that this inequality is false. It implies that L is not a concave function.

One of the most useful dependence ordering is the concordance order. We say that the copula C_1 is smaller than the copula C_2 and we note $C_1 \prec C_2$ if $C_1(u,v) \leqslant C_2(u,v)$ for any (u,v) in the unit square $[0,1]^2$. It turns out that also leading parts of lower tail homogeneous expansions of copulas can be partially ordered. There is implication between the concordance order of copulas and the order between leading parts of their lower tail homogeneous expansions (of course the degree of the compared copulas expansions must be the same). Formulation of this result is stated formally in the next theorem.

THEOREM 1.3. *Assume that C_i, $i = 1,2$, has k-degree homogeneous expansion with decomposition $C_i(u,v) = L_i(u,v) + R_i(u,v)(u+v)^k$, $i = 1,2$. Then the following implication is true:*
$\forall u,v \in [0,1] : C_1(u,v) \leqslant C_2(u,v) \;\Rightarrow\; \forall u,v \in [0,1] : \; L_1(u,v) \leqslant L_2(u,v)$.

Proof. By assumption, for every $u,v \in [0,1]$ and for every $t \in \mathbb{R}^+$ we have $C_1(tu,tv) \leqslant C_2(tu,tv)$. Therefore, it suffices to divide the both sides of the last inequality by t^k and apply formula (1.2.3). ∎

We conclude this subsection with the following theorem concerning the partial derivatives of the leading part L with respect to its variables. The word "almost" is used in the sense of Lebesgue measure.

PROPOSITION 1.4. *Let L be the leading part of the k-degree lower tail homogeneous expansion of some copula C ($k \geqslant 1$). Than for every $v \in [0,1]$ ($u \in [0,1]$) the partial derivative $\partial L/\partial u$ ($\partial L/\partial v$) exists almost everywhere in $u \in [0,1]$ ($v \in [0,1]$). Moreover, functions $u \mapsto \partial L(u,v)/\partial v$ and $v \mapsto \partial L(u,v)/\partial u$ are almost everywhere defined and non-decreasing on $[0,1]$.*

Proof. The method of proving is similar to one used in case of copulas. The details for copulas can be found in [11]. ∎

REMARK 1.1. If there exists at least one point $(u',v') \in (0,1]^2$ such that $L(u',v') = 0$, than $L \equiv 0$. It follows from property i) and iii) of function L from Proposition 1.2 and from the following inequalities:

$$\begin{aligned}\forall u,v \in [0,1] : 0 \leqslant L(u,v) &= L\Big(u'\frac{u}{u'}, v'\frac{v}{v'}\Big)\\ &\leqslant L\Big(\max\Big(\frac{u}{u'},\frac{v}{v'}\Big)u', \max\Big(\frac{u}{u'},\frac{v}{v'}\Big)v'\Big)\\ &\leqslant \max\Big(\frac{u}{u'},\frac{v}{v'}\Big)^k L(u',v') = 0.\end{aligned}$$

2. Characterization of the leading part

2.1. The main result

In this section our main theorem which provides the method of computing the leading part of copula homogeneous expansion of degree higher than one (and the decomposition of the copula function into homogeneous part and the rest) is stated and proved. We will start from the case $k = 2$. Next, we will consider remaining possibilities. The case $k = 1$ was presented in [8] so we will omit it.

THEOREM 2.1. *A copula $C \colon [0,1]^2 \to [0,1]$ has a nontrivial lower tail expansion of degree $k = 2$ iff it has a trivial lower tail expansion of degree* *one and there exists a positive, bounded on cube* $[0,1]^2$ *and homogeneous function* $L\colon R_+^2 \to R$ *of degree two, such that:*

$$\lim_{(u,v)\to(0,0)^+} \frac{|C(u,v)-L(u,v)|}{\|(u,v)\|^2} = 0, \tag{2.1.4}$$

where $\|\vec{u}\| = \sum_i |u_i|$. *Furthermore the value of* L *can be determined as a limit along a ray.*

Proof. "$\Rightarrow$" By the hypothesis, there exists a two-degree homogeneous expansion for copula C which implies that for every point $(u,v) \in [0,1]^2$ copula can be expressed in the form (1.2.2). Then:

$$\begin{aligned}\lim_{(u,v)\to(0,0)^+} \frac{|C(u,v)-L(u,v)|}{\|(u,v)\|^2} &= \lim_{(u,v)\to(0,0)^+} \frac{|R(u,v)(u+v)^2|}{\|(u,v)\|^2} \\ &= \lim_{(u,v)\to(0,0)^+} |R(u,v)| = 0.\end{aligned}$$

It remains to prove that in this case one-degree lower tail homogeneous expansion of copula C is trivial, which means that its limit along a ray is equal to zero:

$$\lim_{t\to 0^+} \frac{C(tu,tv)}{t} = \lim_{t\to 0^+} \left(tL(u,v) + tR(tu,tv)(u+v)^2\right) = 0.$$

"$\Leftarrow$" The procedure is to show that there exist functions L and R such that the copula C can be expressed in the form (1.2.2). Let L be the function which is nonnegative, homogeneous and satisfying limit (2.1.4). For R lets take function $R(u,v) = \frac{C(u,v)-L(u,v)}{(u+v)^2}$. Then:

$$a. \quad \lim_{(u,v)\to(0,0)^+} |R(u,v)| = \lim_{(u,v)\to(0,0)^+} \frac{|C(u,v)-L(u,v)|}{\|(u,v)\|^2} = 0,$$

$$\begin{aligned} b. \quad R(u,v)(u+v) &\geqslant -\frac{L(u,v)}{(u+v)} \geqslant -\frac{L(u,v)}{\max(u,v)} \\ &= -\max(u,v)\, \underbrace{L\Big(\frac{u}{\max(u,v)}, \frac{v}{\max(u,v)}\Big)}_{\text{bounded by the assumption}}, \end{aligned}$$

$$R(u,v)(u+v) \leqslant \frac{C(u,v)}{u+v} \leqslant \frac{\min(u,v)}{u+v} \leqslant 1,$$

which shows that copula C has 2-degree homogeneous expansion by the Definition 1.1. ∎

For the case of one-degree homogeneous expansion we refer interested readers to [8]. Inter alia it is stated that copula has one-degree homogeneous expansion of lower tail iff there exists a homogeneous function $L\colon R_+^2 \to R$

of degree one, such that:

$$\lim_{(u,v)\to(0,0)^+} \frac{|C(u,v)-L(u,v)|}{\|(u,v)\|} = 0.$$

PROPOSITION 2.2. *If copula C has non-trivial m-degree homogeneous expansion of lower tail then homogeneous expansion of copula C of degree $k > m$ does not exist.*

Proof. The statement is obvious from the Proposition 1.1, because:

$$\lim_{t\to 0^+} \frac{C(tu,tv)}{t^k} = \lim_{t\to 0^+} \frac{L(u,v)+R(tu,tv)(u+v)^m}{t^{k-m}} = +\infty. \ \blacksquare$$

It turns out, that conditions from Theorem 2.1 provide not only the possibility of checking when given copula C has 2-degree homogeneous expansion but also what is its form if some additional conditions about copula are required.

THEOREM 2.3. *Assume that a copula C is twice continuously differentiable on $[0,1]^2 \setminus \{(0,0)\}$ and has 2-degree homogeneous expansion. Then the main part L of this expansion is of the form:*

$$L(u,v) = \lim_{t\to 0^+} \left[\frac{1}{2}u^2\frac{\partial^2 C}{\partial x^2}(tu,tv) + uv\frac{\partial^2 C}{\partial x \partial y}(tu,tv) + \frac{1}{2}v^2\frac{\partial^2 C}{\partial y^2}(tu,tv)\right], \tag{2.1.5}$$

assuming, that given limit exists for all $(u,v) \in [0,1]^2 \setminus \{(0,0)\}$.

Proof. Our proof starts with the observation that if a copula C has 2-degree homogeneous expansion then by formula (1.2.3) its leading part can be obtained from the formula $L(u,v) = \lim_{t\to 0^+} K_{(u,v)}(t)/t^2$, where $K_{(u,v)}(t) = C(tu,tv)$. Notice that, using differentiation rule for composite functions, for every fixed $(u,v) \in [0,1]^2 \setminus \{(0,0)\}$ we get:

$$\lim_{t\to 0^+} \frac{\partial K_{(u,v)}(t)/\partial t}{\partial t/\partial t} = \lim_{t\to 0^+} \left(u\frac{\partial C}{\partial x}(tu,tv) + v\frac{\partial C}{\partial y}(tu,tv)\right). \tag{2.1.6}$$

The limit given by formula "(2.1.6)" exists and is equal to the leading part of one-degree homogeneous expansion of copula C, which is therefore trivial by assumption and Theorem 2.1. Moreover:

$$\lim_{t\to 0^+} \frac{\partial\left(u\frac{\partial C}{\partial x}(tu,tv) + v\frac{\partial C}{\partial y}(tu,tv)\right)/\partial t}{\partial t/\partial t} = \lim_{t\to 0^+} \left[u^2\frac{\partial^2 C}{\partial x^2}(tu,tv) + 2uv\frac{\partial^2 C}{\partial x \partial y}(tu,tv) + v^2\frac{\partial^2 C}{\partial y^2}(tu,tv)\right].$$

Using differentiation rule for composite functions we get:

$$\lim_{t\to 0^+}\frac{K_{(u,v)}(t)}{t^2}\overset{H}{=}\lim_{t\to 0^+}\frac{\partial\big(u\frac{\partial C}{\partial x}(tu,tv)+v\frac{\partial C}{\partial y}(tu,tv)\big)/\partial t}{2\partial t/\partial t}$$
$$=\lim_{t\to 0^+}\left[\frac{1}{2}u^2\frac{\partial^2 C}{\partial x^2}(tu,tv)+uv\frac{\partial^2 C}{\partial x\partial y}(tu,tv)+\frac{1}{2}v^2\frac{\partial^2 C}{\partial y^2}(tu,tv)\right],$$

where "$\overset{H}{=}$" stands for the de L'Hospital theorem applied to calculation the limit of indeterminate sign $\left[\frac{0}{0}\right]$ (see [6]). This completes the proof of formula (2.1.5). ∎

Remark 2.1. Notice that, under assumptions of Theorem 2.3, triviality of one-degree lower tail homogeneous expansion of copula implies existence and triviality of its k-degree lower tail homogeneous expansion for $k \in (1,2)$. It follows immediately from the proof of Theorem 2.3 and from the fact, that for $k \in (1,2)$:

$$\lim_{t\to 0^+}\frac{K_{(u,v)}(t)}{t^k}\overset{H}{=}\lim_{t\to 0^+}\frac{\partial\big(u\frac{\partial C}{\partial x}(tu,tv)+v\frac{\partial C}{\partial y}(tu,tv)\big)/\partial t}{\partial(kt^{k-1})/\partial t}$$
$$=\lim_{t\to 0^+}\frac{t^{2-k}}{k(k-1)}\left[u^2\frac{\partial^2 C}{\partial x^2}(tu,tv)+2uv\frac{\partial^2 C}{\partial x\partial y}(tu,tv)+v^2\frac{\partial^2 C}{\partial y^2}(tu,tv)\right]=0.$$

3. The coefficient of lower and upper tail dependence

For financial applications of homogeneous expansions of order one we refer the reader to [8] and [9]. Here we present some examples in order to illustrate applicability of our main theorem in case of homogeneous expansions of order higher than one.

The tail-dependence coefficient is an asymptotic measure of dependence specially focused on bivariate extreme values. For continuous marginal distributions the notion of tail-dependence coefficient is in fact a copula property and so we will follow here the definition in terms of copulas given by Joe [10].

Let X and Y be random variables with continuous distribution functions F and G respectively. The *coefficient of lower tail dependence* of X and Y is

$$\lambda_L=\lim_{q\to 0}P(Y\leqslant G^{-1}(q)|X\leqslant F^{-1}(q))$$

provided a limit exists, and the *coefficient of upper tail dependence* is

$$\lambda_U=\lim_{q\to 1}P(Y> G^{-1}(q)|X> F^{-1}(q))$$

provided a limit exists.

Thus, these coefficients are limiting conditional probabilities that both margins exceed (in the case of coefficient of lower tail dependence) or are less than or equal to (in the case of coefficient of upper tail dependence) a certain quantile level given that one margin does.

If C is the copula of (X,Y), then

$$\lambda_L = \lim_{q\to 0} \frac{P(G(Y)\leqslant q, F(X)\leqslant q)}{P(F(X)\leqslant q)} = \lim_{u\to 0^+} \frac{C(u,u)}{u},$$

and

$$\begin{aligned}\lambda_U &= \lim_{q\to 1} \frac{P(G(Y)>q, F(X)>q)}{1-P(F(X)\leqslant q)} \\ &= \lim_{q\to 1} \frac{1-P(G(Y)>q, F(X)\leqslant q)-P(G(Y)\leqslant q, F(X)>q)}{1-P(F(X)\leqslant q)} \\ &\quad \frac{-P(G(Y)\leqslant q, F(X)\leqslant q)}{} = \lim_{u\to 1^-} \frac{1-2u+C(u,u)}{1-u}.\end{aligned}$$

We say that if the limit λ_L (λ_U) exists and belongs to $(0,1]$, then C has lower (upper) tail dependence. If $\lambda_L = 0$ ($\lambda_U = 0$) we talk of asymptotic independence in the lower (upper) tail.

Another representation of the upper tail dependence is given by the formula $\lambda_U = \lim_{u\to 0^+} \hat{C}(u,u)/u$, where $\hat{C}$ denotes the survival copula of C. It follows from the fact, that:

$$\lambda_U = \lim_{u\to 1^-} \frac{1-2u+C(u,u)}{1-u} = \lim_{u\to 1^-} \frac{\hat{C}(1-u,1-u)}{1-u} = \lim_{u\to 0^+} \frac{\hat{C}(u,u)}{u}.$$

Thus, the upper tail dependence of C equals the lower tail dependence of its survival copula and, vice versa.

Let us now consider connection between the coefficient λ_L and k-degree lower tail homogeneous expansion.

PROPOSITION 3.1. *If copula C has non-trivial k-degree lower tail homogeneous expansion then it is asymptotically dependent in the lower tail for $k=1$ and asymptotically independent in the lower tail for $k>1$.*

Proof. It follows easily that:

$$\begin{aligned}\lambda_L &= \lim_{u\to 0} \frac{C(u,u)}{u} = \lim_{u\to 0} u^{k-1}L(1,1) + \lim_{u\to 0} R(u,u)2^k u^{k-1} \\ &= L(1,1)\lim_{u\to 0} u^{k-1} = \begin{cases} L(1,1) \text{ if } & k=1, \\ 0 \quad\quad\; \text{if } & k>1, \end{cases}\end{aligned}$$

which is the desired conclusion. ∎

Similar conclusion can be drawn for upper tail dependence, but in this case we must consider homogeneous expansion of the survival copula.

Proposition 3.2. *If the survival copula of C has non-trivial k-degree lower tail homogeneous expansion then copula C is asymptotically dependent in the lower tail for $k = 1$ and asymptotically independent in the lower tail for $k > 1$.*

We are now in a position to show that if we know that copula is independent in lower tail, than we can conclude about form of the main part of its one-degree homogeneous expansion. We can explain it on a simple example. For the Gaussian copula, the coefficients of lower tail and upper tail dependence are

$$\lambda_L = \lambda_U = 2 \lim_{x \to -\infty} \Phi\left(x \frac{\sqrt{1-\rho}}{\sqrt{1+\rho}} \right) = 0,$$

where Φ denotes standard Gaussian distribution function and ρ denotes correlation coefficient (see [1]). It means, that the Gaussian copulas do not exhibit tail dependence. Embrechts et al. [4] Remark: "Regardless of how high a correlation we choose, if we go far enough into the tail, extreme events appear to occur independently in each margin."

By the proof of Proposition 3.1 we know that if copula has one-degree homogeneous expansion, then $\lambda_L = L(1,1)$. It means that for Gaussian copula we have $L(1,1) = 0$. Additionally, from Remark 1.1 follows that $L(u,v) \equiv 0$ for all $(u,v) \in [0,1]^2$, which means that the Gaussian copulas has trivial one degree homogeneous expansion.

The example above shows how we can deduce about copula homogeneous expansion from its lower tail dependence coefficient. On the other side, we can deduce about lower tail dependence coefficient from existence and form of copula homogeneous expansion. For example, copulas from FGM and Ali-Mikhail-Haq families are asymptotically independent in the lower tail. It follows from the fact that their homogeneous expansions of degree higher than one exists (see subsection 1.3, Example 4 and 5).

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