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DIRECT THEOREMS FOR MODIFIED BASKAKOV OPERATORS IN Lp-SPACES

Abstract. In the year 1993, Gupta and Srivastava [3] introduced the integral modification of the well known Baskakov operators by taking the weight functions of Szasz basis function, so called Baskakov-Szasz operators. In this paper, we obtain some direct theorems for the linear combination of these Baskakov Szasz type operators. To prove our one of the direct theorems, we use the technique of a mathematical tool which is the linear approximating method and is known as the Steklov means.

1. Introduction

For $f \in L_p[0,\infty)$, $p \geq 1$, Gupta and Srivastava [3] introduced an interesting sequence of linear positive operators to modify the well-known Baskakov operators by considering the weights of Szasz basis functions. The modified Baskakov-Szasz operators, introduced in [3] are defined by

$$(1.1)\qquad S_n(f,x) = n\sum_{v=0}^{\infty} p_{n,v}(x)\int_0^{\infty} q_{n,v}(u)f(u)du,\ x \in [0,\infty),$$

where

$$(1.2)\quad p_{n,v}(x) = \binom{n+v-1}{v} x^v(1+x)^{-n-v} \quad \text{and} \quad q_{n,v}(u) = \frac{e^{-nu}(nu)^v}{v!}.$$

In [3], the authors have estimated asymptotic formula and an error estimates in simultaneous approximation. It is observed from [3] that the rate of convergence for these operators $S_n(f,x)$ is of $O\left(n^{-1}\right)$. To improve the order of approximation, we consider the linear combination of these operators $S_n(f,k,x)$ of the operators $S_{d_jn}(f,x)$, where d_jn, $j = 0,1,2,\dots,k$ are

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arbitrary distinct positive integers, the linear combinations are defined as

$$S_n(f,k,x) = \sum_{j=0}^{k} C(j,k) S_{d_j n}(f,x), \tag{1.3}$$

where

$$C(j,k) = \prod_{\substack{i=0 \\ i \neq j}}^{k} \frac{d_j}{d_j - d_i}, \quad k \neq 0 \text{ and } C(0,0) = 1. \tag{1.4}$$

We may rewrite the operators (1.1) as

$$S_n(f,x) = \int_0^{\infty} V(n,x,u)\, f(u)\, du, \tag{1.5}$$

where $V(n,x,u) = n \sum_{v=0}^{\infty} p_{n,v}(x)\, q_{n,v}(u)$.

Alternately the k-th linear combinations $S_n(f,k,x)$ of the operators $S_{d_j n}(f,x)$ are defined by

$$S_n(f,k,x) = \begin{vmatrix} 1 & d_0^{-1} & \dots & d_0^{-k} \\ 1 & d_1^{-1} & \dots & d_1^{-k} \\ \dots & \dots & \dots & \dots \\ 1 & d_k^{-1} & \dots & d_k^{-k} \end{vmatrix}^{-1} \begin{vmatrix} S_{d_0 n}(f,x) & d_0^{-1} & \dots & d_0^{-k} \\ S_{d_1 n}(f,x) & d_1^{-1} & \dots & d_1^{-k} \\ \dots & \dots & \dots & \dots \\ S_{d_k n}(f,x) & d_k^{-1} & \dots & d_k^{-k} \end{vmatrix}, \tag{1.6}$$

where $d_0, d_1, d_2, \dots, d_k$ are (k+1) arbitrary but fixed distinct natural numbers. Combinations of this type were considered by May [4], to improve the order of approximation of exponential type operators.

Throughout this paper let $0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < \infty$, $0 < a < b < \infty$ and $I_i = [a_i, b_i]$, $i = 1,2,3$. We denote by C, the positive constant not necessarily the same at each occurrence.

For $f \in L_p[0,\infty)$, $1 \le p < \infty$, the Steklov mean $f_{\eta,m}$ of m-th order corresponding to$f$ is defined by

(1.7)

$$f_{\eta,m}(t) = \eta^{-m} \int_{-\eta/2}^{\eta/2} \int_{-\eta/2}^{\eta/2} \dots \int_{-\eta/2}^{\eta/2} [f(t) + (-1)^{m-1} \Delta_u^m f(t) dt_1, dt_2, \dots, dt_m],$$

where $u = \sum_{i=1}^{m} t_i$ and $\Delta_h^m f(t)$ is the m-th order forward difference of the function f with step length h, which is defined as

$$\Delta_h^m f(t) = \Delta_h^{m-1} \Delta_h^1 f(t) = \Delta_h^{m-1}[f(t+h) - f(t)].$$

It follows from [2, 5] that:

(i) $f_{\eta,m}$ has derivative up to order m, $f_{n,m}^{(m-1)} \in AC(I_1)$, and $f_{n,m}^{(m-1)}$ exists a.e. and belong to $L_p(I_1)$;

(ii) $\left\| f_{\eta,m}^{(r)} \right\|_{L_p(I_2)} \leq C\eta^{-r}\omega_r(f,\eta,p,I_1),\ r = 1,2,..,m;$

(iii) $\|f - f_{\eta,m}\|_{L_p(I_2)} \leq C\omega_m(f,\eta,p,I_1);$

(iv) $\|f_{\eta,m}\|_{L_p(I_2)} \leq C\,\|f\|_{L_p(I_1)};$

(v) $\left\| f_{\eta,m}^{(r)} \right\|_{L_p(I_2)} \leq C\eta^{-m}\,\|f\|_{L_p(I_1)},\ r = 1,2,\ldots,m;$

where $\omega_r(f,\eta,p,I_1)$, $r = 1,2,\ldots,m$ is the modulus of continuity of order r on the interval I_1. Also $AC\,[a,\ b]$ stands for the class of absolutely continuous function on [a, b] and C are certain constants which are independent of f and n. Let $BV\,[a,\ b]$ denotes the set of all functions of bounded variation on [a, b]. The semi-norm $\|f\|_{BV[a,b]}$ is defined by the total variation of f on $[a,\ b]$. For $f \in L_p[a,b]$, $1 < p < \infty$, the Hardy-Littlewood majorant of f is defined as:

$$h_f(x) = \sup_{\xi\to x} \frac{1}{\xi - x} \int_x^{\xi} f(t)\,dt,\ \ (a \leq \xi \leq b).$$

In the present article we establish some direct results on Lp-norm for the linear combinations of the modified Baskakov operators.

2. Auxiliary results

This section deals with certain basic lemmas, which are necessary to prove the direct theorem.

Lemma 2.1. [3] *Let the m-th order moment be defined by*

$$T_{n,m}(x) = n\sum_{v=0}^{\infty} p_{n,v}(x) \int_0^{\infty} q_{n,v}(u)(u-x)^m du. \tag{2.1}$$

Then

$$T_{n,0}(x) = 1,\ T_{n,1}(x) = \frac{1}{n},\ T_{n,2} = \frac{2+nx(2+x)}{n^2},$$

and there holds a recurrence relation

(2.2) $nT_{n,m+1}(x) = x(1+x)T_{n,m}^{(1)}(x) + (m+1)T_{n,m}(x) + mx(2+x)T_{n,m-1}(x),$ $m - 1 \in N$.

Consequently for $x \geq 0$,

$$T_{n,m}(x) = O(n^{-[(m+1)/2]}), \tag{2.3}$$

where $[\alpha]$ denotes the integral part of α.

Also by Holder's inequality we conclude from (2.3), for every fixed $x \in [0,\infty)$

$$S_n(|u-x|^r, x) = O(n^{-r/2}),\ \ \forall r > 0. \tag{2.4}$$

Lemma 2.2. *For $p \in N$ and n sufficiently large there holds*

$$S_n\left[(t-x)^p, k, x\right] = n^{-(k+1)}\left\{Q(p,k,x) + o(1)\right\}, \quad t \in [0,\infty)$$

where $Q(p,k,x)$ are certain polynomials in x of degree $p/2$.

Proof. From Lemma 2.1, for sufficiently large n we can write,

$$S_n[(t-x)^p, x] = \frac{P_0(x)}{n^{[(p+1)/2]}} + \frac{P_1(x)}{n^{[(p+1)/2]+1}} + \ldots\ldots\ldots + \frac{P_{[p/2]}(x)}{n^p},$$

where $P_i's$ are certain polynomials in x of degree at most p. Thus

$$S_n\left[(t-x)^p, k, x\right] = \begin{vmatrix} 1 & d_0^{-1} \ldots d_0^{-k} \\ 1 & d_1^{-1} \ldots d_1^{-k} \\ \ldots & \ldots\ldots\ldots\ldots \\ 1 & d_k^{-1} \ldots d_k^{-k} \end{vmatrix}^{-1}$$

$$\begin{vmatrix} \frac{P_0(x)}{(d_0 n)^{[(p+1)/2]}} + \frac{P_1(x)}{(d_0 n)^{[(p+1)/2]+1}} + \cdots + \frac{P_{(p/2)}(x)}{(d_0 n)^p} + \ldots d_0^{-1} \ldots d_0^{-k} \\ \frac{P_0(x)}{(d_1 n)^{[(p+1)/2]}} + \frac{P_1(x)}{(d_1 n)^{[(p+1)/2]+1}} + \cdots + \frac{P_{(p/2)}(x)}{(d_1 n)^p} + \ldots d_1^{-1} \ldots d_1^{-k} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{P_0(x)}{(d_k n)^{[(p+1)/2]}} + \frac{P_1(x)}{(d_k n)^{[(p+1)/2]+1}} + \cdots + \frac{P_{(p/2)}(x)}{(d_k n)^p} + \ldots d_k^{-1} \ldots d_k^{-k} \end{vmatrix}$$

$$= n^{-(k+1)}\{Q(p,k,x) + o(1)\}.$$

Lemma 2.3. [1] *Let $1 \le p \le \infty$, $f \in L_p[a,b]$, $f^{(k)} \in AC[a,b]$ and $f^{(k+1)} \in L_p[a,b]$, then*

$$\|f^{(j)}\|_{L_p[a,b]} \le C(\|f^{(k+1)}\|_{L_p[a,b]} + \|f\|_{L_p[a,b]})$$

$j = 1,2,\ldots,k$, where C are certain constants depending only on j,k,p,a,b.

3. Main results

Theorem 3.1. *Let $f \in L_p[0,\infty)$, $p > 1$. If f has $(2k+2)$ derivatives on I_1, with $f^{(2k+1)} \in AC(I_1)$ then for n sufficiently large*

$$\|S_n(f,k,\cdot) - f\|_{L_p[(I_2)} \le Cn^{-(k+1)}(\|f^{(k+1)}\|_{L_p[(I_2)} + \|f\|_{L_p[0,\infty)}),$$

where the constant C is independent of n and f.

Proof. By the hypothesis, for $x \in I_2$ and $u \in I_1$

$$f(u) = \sum_{j=0}^{2k+1} \frac{(u-x)^j}{j!} f^{(j)}(x) + \frac{1}{(2k+1)!} \int_x^u (u-w)^{2k+1} f^{(2k+2)}(w)\, dw + F(u,x)(1-\Phi(u)), \tag{3.1}$$

where $\Phi(u)$ denotes the characteristic function on I_1.

$$F(u,x) = f(u) - \sum_{j=0}^{2k+1} \frac{(u-x)^j}{j!} f^{(j)}(x),$$

for all $u \in [0, \infty)$ and $x \in I_2$. Using (3.1) in (1.3), we have

$$\begin{aligned} S_n(f,k,x) - f(x) &= \sum_{j=1}^{2k+1} \frac{f^{(j)}(x)}{j!} S_n((u-x)^j, k, x) \\ &+ \frac{1}{(2k+1)!} S_n \Phi(u) \int_x^u ((u-w)^{2k+1} f^{(2k+2)}(w)\, dw, k, x) \\ &+ S_n(F(u,x)(1-\Phi(u), k, x) = \Delta_1 + \Delta_2 + \Delta_3. \end{aligned}$$

In view of Lemma 2.2 and [1]

$$\begin{aligned} \|\Delta_1\|_{L_p(I_2)} &\le Cn^{-(k+1)} \Big(\sum_{j=1}^{2k+1} \left\| f^{(j)} \right\|_{L_p(I_2)} \Big) \\ &\le Cn^{-(k+1)} (\|f^{(j)}\|_{L_p(I_2)} + \|f^{(2k+2)}\|_{L_p(I_2)}). \end{aligned}$$

To estimate I_2, let h_f be the Hardy–Littlewood majorant [6] of $f^{(2k+2)}$ on I_1, using Holder's inequality (2.4), we obtain

$$\begin{aligned} J_1 &= \left| S_n(\Phi(u)) \int_x^u \Big((u-w)^{2k+1} f^{(2k+2)}(w) dw,\ x \Big) \right| \\ &\le S_n(\Phi(u)) \int_t^x \left| (u-w)^{2k+1} \right| \left| f^{(2k+2)}(w) \right| \, |dw|\, , x) \\ &\le S_n(\Phi(u)\, |(u-x)|^{2k+1} \int_x^u \left| f^{(2k+2)}(w) \right| \, |dw|, x) \end{aligned}$$

$$\begin{aligned}
&\le S_n(\Phi(u)(u-x)^{2k+2}|h_f(u)|,x)\\
&\le \{S_n(|u-x|^{(2k+2)q}\,\Phi(u),x\}^{1/q}.\,\{S_n(|h_f(u)|^p\,\Phi(u),x\}^{1/p}\\
&\le Cn^{-(k+1)}\,\{S_n(|h_f(u)|^p\,\Phi(u),x\}^{1/p}\\
&\le Cn^{-(k+1)}\Big(\int_{a_1}^{b_1} V(n,x,u)\,|h_f(u)|^p\,du\Big)^{1/p}.
\end{aligned}$$

Fubini's theorem and [7, chapter 2] imply that

$$\begin{aligned}
\|J_1\|^p_{L_p(I_2)} &\le Cn^{-(k+1)p}\int_{a_2}^{b_2}\int_{a_1}^{b_1} V(n,x,u)\,|h_f(u)|^p\,du\,dx\\
&\le Cn^{-(k+1)p}\int_{a_1}^{b_1}\left[\int_{a_2}^{b_2} V(n,x,u)dx\right]|h_f(u)|^p\,du\\
&\le Cn^{-(k+1)p}\int_{a_1}^{b_1}|h_f(u)|^p\,du \le Cn^{-(k+1)p}\,\|h_f(u)\|^p_{L_p(I_1)}\\
&\le Cn^{-(k+1)p}\left\|f^{(2k+2)}\right\|^p_{L_p(I_1)}.
\end{aligned}$$

Therefore,

$$\|J_1\|_{L_p(I_2)} \le Cn^{-(k+1)p}\|f^{(2k+2)}\|_{L_p(I_1)}.$$

Consequently,

$$\|\triangle_2\|_{L_p(I_2)} \le Cn^{-(k+1)p}\|f^{(2k+2)}\|_{L_p(I_1)}.$$

For $u\in[0,\infty)\backslash[a_1,b_1]$, $x\in I_2$, there exists a $\delta>0$ such that $|u-x|\ge\delta$. Thus

$$\begin{aligned}
&|S_n(F(u,x)(1-\Phi(u),x)| \le \delta^{-(2k+2)}S_n(|F(u,x)|\,(u-x)^{2k+2},x)\\
&\le \delta^{-(2k+2)}S_n(|f(u)|+\sum_{j=0}^{2k+1}\frac{|u-x|^j}{j!}\left|f^{(j)}(x)\right|(u-x)^{2k+2},x)\\
&\le \delta^{-(2k+2)}\Big[S_n(|f(u)|\,(u-x)^{2k+2},x)+\sum_{j=1}^{2k+1}\frac{|f^{(j)}(x)|}{j!}S_n(|u-x|^{2k+2+j},x)\Big]\\
&= J_2+J_3.
\end{aligned}$$

Using Holder's inequality and (2.4), we get

$$\begin{aligned}
|J_2| &\le \delta^{-(2k+2)}S_n(|f(x)|^p,x)^{1/p}S_n(|u-x|^{(2k+2)q},x)^{1/q}\\
&\le Cn^{-(k+1)}S_n(|f(u)|^p,x)^{1/p}.
\end{aligned}$$

Again applying Fubini's theorem, we get

$$\int_{a_2}^{b_2} |J_2|^p \, dt \le Cn^{-(k+1)p} \int_{a_2}^{b_2} \int_0^\infty V(n,x,u) \, |f(u)|^p \, du \, dx$$
$$\le Cn^{-(k+1)} \, \|f\|_{L_p[0,\infty))} \, .$$

Thus

$$\|J_2\|_{L_p(I_2)} \le Cn^{-(k+1)} \, \|f\|_{L_p([0,\infty)} \, .$$

Moreover using (2.4) and [1], we get

$$\|J_1\|_{L_p(I_2)} \le Cn^{-(k+1)} \sum_{j=0}^{2k+1} \|f^{(j)}\|_{L_p(I_2)}$$
$$\le Cn^{-(k+1)} (\|f^{(j)}\|_{L_p(I_2)} + \|f^{(2k+2)}\|_{L_p(I_2)}).$$

Combining the estimates of J_2 and J_1, we are led to

$$\|\Delta_2\|_{L_p(I_2)} \le Cn^{-(k+1)} (\|f\|_{L_p([0,\infty))} + \|f^{(2k+2)}\|_{L_p(I_2)}).$$

Hence we obtain the desired result.

Theorem 3.2. *Let $f \in L_1[0,\infty)$. If f has $(2k+1)$ derivatives in I_1 with $f^{(2k)} \in AC(I_1)$ and $f^{(2k+1)} \in BV(I_1)$, then for all n sufficiently large we have*

$$\|S_n(f,k,\cdot) - f\|_{L_1[(I_2)}$$
$$\le Cn^{-(k+1)} (\|f^{(2k+1)}\|_{BV(I_2)} + \|f^{(2k+1)}\|_{L_1(I_2)} + \|f\|_{L_p[0,\infty)}),$$

where C is a constant independent of f and n.

Proof. By the given assumption on f, for almost all $x \in I_2$ and for all $u \in I_1$, we have

$$f(u) = \sum_{i=0}^{2k+1} \frac{(u-x)^i}{i!} f^{(i)}(x) + \frac{1}{(2k+1)!} \int_x^u ((u-w)^{2k+1} df^{(2k+1)}(w)).$$

We can write

$$f(u) = \sum_{i=0}^{2k+1} \frac{(u-x)^i}{i!} f^{(i)}(x)$$
$$+ \frac{1}{(2k+1)!} \int_x^u ((u-w)^{2k+1} df^{(2k+1)}(w)) \Phi(u) + F(u,x)(1-\Phi(u)).$$

Where $\Phi(u)$ denotes the characteristic function of I_1 and

$$F(u,x) = f(u) - \sum_{i=0}^{2k+1} \frac{(u-x)^i}{i!} f^{(i)}(x).$$

For almost all $x \in I_2$ and for all $u \in [0, \infty)$. Hence we obtain

$$\begin{aligned} S_n(f,k,x) - f(x) &= \sum_{i=0}^{2k+1} \frac{f^{(i)}(x)}{i!} S_n((u-x)^i,\, k,\, x) \\ &+ \frac{1}{(2k+1)!} S_n\Big(\int_x^u (u-w)^{2k+1} df^{(2k+1)}(w))\Phi(u), k, x\Big) \\ &+ S_n(F(u,x)(1-\Phi(u),\, k, x) = J_1 + J_2 + J_3. \end{aligned}$$

Applying Lemma 2.1 and [1] we get

$$\|J_1\|_{L_1(I_2)} \le Cn^{-(k+1)}(\|f\|_{L_1(I_2)} + \|f^{(2k+1)}\|_{L_1(I_2)}).$$

Further, we have

$$\begin{aligned} K &= \Big\| S_n\Big(\int_x^u (u-w)^{2k+1} df^{(2k+1)}(w)\Phi(u), x\Big)\Big\|_{L_1(I_2)} \\ &\le \int_{a_2}^{b_2}\int_{a_1}^{b_1} V(n,x,u)\, |u-x|^{2k+1} \left|\int_x^u \left|df^{(2k+1)}(w)\right|\right| du\, dx. \end{aligned}$$

For each n there exists a nonnegative integer $r = r(n)$ such that

$$rn^{-1/2} < \max(b_1 - a_2,\ b_2 - a_1) \le (r+1)\, n^{-1/2}.$$

Then we have

$$\begin{aligned} K \le \sum_{l=0}^{r} \int_{a_2}^{b_2} \Big\{ &\int_{x+(l)n^{-1/2}}^{x+(l+1)n^{-1/2}} \Phi(u)\, V(n,x,u)\ |u-x|^{2k+1} \\ &\cdot \Big[\int_x^{x+(l+1)n^{-1/2}} \Phi(w) \cdot \left|df^{(2k+1)}(w)\right| \Big] du \\ &+ \int_{x-(l+1)n^{-1/2}}^{x-\ln^{-1/2}} \Phi(u)\, V(n,x,u)\ |u-x|^{2k+1} \\ &\cdot \Big[\int_{x-(l+1)n^{-1/2}}^{x} \Phi(w) \cdot \left|df^{(2k+1)}(w)\right| \Big] du \Big\} dx. \end{aligned}$$

Let $\Phi_{x,c,d}(w)$ denotes the characteristic function of the interval

$$[x - cn^{-1/2},\, x + dn^{-1/2}],$$

where c, d are nonnegative integers. Then we have:

$$K \leq \sum_{l=1}^{r} \int_{a_2}^{b_2} \Bigg\{ \int_{x+(l)n^{-1/2}}^{x+(l+1)n^{-1/2}} \Phi(u)\, V(n,x,u) l^{-4} n^2 \, |u-x|^{2k+5}$$

$$\cdot \Bigg[\int_{x}^{x+(l+1)n^{-1/2}} \Phi(w) \Phi_{x,0,l+1}(w) \cdot \left| df^{(2k+1)}(w) \right| \Bigg] du$$

$$+ \int_{x-(l+1)n^{-1/2}}^{x-(l)n^{-1/2}} \Phi(u)\, V(n,x,u) l^{-4} n^2 \, |u-x|^{2k+5}$$

$$\cdot \Bigg[\int_{x-(l+1)n^{-1/2}}^{x} \Phi(w) \Phi_{x,l+1.0}(w) \cdot \left| df^{(2k+1)}(w) \right| \Bigg] du \Bigg\} dx$$

$$+ \int_{a_2}^{b_2} \int_{-n^{-1/2}}^{a_1+n^{-1/12}} \Phi(u)\, V(n,x,u)\, |u-x|^{2k+1}$$

$$\cdot \Bigg(\int_{x-n^{-1/2}}^{x+n^{-1/2}} \Phi(w) \Phi_{x,l,1}(w) \cdot \left| df^{(2k+1)}(w) \right| \Bigg) du dx$$

$$\leq \sum_{l=1}^{r} \Bigg(l^{-4} n^2 \int_{a_2}^{b_2} \Bigg\{ \int_{x+(l)n^{-1/2}}^{x+(l+1)n^{-1/2}} \Phi(u)\, V(n,x,u) \, |u-x|^{2k+5}$$

$$\cdot \Bigg(\int_{a_1}^{b_1} \Phi_{x,0,l+1}(w) \left| df^{(2k+1)}(w) \right| \Bigg) du \Bigg)$$

$$+ \int_{x-(l+1)n^{-1/2}}^{x-(l)n^{-1/2}} \Phi(u)\, V(n,x,u) \, |u-x|^{2k+5}$$

$$\cdot \Bigg(\int_{a_1}^{b_1} \Phi_{x,l+1,0}(w) \left| df^{(2k+1)}(w) \right| \Bigg) du \Bigg) dx$$

$$+ \int_{a_2}^{b_2} \int_{-n^{-1/2}}^{a_1+n^{-1/12}} \Phi(u)\, V(n,x,u)\, |u-x|^{2k+1}$$

$$\cdot \Bigg(\int_{a_1}^{b_1} \Phi_{x,l,1}(w) \left| df^{(2k+1)}(w) \right| du\, dx \Bigg).$$

In the next step, using Lemma 2.1 and Fubini's theorem, we obtain

$$
\begin{aligned}
K &\le C n^{-(2k+1)/2}\Big(\sum_{l=1}^{r} l^{-4}\Big(\int_{a_2}^{b_2}\int_{a_1}^{b_1} \Phi_{x,0,l+1}(w)\left|df^{(2k+1)}(w)\right| dx \\
&\quad + \int_{a_2}^{b_2}\int_{a_1}^{b_1} \Phi_{x,l+1,0}(w)\left|df^{(2k+1)}(w)\right| dx \\
&\quad + \int_{a_2}^{b_2}\int_{a_1}^{b_1} \Phi_{x,l,1}(w)\left|df^{(2k+1)}(w)\right| dx\Big) \\
&= C n^{-(2k+1)/2}\Big(\sum_{l=1}^{r} l^{-4}\Big(\int_{a_1}^{b_1}\int_{a_2}^{b_2} \Phi_{x,0,l+1}(w)dx\left|df^{(2k+1)}(w)\right|\Big) \\
&\quad + \int_{a_1}^{b_1}\Big(\int_{a_2}^{b_2} \Phi_{x,0,l+1}(w)dx\Big)\left|df^{(2k+1)}(w)\right|\Big) \\
&\quad + \int_{a_1}^{b_1}\Big(\int_{a_2}^{b_2} \Phi_{x,l,1}(w)dx\Big)\left|df^{(2k+1)}(w)\right| \\
&\le C n^{-(2k+1)/2}\Big(\sum_{l=1}^{r} l^{-4}\Big(\int_{a_2}^{b_2}\int_{w-(l+1)n^{-1/2}}^{w} dx\Big)\left|df^{(2k+1)}(w)\right| \\
&\quad + \int_{a_1}^{b_1}\Big(\int_{w}^{w+(l+1)n^{-1/2}} dx\Big)\left|df^{(2k+1)}(w)\right|\Big) \\
&\quad + \int_{a_1}^{b_1}\Big(\Big(\int_{w-n^{-1/2}}^{w+n^{-1/2}} dx\Big)\left|df^{(2k+1)}(w)\right|\Big) \\
&\le C n^{-(k+1)}\|f^{(2k+1)}\|_{BV(I_1)}.
\end{aligned}
$$

Hence, $\|J_2\|_{L_1(I_2)} \le C n^{-(k+1)}\left\|f^{(2k+1)}\right\|_{BV(I_1)}$, where the constant on the right side depends on k.

For all $u \in [0,\infty)\backslash[a_1,b_1]$, $x \in I_2$, we choose a $\delta > 0$ such that $|u-x| \ge \delta$. Then

$$
\begin{aligned}
\|S_n(F(u,x)(1-\Phi(u)),\, x\|_{L_1(I_2)} &\le \int_{a_2}^{b_2}\int_{0}^{\infty} V(n,x,u)\,|f(u)|(1-\Phi(u))dudx \\
&\quad + \sum_{i=0}^{2k+1}\frac{1}{i!}\int_{a_2}^{b_2}\int_{0}^{\infty} V(n,x,u)\left|f^{(i)}(x)\right|\,|u-x|^i(1-\Phi(u))dudx = J_4 + J_5.
\end{aligned}
$$

For sufficiently large u, there exist positive constants M_0 and C such that $\frac{(u-x)^{2k+2}}{u^{2k+2}+1} > C$, for all $u \ge M_0$, $x \in I_2$. Also by Fubini's theorem

$$J_4 = \left(\int_0^{M_0}\int_{a_2}^{b_2} + \int_{M_0}^{0}\int_{a_2}^{b_2}\right) V(n,x,u)\,|f(u)|\,(1-\Phi(u))dx\,du = J_6 + J_7.$$

Now, using Lemma 2.1, we have

$$\begin{aligned} J_6 &= \delta^{-(2k+2)} \int_0^{M_0}\int_{a_2}^{b_2} V(n,x,u)\,|f(u)|\,(u-x)^{2k+2}dx\,du \\ &\le Cn^{-(k+1)}\left(\int_0^{M_0} |f(u)|\,du\right). \end{aligned}$$

And

$$J_7 = \frac{1}{C}\int_{M_0}^{\infty}\int_{a_2}^{b_2} V(n,x,u)\frac{(u-x)^{2k+2}}{u^{2k+2}+1}\,|f(u)|\,dx\,du \le Cn^{-(k+1)}\left(\int_{M_0}^{\infty} |f(u)|\,du\right).$$

Combining the estimates of J_6 and J_7, we get

$$J_4 \le Cn^{-(k+1)}\,\|f\|_{L_1[0,\infty)}\,.$$

Further, using (2.4) and [1], we get

$$\begin{aligned} J_5 &\le \delta^{-(2k+2)}\sum_{i=0}^{2k+1}\frac{1}{i!}\int_{a_2}^{b_2}\int_0^{\infty} V(n,x,u)\left|f^{(i)}(x)\right|(u-x)^{2k+i+2}du\,dx \\ &\le Cn^{-(k+1)}\left(\sum_{i=0}^{2k+1}\left\|f^{(i)}\right\|_{L_1(I_2)}\right) \\ &\le Cn^{-(k+1)}\left(\|f\|_{L_1(I_2)} + \|f^{(2k+1)}\|_{L_1(I_2)}\right). \end{aligned}$$

From above estimates of J_4 and J_5, we get

$$\|S_n(F(u,x)(1-\Phi(u)),x\|_{L_1(I_2)} \le Cn^{-(k+1)}(\|f\|_{L_1[0,\infty)} + \|f^{(2k+1)}\|_{L_1(I_2)}).$$

Consequently we obtain

$$\|J_3\|_{L_1(I_2)} \le Cn^{-(k+1)}(\|f\|_{L_1[0,\infty)} + \|f^{(2k+1)}\|_{L_1(I_2)}).$$

Finally combining the estimates of J_1, J_2 and J_3, we obtain the required result.

Theorem 3.3. *Let $f \in L_p[0,\infty)$, $p \ge 1$, then for n sufficiently large*

$$\|S_n(f,k,\cdot) - f\|_{L_p(I_2)} \le C(w_{2k+2}(f,n^{-1/2},p,I_1) + n^{-(k+1)}\|f\|_{L_p[0,\infty)}),$$

where C is a constant independent of f and n.

Proof. Let $f_{\eta,2k+2}(u)$ be the Steklov mean of $(2k+2)$-th order corresponding to $f(u)$ where $\eta > 0$ is sufficiently small and $f(u)$ is defined as zero outside $[0,\infty)$. Then we have

$$\begin{aligned}\|S_n(f,k,\cdot)-f\|_{L_p(I_2)} &\le \|S_n(f-f_{\eta,2k+2},k,\cdot)\|_{L_p(I_2)}+\|S_n(f-f_{\eta,2k+2},k,\cdot)\\ &\quad -f_{\eta,2k+2}\|_{L_p(I_2)}+\|f_{\eta,2k+2}-f\|_{L_p(I_2)}\\ &=\Delta_1+\Delta_2+\Delta_3 \text{ (Say)}.\end{aligned}$$

To estimate Δ_1, let $\Phi(u)$ be the characteristic function of I_3; then

$$S_n((f-f_{\eta,2k+2})(u),x)=S_n(\Phi(u)(f-f_{\eta,2k+2})(u),x)=\Delta_4+\Delta_5.$$

The following is true for $p=1$, and it is also same for p>1 follows from Holder's inequality

$$\int_{a_2}^{b_2}|\Delta_4|^p\,du\le\int_{a_2}^{b_2}\int_{a_3}^{b_3}V(n,x,u)\,|(f-f_{\eta,2k+2})(u)|^p du\,dx.$$

On applying Fubini's theorem, we get

$$\begin{aligned}\int_{a_2}^{b_2}|\Delta_4|^p\,du&\le\int_{a_2}^{b_2}\int_{a_3}^{b_3}V(n,x,u)\,|(f-f_{\eta,2k+2})(u)|^p dx\,du\\ &\le\|f-f_{\eta,2k+2}\|^p_{L_p(I_3)}.\end{aligned}$$

Hence

$$\|\Delta_4\|_{L_p(I_2)}\le\|f-f_{\eta,2k+2}\|^p_{L_p(I_3)}.$$

Using Holder's inequality, (2.4) and Fubini's theorem, we get the following for $p\ge1$

$$\|\Delta_5\|_{L_p(I_2)}\le Cn^{-(k+1)}\|f-f_{\eta,2k+2}\|_{L_p[0,\infty)}.$$

By using Jenson's inequality and Fubini's theorem, we obtain

$$\|f_{\eta,2k+2}\|_{L_p[0,\infty)}\le C\,\|f\|_{L_p[0,\infty)}.$$

Hence

$$\|\Delta_5\|_{L_p[0,\infty)}\le Cn^{-(k+1)}\|f\|_{L_p[0,\infty)}.$$

Now using third property of Steklov means, we get

$$\Delta_1\le C(w_{2k+2}(f,n,p,I_1)+n^{-(k+1)}\|f\|_{L_p[0,\infty)}).$$

And we know that,

$$\left\|f^{(2k+1)}_{\eta,2k+2}\right\|_{BV(I_3)}=\left\|f^{(2k+2)}_{\eta,2k+2}\right\|_{L_1(I_3)}.$$

Hence by virtue of Theorem 3.1 (p>1), Theorem 3.2 (p=1) and Lemma 2.3, we have

$$\begin{aligned}\Delta_2&\le Cn^{-(k+1)}(\|f^{(2k+2)}_{\eta,2k+2}\|_{L_p(I_3)}+\|f_{\eta,2k+2}\|_{L_p[0,\infty)})\\ &\le C(\eta^{-(2k+2)}\omega_{2k+2}(f,n,p,I_1)+n^{-(k+1)}\|f\|_{L_p[0,\infty)}),\end{aligned}$$

in view of the properties of Steklov means.

To estimate Δ_3, we use the Steklov means property third, and obtain that

$$\Delta_3 \le C\omega_{2k+2}(f, n, p, I_1).$$

Hence the required result follows. This completes the proof of Theorem 3.3.

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