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# BOUNDED SOLUTIONS OF A GENERALIZED GOŁĄB–SCHINZEL EQUATION

**Abstract.** Let $X$ be a linear space over the field $\mathbb{K}$ of real or complex numbers. We characterize solutions $f : X \to \mathbb{K}$ and $M : \mathbb{K} \to \mathbb{K}$ of the equation

$$f(x + M(f(x))y) = f(x)f(y)$$

in the case where the set $\{x \in X : f(x) \neq 0\}$ has an algebraically interior point. As a consequence we give solutions of the equation such that $f$ is bounded on this set.

## 1. Introduction

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of positive integers, reals and complex numbers, respectively, and let $X$ be a linear space over a field $\mathbb{K}$. The following two classical functional equations, the exponential one

$$f(x+y) = f(x)f(y) \tag{1}$$

and the Gołąb–Schinzel equation

$$f(x + f(x)y) = f(x)f(y), \tag{2}$$

(for $f : X \to \mathbb{K}$) seem to be of a quite different nature. However, it is easily seen that the both equations are particular cases of the following general equation

$$f(x + M(f(x))y) = f(x)f(y) \tag{3}$$

(for $f : X \to \mathbb{K}$ and $M : \mathbb{K} \to \mathbb{K}$); with $M = 1$ and $M = \mathrm{id}_{\mathbb{K}}$, respectively. So we may say that equation (3) connects equations (1) and (2).

Equation (1) is very well known; for results and further references see e.g. the monograph [1, pp. 25–33, 52–57]. Equation (2) has been first studied by S. Gołąb and A. Schinzel in [8]. For further information on (2) we refer to a survey paper [6].

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J. Brzdęk has considered in [4] the generalized Gołąb–Schinzel equation

$$f(x + f(x)^k y) = f(x)f(y), \tag{4}$$

where $k \in \mathbb{N}$, $X$ is a linear space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $f : X \to \mathbb{K}$. He has assumed that supp $f$ has an algebraically interior point. An analogous result for equation (3) in the case when $\mathbb{K} = \mathbb{R}$ has been proved in author's papers [10] and [11]. In Section 3 we consider the more complicated case $\mathbb{K} = \mathbb{C}$. In Section 4 we characterize solutions of (3) under the assumption that $f$ is bounded on a set having an algebraically interior point; our main theorem corresponds also to results found in the papers by J. Brzdęk [5] and by the author [9].

Throughout the paper we assume that

$$X \textit{ is a linear space over } \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$$

(unless explicitly stated otherwise).

**Definition 1.** A point $x \in B \subset X$, $B \neq \emptyset$, is said to be an algebraically interior point (*a.i.p.*) of $B$ provided, for each $y \in X$, there is a $c \in \mathbb{R}$, $c > 0$, such that $x + ay \in B$ for $a \in \mathbb{K}$, $|a| < c$.

In the whole paper, for $f : X \to \mathbb{K}$, we shall use the notations:

$$A := f^{-1}(\{1\}), \quad W := f(X) \setminus \{0\}, \quad F := \{x \in X : f(x) \neq 0\}.$$

## 2. Preliminary lemmas

First we recall some lemmas which will be useful in the sequel.

**Lemma 1.** (cf. [4, Theorem 3]) A function $f : X \to \mathbb{K}$ satisfies (2) and the set $F$ has an a.i.p. if and only if the following two conditions hold:

(i) if $f(X) \subset \mathbb{R}$, then there exists an $\mathbb{R}$–linear functional $g : X \to \mathbb{R}$ such that either

$$f(x) = g(x) + 1 \quad \text{for } x \in X$$

or

$$f(x) = \max\{g(x) + 1, 0\} \quad \text{for } x \in X;$$

(ii) if $f(X) \not\subset \mathbb{R}$, then there exists a $\mathbb{C}$–linear functional $g : X \to \mathbb{C}$ such that $f(x) = g(x) + 1$ for $x \in X$.

**Lemma 2.** ([10, Lemma 2, Lemma 3]) *Let $f : X \to \mathbb{K}$, $M : \mathbb{K} \to \mathbb{K}$, $f \neq 1$ and $f \neq 0$. If $f$, $M$ satisfy equation* (3), *then the following properties hold:*

(i) $f(M(f(x))^{-1}(z - x)) = f(z)f(x)^{-1}$ *for* $x, z \in X$, $f(x) \neq 0$;
(ii) $(M \circ f)^{-1}(\{0\}) = f^{-1}(\{0\})$;
(iii) $M(a)A = A$ *for* $a \in W$;
(iv) $A$ *is a subgroup of* $(X, +)$;

(v) $A \setminus \{0\}$ *is the set of periods of* $f$ (*i.e.* $f(x+z) = f(x)$ *for every* $x \in X$ *and* $z \in A \setminus \{0\}$);
(vi) $W$ *is a subgroup of* $(\mathbb{K} \setminus \{0\}, \cdot)$;
(vii) $y - x \in A$ *for every* $x, y \in X$ *with* $f(x) = f(y) \neq 0$.

From Proposition 1 and Proposition 2, all in [10], we have the following

**Lemma 3.** *Let* $f : X \to \mathbb{K}$, $M : \mathbb{K} \to \mathbb{K}$, $f \neq 1$ *and* $f \neq 0$. *If* $f$, $M$ *satisfy equation* (3), *then:*

(i) *there exists a function* $w : W \to X$ *such that* $x \in (w(f(x)) + A)$ *for each* $x \in F$;
(ii) $f$ *and* $\widetilde{M}$ *satisfy* (3), *where*

$$\widetilde{M}(a) = \frac{M(a)}{M(1)} \quad \text{for each } a \in \mathbb{K}; \tag{5}$$

(iii) *if, moreover,* $M(1) = 1$ *and* $M \circ f \neq 1$, *then* $0 \in f(X)$.

**Proof.** From [10, Proposition 1 and Proposition 2] we have conditions (i) and (iii), respectively. To prove condition (ii), in the same way as in the proof of [10, Corollary 1], we put $x = 0$ in (3). Then, in view of Lemma 2 (iv), we obtain $f(M(1)y) = f(y)f(0) = f(y)$ for each $y \in X$. By Lemma 2 (ii) we have $M(1) \neq 0$. Whence, replacing $y$ by $\frac{z}{M(1)}$, we obtain $f(\frac{z}{M(1)}) = f(z)$ for $z \in X$. Consequently, for every $x, z \in X$,

$$f(x + \widetilde{M}(f(x))z) = f(x + M(f(x))\frac{z}{M(1)}) = f(x)f(\frac{z}{M(1)}) = f(x)f(z),$$

what ends the proof. ∎

**Lemma 4.** (cf. [10, Lemma 4]) *Let* $f : X \to \mathbb{K}$ *and* $M : \mathbb{K} \to \mathbb{K}$ *satisfy equation* (3), $f \neq 0$, $M(1) = 1$ *and* $M(W) \setminus \{1\} \neq \emptyset$. *Then there exists an* $x_0 \in X$ *such that*

$$F \subset (M(W) - 1)x_0 + A_0, \tag{6}$$

*where* $A_0$ *denotes the linear subspace of* $X$ *spanned by* $A$ *over the field*

$$\mathbb{K}_0 = \begin{cases} \mathbb{R}, & \text{if } M(W) \subset \mathbb{R}; \\ \mathbb{C}, & \text{otherwise}. \end{cases} \tag{7}$$

*Furthermore, if* $A_0 = A$, *then* $x_0 \notin A$,

$$x \in (M(f(x)) - 1)x_0 + A \quad \text{for each } x \in F \tag{8}$$

*and the function* $M|_{f(X)}$ *is injective and multiplicative.*

**Proof.** Because of Lemma 4 in [10] only the multiplicativity of $M|_{f(X)}$ requires a proof.

By Lemma 2 (ii), (vi), it is easy to see that

$$M(f(x)f(y)) = 0 = M(f(x))M(f(y))$$

for $x, y \in X$ with $f(x)f(y) = 0$.

Now, take $x, y \in X$ such that $f(x)f(y) \neq 0$. Then, by (8),

$$x = (M(f(x)) - 1)x_0 + z_1 \text{ and } y = (M(f(y)) - 1)x_0 + z_2$$

for some $z_1, z_2 \in A$. According to equation (3)

$$\begin{aligned} f(x)f(y) &= f(x + M(f(x))y) \\ &= f((M(f(x)) - 1)x_0 + z_1 + M(f(x))((M(f(y)) - 1)x_0 + z_2)) \\ &= f((M(f(x))M(f(y)) - 1)x_0 + z_1 + M(f(x))z_2). \end{aligned}$$

Since $A$ is a linear subspace of $X$ over $\mathbb{K}_0$, $z_1 + M(f(x))z_2 \in A$. Thus, in view of Lemma 2 (v),

$$f(x)f(y) = f((M(f(x))M(f(y)) - 1)x_0) \neq 0.$$

Next, by (8),

$$(M(f(x))M(f(y)) - 1)x_0 \in (M(f(x)f(y)) - 1)x_0 + A$$

and hence

$$[M(f(x))M(f(y)) - M(f(x)f(y))]x_0 \in A.$$

Consequently, since $x_0 \notin A$ and $A = A_0$ is a linear subspace of $X$,

$$M(f(x))M(f(y)) = M(f(x)f(y)),$$

what completes the proof. ■

## 3. An algebraically interior point in F

Here we will prove a theorem generalizing Theorem 1 in [11] and Theorem 3 in [5].

To prove the main result we need two lemmas.

**Lemma 5.** *Let $X$ be a linear space over $\mathbb{C}$, $f : X \to \mathbb{C}$, $M : \mathbb{C} \to \mathbb{C}$, $M(1) = 1$ and $M \circ f \neq 1$. If $f$, $M$ satisfy* (3) *and* 0 *is an a.i.p. of $F$, then $M(W)$ contains a point which is not a root of unity.*

**Proof.** For the proof by contradiction suppose that

(9) for every $b \in W$ there exists a $k \in \mathbb{N}$ such that $(M(b))^k = 1$.

As in the proof of [10, Lemma 6] we obtain that $M(W) \setminus \{-1, 1\} \neq \emptyset$. Thus $k \geq 3$; i.e. $M(f(y))^k = 1$ for some $y \in X$ and $k \in \mathbb{N}$ (the smallest possible). Using (3) we can prove by mathematical induction that for every $n \in \mathbb{N} \setminus \{1\}$

and $x \in X$

(10) $$(f(x))^n = f\Big(x\Big(1 + \sum_{k=1}^{n-1}(M(f(x)))^k\Big)\Big),$$

and hence, for each $x \in X$ with $M(f(x)) \neq 1$,

(11) $$(f(x))^n = f\Big(\frac{1-(M(f(x)))^n}{1-M(f(x))} \cdot x\Big).$$

By Lemma 2 (iv) we know that $0 \in A$, whence $f(0) = 1$. Thus, in view of (9), (11), we obtain that for each $b \in W$ with $M(b) \neq 1$ there exists a $k \in \mathbb{N}$ fulfilling $b^k = 1$. Hence $\operatorname{card}\{a \in W : M(a) \neq 1\} \leq \aleph_0$.

Now we show that $\operatorname{card}\{a \in W : M(a) = 1\} \leq \aleph_0$. Let $x \in X$ be such that $M(f(x)) = 1$. In view of (3)

$$f(y + M(f(y))x) = f(x)f(y) = f(x + M(f(x))y) = f(x+y).$$

Since $M(f(y))^k = 1$ for some $k \geq 3$ and $M(f(x)) = 1$, by Lemma 2 (ii), $f(x)f(y) \neq 0$. Hence, according to Lemma 2 (vii), we obtain that

(12) $$x - M(f(y))x \in A.$$

Now we prove by mathematical induction that

(13) $$x - M(f(y))^n x \in A \text{ for each } n \in \{1, 2, \ldots, k-1\}.$$

For $n = 1$ (13) coincides with (12). Assume that

$$x - M(f(y))^n x \in A \text{ for some } n \in \{2, \ldots, k-2\}.$$

Using Lemma 2 (ii), (iii) we have

$$M(f(y))x - M(f(y))^{n+1}x \in M(f(y))A = A.$$

Now, in view of (12) and Lemma 2 (iv),

$$x - M(f(y))^{n+1}x \in A + A = A.$$

This ends the proof of condition (13).

By (13) and Lemma 2 (iv) we obtain

$$A \ni \sum_{n=1}^{k-1}(x - M(f(y))^n x) = kx - x\sum_{n=0}^{k-1} M(f(y))^n = kx$$

as $M(f(y))^k = 1$. But $M(f(x)) = 1$. Thus, in view of (10), $1 = f(kx) = f(x)^k$. Hence, by Lemma 2 (ii), for each $b \in W$ such that $M(b) = 1$, there exists a $k \in \mathbb{N}$ fulfilling $b^k = 1$. So we have proved that $\operatorname{card}\{a \in W : M(a) = 1\} \leq \aleph_0$.

In this way we obtain that $\operatorname{card} W \leq \aleph_0$. Now, fix a $z \in X \setminus A$ and put $F_z = \{a \in \mathbb{C} : az \in F\}$. Then the functions $f_z : \mathbb{C} \to \mathbb{C}$, $f_z(a) = f(az)$, and $M$ satisfy (3), $f_z \neq \text{const}$ and $F_z = f_z^{-1}(\mathbb{C} \setminus \{0\})$. Note that $0 \in \operatorname{int} F_z$, because 0 is an *a.i.p.* of $F$. Put $W_z = f_z(F_z)$ and $A_z = f_z^{-1}(\{1\})$. Then, by Lemma 3 (i),

$$F_z = \bigcup_{a \in W_z} (w(a) + A_z)$$

for a function $w : W_z \to \mathbb{C}$. Since $W_z \subset W$ (i.e. $\operatorname{card} W_z \leq \aleph_0$), the set $A_z$ is of second category. This implies, in view of Theorem of S. Piccard (see [12, Theorem 1, p. 48]), that $0 \in \operatorname{int}(\operatorname{cl} A_z - \operatorname{cl} A_z)$ and consequently, by Lemma 2 (iv), $\operatorname{int} \operatorname{cl} A_z = \mathbb{C}$. Hence, $A_z$ is dense in $\mathbb{C}$ and, according to Lemma 2 (v), we have $\mathbb{C} = F_z + A_z = F_z$. Consequently $0 \notin f_z(\mathbb{C})$. Now, by Lemma 3 (iii), we obtain that $M \circ f_z = 1$ (because $M(1) = 1$). Clearly, for $z \in A$, $M \circ f_z = 1$, too. Lemma 2 (iv) $f(0) = 1$. Hence $M \circ f = 1$. This contradiction ends the proof. ■

**Lemma 6.** *Let $X$ be a linear space over $\mathbb{C}$, $f : X \to \mathbb{C}$, $M : \mathbb{C} \to \mathbb{C}$, $M(1) = 1$ and $M \circ f \neq 1$. If $f$ and $M$ satisfy* (3) *and the set $F$ has an a.i.p., then $M|_{f(X)}$ is injective and multiplicative and the following conditions hold:*

(i) *if $M(f(X)) \subset \mathbb{R}$, then there exists an $\mathbb{R}$-linear functional $g : X \to \mathbb{R}$, $g \neq 0$, such that either*

(14) $$M(f(x)) = g(x) + 1 \quad \text{for } x \in X$$

*or*

(15) $$M(f(x)) = \max\{0, g(x) + 1\} \quad \text{for } x \in X;$$

(ii) *if $M(f(X)) \not\subset \mathbb{R}$, then there exists a $\mathbb{C}$-linear functional $g : X \to \mathbb{C}$, $g \neq 0$, such that $f$, $M$ fulfill* (14).

**Proof.** Suppose that $f$ and $M$ satisfy equation (3) and $x_0$ is an *a.i.p.* of $F$. By Lemma 2 (i), $M(f(x_0))^{-1}(F - x_0) \subset F$. Thus 0 is an *a.i.p.* of $F$.

First we prove that $A$ is a linear subspace of $X$ over the field $\mathbb{K}_0$ given by (7). Let $A_0$ denote the linear subspace of $X$ spanned by $A$ over $\mathbb{K}_0$. Fix $c \in \mathbb{K}_0$ and $w \in A \setminus \{0\}$. On account of Lemma 2 (iii), $aA = A$ for $a \in M(W)$, whence also for $a \in W_0$, where $W_0$ is the multiplicative group generated by $M(W)$. According to Lemma 5, $W_0$ is the infinite subgroup of $(\mathbb{K}_0 \setminus \{0\}, \cdot)$ and hence, by [4, Lemma 2 and Lemma 3], the set $A_w = \{a \in \mathbb{K}_0 : aw \in A\}$ is dense in $\mathbb{K}_0$. Since 0 is an *a.i.p.* of $F$, there is a $d > 0$ such that $aw \in F$ for $|a| < d$. Fix a $b \in A_w$ with $|b + c| < d$. Then $(c + b)w \in F$ and we get

$$0 \neq f((b + c)w) = f(bw + M(f(bw))cw) = f(bw)f(cw).$$

So we have proved that $f(cw) \neq 0$ for $w \in A \setminus \{0\}$ and $c \in \mathbb{K}_0$. Since, by Lemma 2 (iv), $f(0) = 1$, the condition $f(cw) \neq 0$ holds for every $w \in A$ and $c \in \mathbb{K}_0$. Moreover, for each $w \in A$, the functions $f|_{\mathbb{K}_0 w}$ and $M$ satisfy equation (3) for $x, y \in \mathbb{K}_0 w$. Hence, by Lemma 3 (iii), $M \circ f|_{\mathbb{K}_0 w} = 1$ for $w \in A$.

To prove that $A_0$ is a proper subspace of $X$, suppose that $A_0 = X$. Then $A$ must contain a basis for $X$. Thus each $x \in X$ is given by $x = \sum_{i=1}^{n} \alpha_i z_i$ for some $n \in \mathbb{N}$, $\alpha_i \in \mathbb{K}_0$ and $z_i \in A$. But $(M \circ f)(\alpha_i z_i) = 1$ for each $i \in \{1, \dots, n\}$. Hence, according to (3), we obtain

$$\begin{aligned} f(x) &= f\Big(\sum_{i=1}^{n} \alpha_i z_i\Big) = f\Big(\alpha_1 z_1 + M(f(\alpha_1 z_1)) \sum_{i=2}^{n} \alpha_i z_i\Big) \\ &= f(\alpha_1 z_1) f\Big(\sum_{i=2}^{n} \alpha_i z_i\Big) = f(\alpha_1 z_1) f\Big(\alpha_2 z_2 + M(f(\alpha_2 z_2)) \sum_{i=3}^{n} \alpha_i z_i\Big) \\ &= f(\alpha_1 z_1) f(\alpha_2 z_2) f\Big(\sum_{i=3}^{n} \alpha_i z_i\Big) = \ldots = \prod_{i=1}^{n} f(\alpha_i z_i) \neq 0, \end{aligned}$$

what contradicts Lemma 3 (iii). This proves that $A_0 \neq X$.

Now, by Lemma 4, there exists an $x_0 \in X$ such that

$$F \subset (M(W) - 1)x_0 + A_0. \tag{16}$$

We show that (16) implies $x_0 \notin A_0$. Otherwise, by linearity of $A_0$ and condition (16), $F \subset A_0$ and hence we would obtain that 0 is an *a.i.p.* of $A_0$. Consequently (by linearity of $A_0$ once again) $X = A_0$, which leads to a contradiction. Thus $x_0 \notin A_0$.

Moreover, since 0 is an *a.i.p.* of $F$, there exists a $c_0 > 0$ such that $ax_0 \in F$ for $a \in \mathbb{C}$, $|a| < c_0$. Thus, in view of (16), for each $a \in \mathbb{C}$, $|a| < c_0$, there exists a $w \in W$ fulfilling condition $(a - M(w) + 1)x_0 \in A_0$. Since $A_0$ is a linear space and $x_0 \notin A_0$, we obtain $a - M(w) + 1 = 0$. In this way we have proved that

$$M(W) - 1 \supset \{a \in \mathbb{K}_0 : |a| < c_0\}.$$

But $M(W) \subset W_0$. Hence, if $M(W) \subset \mathbb{R}$, then $W_0 \supset (0, \infty)$; in the other case $W_0 = \mathbb{C} \setminus \{0\}$. Finally, $W_0 A = A$ so, in view of Lemma 2 (iv), $\mathbb{K}_0 A \subset A$ and we obtain that $A_0 = A$.

Now, according to Lemma 4, there is an $x_0 \notin A$ such that $f$ is of form (8) and $M|_{f(X)}$ is injective and multiplicative. From the multiplicativity of $M|_{f(X)}$ and in view of (3) we have that the function $M \circ f : X \to \mathbb{C}$ satisfies (2) and, by Lemma 2 (ii), $\{x \in X : (M \circ f)(x) \neq 0\} = F$. Thus, according to Lemma 1, if $M(f(X)) \subset \mathbb{R}$, then there exists an $\mathbb{R}$–linear functional $g : X \to \mathbb{R}$ such that $M(f(x)) = g(x) + 1$ or $M(f(x)) = \max\{0, g(x) + 1\}$ for $x \in X$, if $M(f(X)) \not\subset \mathbb{R}$, then there exists a $\mathbb{C}$–linear functional $g : X \to \mathbb{C}$ such that $M(f(x)) = g(x) + 1$ for $x \in X$. Since $M \circ f \neq 1$, $g \neq 0$. This completes the proof. ∎

Now we can prove the announced theorem.

**THEOREM 1.** *Functions $f : X \to \mathbb{K}$, $M : \mathbb{K} \to \mathbb{K}$ satisfy equation* (3) *and the set $F$ has an a.i.p. if and only if one of the following conditions holds:*

(i) $f = 1$;

(ii) $f : X \to \mathbb{K} \setminus \{0\}$ *is a nontrivial exponential function and $M$ is any function such that* $M \circ f = 1$;

(iii) $\mathbb{K} = \mathbb{R}$ *and there exist a multiplicative injection $H : \mathbb{R} \to \mathbb{R}$ and a nontrivial $\mathbb{R}$-linear functional $g : X \to \mathbb{R}$ such that either*

$$\begin{aligned} f(x) &= H(g(x)+1) \ \textit{for } x \in X, \\ M(y) &= H^{-1}(y) \ \textit{for } y \in H(\mathbb{R}) \end{aligned} \tag{17}$$

*or*

$$\begin{aligned} f(x) &= H(\max\{0, g(x)+1\}) \ \textit{for } x \in X, \\ M(y) &= H^{-1}(y) \ \textit{for } y \in H([0,\infty)); \end{aligned} \tag{18}$$

(iv) $\mathbb{K} = \mathbb{C}$ *and one of the following two conditions holds:*

(1) *there exist a multiplicative injection $H : \mathbb{C} \to \mathbb{C}$ and a nontrivial $\mathbb{C}$-linear functional $g : X \to \mathbb{C}$ such that*

$$\begin{aligned} f(x) &= H(g(x)+1) \ \textit{for } x \in X, \\ M(y) &= H^{-1}(y) \ \textit{for } y \in H(\mathbb{C}); \end{aligned} \tag{19}$$

(2) *there exist a multiplicative function $H : \mathbb{R} \to \mathbb{C}$ and a nontrivial $\mathbb{R}$-linear functional $g : X \to \mathbb{R}$ such that either $f$ and $M$ are given by* (17) *and $H$ is injective, or $f$ and $M$ are given by* (18) *and $H|_{[0,\infty)}$ is injective.*

**Proof.** First assume that functions $f$ and $M$ satisfy the equation (3) and the set $F$ has an *a.i.p.* The constant function $f = 1$ obviously satisfies (3), so assume that $f \neq 1$. If $M \circ f = c$, then, by Lemma 3 (ii), $f$ and $\widetilde{M}$ (given by (5)) fulfill (3) and $\widetilde{M} \circ f = 1$. Hence, by equation (3), $f$ is an exponential function. Since $f \neq \text{const}$, $f : X \to \mathbb{K} \setminus \{0\}$ and, putting $x = 0$ in (3), we have $f((c-1)y) = 1$ for each $y \in X$. Thus $c = 1$.

Now suppose that $M \circ f$ is not constant. If $\mathbb{K} = \mathbb{R}$, then condition (iii) holds by Theorem 1 in [11]. Now, let $\mathbb{K} = \mathbb{C}$. Then, by Lemma 3 (ii), $\widetilde{M} \circ f \neq 1$, $\widetilde{M}(1) = 1$ and, in view of Lemma 6, $\widetilde{M}|_{f(X)}$ is injective and multiplicative and conditions (i)-(ii) of Lemma 6 holds with function $\widetilde{M}$ instead of $M$. Consequently, from (5), $M|_{f(X)}$ is injective,

$$M(1)M(ab) = M(1)^2\widetilde{M}(ab) = M(1)^2\widetilde{M}(a)\widetilde{M}(b) = M(a)M(b) \tag{20}$$

for $a, b \in f(X)$ and the following conditions hold:

(a) if $\frac{M(f(X))}{M(1)} \subset \mathbb{R}$, then there exists a nontrivial $\mathbb{R}$–linear functional $g : X \to \mathbb{R}$ such that

$$M(f(x)) = M(1)(g(x) + 1) \quad \text{for } x \in X \tag{21}$$

or

$$M(f(x)) = M(1) \max\{0, g(x) + 1\} \quad \text{for } x \in X; \tag{22}$$

(b) if $\frac{M(f(X))}{M(1)} \not\subset \mathbb{R}$, then there exists a nontrivial $\mathbb{C}$–linear functional $g : X \to \mathbb{C}$ such that $f$ and $M$ fulfill (21).

Then, by Lemma 2 (ii), $F = \{x \in X : g(x) > -1\}$, when $M \circ f$ is given by (22), and $F = \{x \in X : g(x) \neq -1\}$ in the other cases. From (20) we obtain that for every $x, y \in F$

$$M(f(x)f(y)) = \frac{M(f(x))M(f(y))}{M(1)} = (g(x) + 1)(g(y) + 1)M(1).$$

On the other hand, in view of (3) and Lemma 2 (ii), $M(f(x+M(f(x))y)) \neq 0$ for $x, y \in F$ and hence

$$\begin{aligned} &M(f(x + M(f(x))y)) = M(1)(g(x + M(f(x))y) + 1) \\ &= M(1)(g(x) + g(M(f(x))y) + 1) = M(1)\left(g(x) + 1 + \tfrac{M(f(x))}{M(1)} g(M(1)y)\right) \\ &= M(1)(g(x) + 1)(g(M(1)y) + 1). \end{aligned}$$

Now, from (3), we obtain $g(y) = g(M(1)y)$ for each $y \in F$. Thus

$$g((1 - M(1))y) = 0 \quad \text{for } y \in F. \tag{23}$$

Suppose that $M(1) \neq 1$. Since $M \circ f \neq \text{const}$, there exists $z \in X$ such that $g(z) \neq 0$. Let $w = (1 - M(1))^{-1}z$. Then, in view of (23) and the homogeneity of $g$, $w \notin F$. Hence $g(w) \leq -1$, if (21) holds, and $g(w) = -1$ in the other cases. Now, by $\mathbb{R}$–homogeneity of $g$, there exists an $r \in \mathbb{R} \setminus \{0\}$ such that $rw \in F$. It means that

$$0 = g((1 - M(1))rw) = g(rz) = rg(z) \neq 0.$$

This contradiction proves that $M(1) = 1$. Consequently $M|_{f(X)}$ is injective and multiplicative and conditions (i), (ii) of Lemma 6 hold. Hence we obtain that if $M(f(X)) = \mathbb{L} \in \{\mathbb{R}, \mathbb{C}\}$, then the function $H = (M|_{f(X)})^{-1}$ is multiplicative and injective on $\mathbb{L}$. In the case when $M(f(X)) = [0, \infty)$ the function $H : \mathbb{R} \to \mathbb{C}$ given by

$$H(z) = \begin{cases} (M|_{f(X)})^{-1}(z), & z \in [0, \infty) \\ -(M|_{f(X)})^{-1}(-z), & z \in (-\infty, 0) \end{cases}$$

is multiplicative on $\mathbb{R}$ and injective on $[0, \infty)$. Hence condition (iv) holds.

It remains to prove the converse statement. If $f$, $M$ are given by the condition (i) or (ii) of this theorem, one checks that $f$, $M$ satisfy (3) and $F = X$ has an *a.i.p.* So consider the case when $f$, $M$ are given by (iii) or (iv). Then,

$$(M \circ f)(x) = (H^{-1} \circ f)(x) = g(x) + 1 \text{ for } x \in X$$

or

$$(M \circ f)(x) = (H^{-1} \circ f)(x) = \max\{g(x) + 1, 0\} \text{ for } x \in X$$

and hence, according to Lemma 1, $M \circ f$ satisfies equation (2). Since $f(X) \in \{H(\mathbb{C}), H(\mathbb{R}), H([0,\infty))\}$, $M|_{f(X)} = H^{-1}|_{f(X)}$ and thus $M|_{f(X)}$ is multiplicative and injective. Consequently, functions $f$ and $M$ fulfill (3). Hence, according to Lemma 2 (ii), $F = \{x \in X : g(x) > -1\}$, when $f$, $M$ are given by (22), and $F = \{x \in X : g(x) \neq -1\}$ in the other cases. It is easy to see that 0 is an *a.i.p.* of $F$, when $g : X \to \mathbb{K}$ is $\mathbb{K}$–linear. To complete the proof we consider the case when $\mathbb{K} = \mathbb{C}$ and $g : X \to \mathbb{R}$ is $\mathbb{R}$–linear. Take $x \in X \setminus \{0\}$ and $a = a_1 + a_2 i \in \mathbb{C} \setminus \{0\}$. We have

$$\begin{aligned} |g(ax)| &= |a_1 g(x) + a_2 g(ix)| \\ &\le \sqrt{a_1^2 + a_2^2}\sqrt{g(x)^2 + g(ix)^2} = |a|\sqrt{g(x)^2 + g(ix)^2}. \end{aligned}$$

If $g(x) = g(ix) = 0$, then $g(ax) = 0$, $f(ax) = 1$ and $ax \in F$ for each $a \in \mathbb{C}$. Otherwise $|g(ax)| < 1$ and $ax \in F$ whenever $a \in \mathbb{C}$, $|a| < \frac{1}{\sqrt{g(x)^2 + g(ix)^2}}$. In this way we obtain that 0 is an *a.i.p.* of $F$, what ends the proof. ∎

In the case when $\mathbb{K} = \mathbb{C}$ there exist solutions of (3) such that $M(f(X)) \subset \mathbb{R}$ and $f(X) \not\subset \mathbb{R}$.

**Example 1.** Let $X$ be a linear space over $\mathbb{C}$ and $g : X \to \mathbb{R}$ be a nontrivial $\mathbb{R}$–linear functional. Let $f : X \to \mathbb{C}$ be given by

$$f(x) = \begin{cases} (g(x) + 1)e^{i \ln |g(x)+1|}, & g(x) \neq -1; \\ 0, & g(x) = -1. \end{cases}$$

Then $f(X) \setminus \{0\} = \{te^{i \ln |t|} : t \in \mathbb{R} \setminus \{0\}\}$. Now define $M : f(X) \to \mathbb{R}$ as follows:

$$M(0) = 0 \text{ and } M(te^{i \ln |t|}) = t \text{ for } t \in \mathbb{R} \setminus \{0\}.$$

Note that such functions $f$, $M$ fulfill (3) and 0 is an *a.i.p.* of $F$. Moreover $f(X) \not\subset \mathbb{R}$, $M(f(X)) = \mathbb{R}$ and $M(f(x)) = g(x) + 1$ for each $x \in X$.

## 4. Bounded solutions

In this section we characterize solutions of (3) under assumption that $f$ is bounded on a set having an *a.i.p.*

Let $\arg z \in [-\pi, \pi)$ be an argument of $z \in \mathbb{C} \setminus \{0\}$. First we prove a lemma on multiplicative functions.

**Lemma 7.** *If $m : \mathbb{C} \to \mathbb{R}$ is multiplicative and bounded on a set of positive inner Lebesgue measure in $\mathbb{C}$, then $m$ is continuous on $\mathbb{C} \setminus \{0\}$.*

**Proof.** Assume that $m \neq \text{const}$. Let $U$ be a set of positive inner Lebesgue measure in $\mathbb{C}$. Without loss of generality we may assume that $0 \notin U$ and $1 \in U$. Consider a diffeomorphism $d : \mathbb{C} \setminus \{z \in \mathbb{C} : \arg z = -\pi \text{ or } z = 0\} \to (0, \infty) \times (-\pi, \pi)$ given by $d(te^{i\alpha}) = (t, \alpha)$. Since $U_0 = U \setminus \{z \in \mathbb{C} : \arg z = -\pi \text{ or } z = 0\}$ is a set of positive inner Lebesgue measure in $\mathbb{C}$ and $1 \in U_0$, the set $d(U_0)$ includes a subset $V_1 \times V_2$ of positive Lebesgue measure in $\mathbb{R}^2$ such that $(1, 0) \in V_1 \times V_2$.

Define a function $s : (0, \infty) \times [-\pi, \pi) \to \mathbb{R}$ by $s(t, \alpha) = m(te^{i\alpha})$. Then

$$s(t_1t_2, \alpha_1 + \alpha_2) = s(t_1, \alpha_1)s(t_2, \alpha_2)$$

for every $(t_1, \alpha_1), (t_2, \alpha_2) \in (0, \infty) \times [-\pi, \pi)$ such that $\alpha_1 + \alpha_2 \in [-\pi, \pi)$. The function $s_1 : (0, \infty) \to \mathbb{R}$ given by $s_1(t) = s(t, 0)$ is multiplicative on $(0, \infty)$ and bounded on the set $V_1$ of positive Lebesgue measure in $\mathbb{R}$, hence continuous, by [1, Corollary 7 and Corollary 8, pp. 30–31]. Now consider $s_2 : [-\pi, \pi) \to \mathbb{R}$ given by $s_2(\alpha) = s(1, \alpha)$. Then

$$s_2(\alpha_1 + \alpha_2) = s_2(\alpha_1)s_2(\alpha_2)$$

for every $\alpha_1, \alpha_2 \in [-\pi, \pi)$ such that $\alpha_1 + \alpha_2 \in [-\pi, \pi)$. We prove that there exists a unique exponential function $\widetilde{s_2} : \mathbb{R} \to \mathbb{R}$ such that $\widetilde{s_2}|_{[-\pi,\pi)} = s_2$.

First note that $s_2(\alpha) = s_2(2 \cdot \frac{\alpha}{2}) = s_2(\frac{\alpha}{2})^2 \geq 0$ for each $\alpha \in (-\pi, \pi)$. Moreover, we have that either $s_2 = 0$ or $s_2(\alpha) > 0$ for each $\alpha \in (-\pi, \pi)$. Indeed, if $s_2(\alpha_0) = 0$ for an $\alpha_0 \in (-\pi, \pi)$, then

$$s_2(0) = s_2(\alpha_0 - \alpha_0) = s_2(\alpha_0)s_2(-\alpha_0) = 0$$

and, consequently, $s_2(\alpha) = s_2(\alpha + 0) = s_2(\alpha)s_2(0) = 0$ for each $\alpha \in (-\pi, \pi)$.

If $s_2 = 0$, then $\widetilde{s_2} = 0$. So consider the case when $s_2(\alpha) > 0$ for each $\alpha \in (-\pi, \pi)$. Then $p : (-\pi, \pi) \to \mathbb{R}$ given by $p(\alpha) = \ln s_2(\alpha)$ is well defined and $p(\alpha_1 + \alpha_2) = p(\alpha_1) + p(\alpha_2)$ for every $\alpha_1, \alpha_2 \in (-\pi, \pi)$ with $\alpha_1 + \alpha_2 \in (-\pi, \pi)$. Hence, by [3, Lemma 1], $p$ can be extended to a unique additive function $\widetilde{p} : \mathbb{R} \to \mathbb{R}$. Thus $\widetilde{s_2} : \mathbb{R} \to \mathbb{R}$ given by $\widetilde{s_2}(\alpha) = \exp \widetilde{p}(\alpha)$ is the unique extension of $s_2|_{(-\pi,\pi)}$. Moreover, since $-\frac{\pi}{2} \in (-\pi, \pi)$,

$$s_2(-\pi) = s_2\left(2 \cdot \left(-\frac{\pi}{2}\right)\right) = s_2\left(-\frac{\pi}{2}\right)^2 = \widetilde{s_2}\left(-\frac{\pi}{2}\right)^2 = \widetilde{s_2}(-\pi).$$

Hence $\widetilde{s_2}$ is also the unique extension of $s_2$.

But $\widetilde{s_2}$ is bounded on the set $V_2$ of positive Lebesgue measure in $\mathbb{R}$. Thus, by [1, Theorem 5, p. 29], $\widetilde{s_2}$ is continuous. Since

$$s(t,\alpha) = s(t,0) \cdot s(1,\alpha) = s_1(t) \cdot s_2(\alpha),$$

$s$ is continuous on the domain and, consequently, $m$ is continuous on $\mathbb{C} \setminus \{0\}$, what ends the proof. ∎

Now, we are in a position to prove our main result.

**Theorem 2.** *Let $B$ be a set having an a.i.p., $f : X \to \mathbb{K}$, $M : \mathbb{K} \to \mathbb{K}$,*

$$|f(B)| = \{|f(z)| : z \in B\} \subset (0,a) \text{ for some } a > 0 \tag{24}$$

*and, moreover, in the case $\mathbb{K} = \mathbb{C}$,*

$$\left|\arg \frac{f(z_1)}{f(z_2)}\right| < \frac{2\pi}{3} \text{ for every } z_1, z_2 \in B. \tag{25}$$

*Functions $f$ and $M$ satisfy (3) if and only if one of the following conditions holds:*

(a) *$f = 1$;*
(b) *$M \circ f = 1$ and there exists a nontrivial $\mathbb{R}$–linear functional $g : X \to \mathbb{K}$ such that $f(x) = \exp g(x)$ for each $x \in X$;*
(c) *$\mathbb{K} = \mathbb{R}$ and there exist a multiplicative function $H : \mathbb{R} \to \mathbb{R}$ continuous on $\mathbb{R} \setminus \{0\}$ and a nontrivial $\mathbb{R}$–linear functional $g : X \to \mathbb{R}$ such that either (17) or (18) holds;*
(d) *$\mathbb{K} = \mathbb{C}$ and one of the following conditions holds:*
   (d1) *there exist a multiplicative injection $H : \mathbb{C} \to \mathbb{C}$ continuous on $\mathbb{C} \setminus \{0\}$ and a nontrivial $\mathbb{C}$–linear functional $g : X \to \mathbb{C}$ such that (19) holds;*
   (d2) *there exist a multiplicative function $H : \mathbb{R} \to \mathbb{C}$ continuous on $\mathbb{R} \setminus \{0\}$ and a nontrivial $\mathbb{R}$–linear functional $g : X \to \mathbb{R}$ such that either (17) or (18) holds.*

**Proof.** Assume that $f$ and $M$ satisfy (3). Let $x_0$ be an *a.i.p.* of $B$ and $|f(B)| \subset (0,a)$ for an $a > 0$. Since $B \subset F$, this $x_0$ is also an *a.i.p.* of $F$. Thus one of conditions (i)–(iv) of Theorem 1 holds.

Condition (a) coincides with (i). To get condition (b) we use (ii) and [12, Theorem 1, p. 308], which says that for such an $f$ there is an additive function $g : X \to \mathbb{R}$, $g \neq 0$, such that $f = \exp g$. The thing to show is $\mathbb{R}$–homogeneity of $g$. Since $x_0$ is an *a.i.p.* of $B$, for each $x \in X \setminus \{0\}$ there exists a $c > 0$ such that $B_x = \{x_0 + ax : a \in \mathbb{K}, |a| < c\} \subset B$. Since $|f(B)| \subset (0,a)$, $g(y) \leq \ln a$ for every $y \in B_x$. For each $x \in X \setminus \{0\}$ define $g_x : \mathbb{K} \to \mathbb{K}$ by $g_x(\alpha) = g(\alpha x)$ for $\alpha \in \mathbb{K}$. Obviously the function $g_x$ is additive and $g_x(\alpha) \leq \ln a - g(x_0)$ for each $\alpha \in (-c,c)$. Thus, according to [12, Lemma 1, p. 210], $g_x$ is continuous.

Hence, in the case $\mathbb{K} = \mathbb{R}$, there is an $m \in \mathbb{R}$ with $g_x(\alpha) = m\alpha$ and, in the case $\mathbb{K} = \mathbb{C}$, $g_x(\alpha) = m_1\alpha + m_2\bar{\alpha}$ for some $m_1, m_2 \in \mathbb{C}$ (see [12, pp. 121, 132]). Consequently, for every $x \in X \setminus \{0\}$, $g_x$ is $\mathbb{R}$–homogenous and so does $g$, as claimed in (b).

Next we prove that if one of conditions (iii), (iv) holds, then $H$ is continuous on its domain except of 0. To this end define the field $\mathbb{L}$ as follows:

$$\mathbb{L} = \begin{cases} \mathbb{R}, \text{ if } g \text{ is } \mathbb{R} - \text{linear;} \\ \mathbb{C}, \text{ if } g \text{ is } \mathbb{C} - \text{linear} \end{cases}$$

and fix an $x \in X$ with $g(x) \neq 0$. Then, by Definition 1, we have

$$B_0 = \{x_0 + ax : a \in \mathbb{L},\ |a| < c\} \subset B$$

for some $c > 0$. Since functional $g$ is $\mathbb{L}$–linear,

$$g(B_0) = \{g(x_0) + ag(x) : a \in \mathbb{L},\ |a| < c\}.$$

Hence the set

$$I := g(B_0) + 1$$

is open in $\mathbb{L}$. We show that $H(I) = f(B_0)$. This equality is clear when $f(x) = H(g(x)+1)$. So consider the case when $f(x) = H(\max\{0, g(x)+1\})$. Then $\mathbb{L} = \mathbb{R}$ and the set $I$ is an open interval in $\mathbb{R}$. Moreover $I \subset (0, \infty)$, because otherwise $0 \in \{\max\{0, g(x) + 1\} : x \in B_0\}$ and, consequently,

$$0 = H(0) \in H(\{\max\{0, g(x) + 1\} : x \in B_0\}) = f(B_0) \subset f(B),$$

what contradicts the assumption. Thus $\{\max\{0, g(x) + 1\} : x \in B_0\} = I$.

Hence the set $I$ is open in $\mathbb{L}$ and

$$|H(I)| = |f(B_0)| \subset |f(B)| \subset (0, a).$$

Moreover, in the case $\mathbb{K} = \mathbb{C}$,

$$\left|\arg \frac{H(z_1)}{H(z_2)}\right| < \frac{2\pi}{3} \quad \text{for every} \quad z_1, z_2 \in I.$$

If $H : \mathbb{L} \to \mathbb{R}$, then, by [1, Corollary 7 and Corollary 8, pp. 30–31] and Lemma 7, we infer that $H$ is continuous on $\mathbb{L} \setminus \{0\}$.

If $H : \mathbb{L} \to \mathbb{C}$, then

$$H(x) = r(x)e^{i\varphi(x)},$$

where $r : \mathbb{L} \to [0, \infty)$, $\varphi : \mathbb{L} \to [-\pi, \pi)$ and $\varphi(x) = \arg H(x)$. Since the function $H$ is multiplicative, so is $r$. Moreover, the boundedness of $H$ on the set of positive inner Lebesgue measure in $\mathbb{L}$ implies the boundedness of $r$ on the same set. Hence, by [1, Corollary 7 and Corollary 8, pp. 30–31] and Lemma 7, $r$ is continuous on $\mathbb{L} \setminus \{0\}$.

It remains to prove that the function $h : \mathbb{L} \setminus \{0\} \to \mathbb{C}$ given by $h(x) = e^{i\varphi(x)}$ is continuous. We know that $h$ is a character on $\mathbb{L} \setminus \{0\}$ and

$$\left|\arg \frac{h(z_1)}{h(z_2)}\right| < \frac{2\pi}{3} \text{ for every } z_1, z_2 \in I,$$

because $\arg h(z) = \arg H(z)$ for each $z \in \mathbb{L} \setminus \{0\}$. Since $0 = H(0) \notin H(I)$, we have $0 \notin I$. Consequently $I \subset \mathbb{L}\setminus\{0\}$ is a set of positive Lebesgue measure in $\mathbb{L}$ and hence, by [7, Theorem 2], $1 \in \operatorname{int}(I^{-1}\cdot I)$, where $I^{-1} = \{z^{-1} : z \in I\}$. Moreover,

$$|\arg h(z_1^{-1}z_2)| = \left|\arg \frac{h(z_2)}{h(z_1)}\right| < \frac{2\pi}{3} \text{ for every } z_1, z_2 \in I$$

and thus $|\arg h(w)| < \frac{2\pi}{3}$ for each $w \in I^{-1} \cdot I$. Hence

$$|h(w) - 1| < \sqrt{3} \text{ for } w \in I^{-1} \cdot I$$

and, according to [2, Theorem 2], $h$ is continuous, what ends the proof of the continuity of $H$ on its domain without 0. ∎

**Remark 1.** It is easy to see that if in Theorem 1 function $f : X \to \mathbb{C}$ satisfies (24) but not (25), then $H : \mathbb{C} \to \mathbb{C}$ need not be continuous on its domain except of 0. To see this, take

$$H(x) = \begin{cases} \exp^{2\pi i a(\ln|x|)}, & x \in \mathbb{C} \setminus \{0\}; \\ 0, & x = 0, \end{cases}$$

where $a : \mathbb{C} \to \mathbb{R}$ is a discontinuous additive function.

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