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# APPLICATION OF MEASURES OF WEAK NONCOMPACTNESS TO A NONLOCAL DARBOUX PROBLEM

**Abstract.** In this paper we study the existence of pseudosolutions of a nonlocal hyperbolic Darboux problem for the equation

$$\frac{\partial^2 u}{\partial x \partial y}(x,y) = f\left((x,y), u(x,y)\right)$$

with nonlocal boundary conditions $u(x,0) + h_1(x,u) = g_1(x), u(0,y) + h_2(y,u) = g_2(y)$, on the bounded region. The functions considered have values in a Banach space and are weakly-weakly sequentially continuous, and the relevant integrals are Pettis integrals.

## Introduction

In this paper we study the existence of pseudosolutions of a nonlocal hyperbolic Darboux problem for functional-differential equations. Methods of functional analysis together with measures of weak noncompactness and Sadovskii's fixed point theorem are applied.

We consider the problem

$$(1) \qquad \begin{cases} \frac{\partial^2 u}{\partial x \partial y}(x,y) = f\left((x,y), u(x,y)\right), (x,y) \in \Delta, \\ u(x,0) + h_1(x,u) = g_1(x), \quad x \in [0,a_1], \\ u(0,y) + h_2(y,u) = g_2(y), \quad y \in [0,a_2], \end{cases}$$

where $\Delta = [0,a_1] \times [0,a_2] \subset R^2$, $a_1, a_2 > 0$, $f : \Delta \times E \to E$, $g_i \in C^1([0,a_i],E)$, $h_i : C^1(\Delta, E) \to E$ $(i = 1,2)$ ($E$ is a Banach space and $E^*$ its topological dual) are continuous functions.

When the functions $h_i, g_i$ are equal to zero, this problem can be found in [9], [10]. In [9] the properties of the set of solutions of problem (1) are also considered.

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Existence and uniqueness theorems for solutions of nonlocal hyperbolic problems were proved by Byszewski ([4], [5]). Our results extend those of [4], [5], [7], [8], [10], [17], [18].

The theory of differential equations with nonlocal conditions is an interesting and important theory elaborated on in the literature ([3–7, 10-11]) as it can be applied in many real world problems.

The hyperbolic Darboux problem described here can be applied in physics —the nonlocal condition can determine the way of testing a given phenomenon or disturbance affecting the phenomenon being examined. Some physical phenomena which require application of nonlocal conditions are analysed in [4], [7], [8].

## 1. Preliminaries

Let $E$ be an infinite dimensional Banach space. In the present paper $C^1(\Delta, E)$ will denote the space of all continuously differentiable functions defined on $\Delta$ and taking values in $E$, with the norm

$$\|u\|_{C^1} = \max\left\{ \sup_{(x,y)\in\Delta} \|u(x,y)\|,\ \sup_{(x,y)\in\Delta} \left\|\frac{\partial u}{\partial x}(x,y)\right\|,\ \sup_{(x,y)\in\Delta} \left\|\frac{\partial u}{\partial y}(x,y)\right\| \right\}.$$

For any subsets $V \subset C^1(\Delta, E)$ and $P \subset \Delta$, we set

$$V(x,y) = \{u(x,y) : u \in V\},\quad V(P) = \{u(x,y) : u \in V,\ (x,y) \in P\}.$$

Let $C(I, E)$ be the space of all continuous functions defined on $I = (t_0, t_0+a]$, $a > 0$, with values in $E$ and with supremum norm $\|\cdot\|_C$. Moreover, $B(x, r)$ is the closed ball in $E$ with center at $x$ and radius $r$, and $\mu(A)$ denotes the Lebesgue measure of the set $A$.

**Definition 1.** ([13]) Given a bounded subset $A \subset E$, we define the measure of weak noncompactness $\omega(A)$ as follows:

$$\omega(A) = \inf\{\varepsilon > 0 : A \subset C + B(0,\varepsilon),\ C \in K^\omega\},$$

where $K^\omega$ is the family of all weakly compact subsets of $E$.

For the properties of $\omega$, see [2], [3], for instance.

**Definition 2.** ([2], [3]) Let $\aleph$ denote the set of all bounded subsets of $E$. An axiomatic measure of weak noncompactness is a function $\Phi : \aleph \to [0, \infty)$ satisfying for all $A, B \in \aleph$ the following conditions:

1° $\Phi(A) = 0$ if and only if $\bar{A}^\omega$ is a weakly compact set;
2° if $A \subset B$, then $\Phi(A) \le \Phi(B)$;
3° $\Phi(A \cup \{x\}) = \Phi(A)$, $x \in E$;
4° $\Phi(\lambda A) = |\lambda|\, \Phi(A)$, $\lambda \in R$;
5° $\Phi(A \cup B) = \max\{\Phi(A), \Phi(B)\}$;

6° $\Phi(A+B) \leq \Phi(A) + \Phi(B)$;
7° $\Phi(\overline{conv}A) = \Phi(A)$.

**DEFINITION 3.** A Banach space $E$ is weakly compactly generated (WCG) if there exists a weakly compact subset of $E$ with dense linear hull in $E$.

**DEFINITION 4.** A Banach space $E$ is a Fubini-Pettis space (an $FP$-space) if there exists a WCG space $X$ containing no isomorphic copy of the space $l^1$.

Examples of $FP$ spaces can be found in [14].

We will need the following lemmas and theorems:

**LEMMA 5.** ([20]) *Let $V \subset C(I,E)$ be a family of strongly equicontinuous functions. Then the function $t \mapsto \omega(V(t))$ is strongly continuous and*

$$\omega_C(V) = \sup_{t\in I} \omega(V(t)) = \omega(V(I))$$

*defines a measure of weak noncompactness in $C(I,E)$.*

**LEMMA 6.** ([16]) *Let $V \subset C_\omega(I,E)$ be a family of strongly equicontinuous functions. Then the function $t \mapsto \Phi(V(t))$ is continuous and*

$$\Phi(V(I)) = \sup_{t\in I} \Phi(V(t)).$$

**THEOREM 7.** ([19]) *Let $E$ be an $FP$ space. Then for every bounded function $f : \Delta \to E$ there exists a function $f_1 : \Delta \to E$ scalarly equivalent to $f$ such that*

(i) *the function $s \mapsto f_1(s,t)$ is Pettis integrable for almost all $t \in [0,a_2]$,*
(ii) *the function $t \mapsto f_1(s,t)$ is Pettis integrable for almost all $s \in [0,a_1]$,*
(iii) *moreover,*

$$\iint_{A\times B} f(s,t)\,dsdt = \iint_{A\times B} f_1(s,t)\,dsdt = \int_B \left( \int_A f_1(s,t)\,ds \right) dt = \int_A \left( \int_B f_1(s,t)\,dt \right) ds$$

*for any measurable subsets $A \subset [0,a_1]$ and $B \subset [0,a_2]$.*

**DEFINITION 8.** Let $E_1, E_2$ be Banach spaces. We say that a function $f : E_1 \to E_2$ is weakly-weakly sequentially continuous if for every sequence $(x_n)$, $x_n \in E_1$, weakly convergent to $x \in E_1$ the sequence $(f(x_n))$ is weakly convergent to $f(x)$.

A weakly-weakly continuous function is weakly-weakly sequentially continuous but the converse is not true. Some comparison results for this type of continuity can be found in [1].

Now, let us present a fixed point theorem for such functions:

**THEOREM 9.** ([15]) *Let $X$ be a bounded, closed and convex subset of $C(I, E)$ and $\Phi$ an axiomatic measure of weak noncompactness on $X$. Let $F : X \to X$ be a weakly-weakly sequentially continuous function such that*

$$\Phi(F(V)) < \Phi(V)$$

*for every $V \subset X$ with $\Phi(V) > 0$. Then $F$ has a fixed point.*

## 2. Existence of a pseudosolution

**DEFINITION 10.** Suppose that the function $(t, s) \mapsto f((t, s), u(t, s))$ is Pettis integrable for each $u \in C(\Delta, E)$. A continuous function $u : \Delta \to E$ satisfying the equation

$$u(x, y) = g_1(x) - g_1(0) + g_2(y) - h_1(x, u) - h_2(y, u) + \int_0^x \int_0^y f((t, s), u(t, s))\, dt ds, \qquad (x, y) \in \Delta,$$

is said to be a pseudosolution of the nonlocal Darboux problem (1).

We will need the following assumptions:

(F1) For each continuous function $u : \Delta \to E$ the function $(x, y) \mapsto f((x, y), u(x, y))$ is Pettis integrable.

(F2) For each $(x, y) \in \Delta$ the function $z \mapsto f(x, y, z)$ is weakly-weakly sequentially continuous.

(F3) There are functions $v_i \in L^1([0, a_i], R)$ and $v \in L^\infty(\Delta, R)$ such that for each bounded subset $A \subset E$,

$$\omega(f(\{(x, y)\} \cdot A)) \le v_1(x) \cdot v_2(y) \cdot v(x, y) \cdot \omega(A)$$

for almost all $(x, y) \in \Delta$; here $\omega$ is the measure of weak noncompactness.

(H1) The functions $h_i : [0, a_i] \cdot C(\Delta, E) \to E$ are continuous and weakly differentiable with respect to the first variable and weakly-weakly sequentially continuous with respect to the second variable. Moreover, there are functions $\phi_i : R_+ \to R_+$ right-continuous at zero such that for any $z_1^i \in [0, a_i]$ and $u \in C(\Delta, E)$,

$$\left\| h_i(z_1^i, u) - h_i(z_2^i, u) \right\| \le \phi_i\left( \left| z_1^i - z_2^i \right| \right) (1 + \|u\|_C).$$

(H2) There are constants $C_i^j$ $(i, j = 1, 2)$ with $1 - (C_1^1 + C_1^2) > 0$ and $C_2^i > 0$ such that for each $u \in C(\Delta, E)$,

$$\|h_i(z_i, u)\| \le C_1^i \|u\|_{C^1} + C_2^i, \quad z_i \in [0, a_i].$$

(H3) There are constants $C^1, C^2$ with $C^1 + C^2 < 1$ such that for each bounded equicontinuous subset $V \subset C(\Delta, E)$ and $z_i \in [0, a_i]$

$$\omega(h_i(\{z_i\} \cdot V)) \le C^i \cdot \omega(V) \quad (i = 1, 2).$$

(G1) The functions $g_i$ are continuous and weakly differentiable on $[0, a_i]$ $(i = 1, 2)$.

(G2) $g_1(0) = g_2(0)$.

Now we are in a position to formulate our main result.

**THEOREM 11.** *Suppose that the bounded function $f$ and the functions $g_i, h_i$ satisfy the assumptions* (F1)–(F3), (H1)–(H3) *and* (G1), (G2). *Then there exists a pseudosolution of the problem* (1).

**Proof.** The existence of a solution of (1) is equivalent to the existence of a fixed point of the operator defined by

$$(Fu)(x,y) = g_1(x) - g_1(0) + g_2(y) - h_1(x,u) - h_2(y,u) + \int_0^x \left( \int_0^y f\left((t,s), u(t,s)\right) dt \right) ds.$$

Let $x^* \in E^*$ and $\|x^*\| \leq 1$. If $u$ is a solution of (1) then by Theorem 7 and the assumptions (H2), (G1) we have the estimate

$$\begin{aligned} |x^* u(x,y)| &\leq |x^* g_1(x)| + |x^* g_1(0)| + |x^* g_2(y)| + |x^* h_1(x,u)| + |x^* h_2(y,u)| \\ &\quad + \left| x^* \int_0^x \left( \int_0^y f\left((t,s), u(t,s)\right) dt ds \right) \right| \\ &\leq \|g_1(x)\| + \|g_1(0)\| + \|g_2(y)\| + \|h_1(x,u)\| + \|h_2(y,u)\| \\ &\quad + \iint_{[0,a_1]\cdot[0,a_2]} |x^* f\left((t,s), u(t,s)\right)| dtds \\ &\leq 2G_1 + G_2 + C_1^1 \|u\|_C + C_2^1 + C_1^2 \|u\|_C + C_2^2 + a_1 a_2 C, \end{aligned}$$

and so

$$\|u\|_C \leq \frac{2G_1 + G_2 + C_2^1 + C_2^2 + a_1 a_2 C}{1 - \left(C_1^1 + C_1^2\right)},$$

where $G_i$ and $C$ are the supremum norms of $g_i$ and $f$ $(i = 1, 2)$, respectively. Denote the right side of the above inequality by $S$. Let $X_S$ be the set of all continuous functions $u$ bounded by $S$ and satisfying

$$\begin{aligned} \|u(x,y) - u(x_1,y_1)\| \leq \|g_1(x) - g_1(x_1)\| + \|g_2(y) - g_2(y_1)\| \\ + \phi_1(|x - x_1|)(1 + S) + \phi_2(|y - y_1|)(1 + S) \\ + Ca_1 |y - y_1| + Ca_2 |x - x_1|. \end{aligned}$$

The set $X_S$ is bounded, closed and convex. By the continuity of $g_i$ and the right-hand continuity of $\phi_i$ at zero, it is a family of strongly equicontinuous functions.

Now, we will show that $F$ is weakly-weakly sequentially continuous and $F(X_S) \subset X_S$. Indeed, by Theorem 7 and the assumption (G1) we obtain

$$
\begin{aligned}
|x^*(Fu(x,y) - Fu(x_1,y_1))| &\le \|g_1(x) - g_1(x_1)\| + \|g_2(y) - g_2(y_1)\| \\
&\quad + \|h_1(x,u) - h_1(x_1,u)\| + \|h_2(y,u) - h_2(y_1,u)\| \\
&\quad + x^*\Big(\int_0^x\Big(\int_0^y f((t,s),u(t,s))\,dt\Big)ds - \int_0^x\Big(\int_0^y f((t,s),u(t,s))\,dt\Big)ds\Big) \\
&\le \|g_1(x) - g_1(x_1)\| + \|g_2(y) - g_2(y_1)\| + (1+S)\,\phi(|x-x_1|) \\
&\quad + (1+S)\,\phi(|y-y_1|) \\
&\quad + \Big|x^*\Big(\iint_{[0,x]\cdot[0,y]} f((t,s),u(t,s))\,dtds - \iint_{[0,x_1]\cdot[0,y_1]} f((t,s),u(t,s))\,dtds\Big)\Big| \\
&\le \|g_1(x) - g_1(x_1)\| + \|g_2(y) - g_2(y_1)\| \\
&\quad + (1+S)\,\phi(|x-x_1|) + (1+S)\,\phi(|y-y_1|) \\
&\quad + \Big|x^*\Big(\iint_{[x_1,x]\cdot[0,y]} f((t,s),u(t,s))\,dtds + \iint_{[0,x_1]\cdot[y_1,y]} f((t,s),u(t,s))\,dtds\Big)\Big| \\
&\le \|g_1(x) - g_1(x_1)\| + \|g_2(y) - g_2(y_1)\| \\
&\quad + (1+S)\,\phi(|x-x_1|) + (1+S)\,\phi(|y-y_1|) \\
&\quad + \iint_{[x_1,x]\cdot[0,y]} |x^*f((t,s),u(t,s))|\,dtds + \iint_{[0,x_1]\cdot[y_1,y]} |x^*f((t,s),u(t,s))|\,dtds \\
&\le \|g_1(x) - g_1(x_1)\| + \|g_2(y) - g_2(y_1)\| + (1+S)\,\phi(|x-x_1|) \\
&\quad + (1+S)\,\phi(|y-y_1|) + Ca_2\,|x-x_1| + Ca_1\,|y-y_1|,
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\|Fu(x,y) - Fu(x_1,y_1)\| \\
&\quad \le \|g_1(x) - g_1(x_1)\| + \|g_2(y) - g_2(y_1)\| + (1+S)\,\phi(|x-x_1|) \\
&\quad\quad + (1+S)\,\phi(|y-y_1|) + Ca_2\,|x-x_1| + Ca_1\,|y-y_1|.
\end{aligned}
$$

By the weak-weak continuity of $h_i$ and from the Lebesgue theorem for the Pettis integral, for each weakly convergent sequence $(u_n)$, $u_n \in X_S$, $u_n \xrightarrow{\omega} u \in X_S$ we get, for each $x^* \in E^*$,

$$
\begin{aligned}
&|x^*(Fu_n(x,y) - Fu(x,y))| \\
&\quad = |x^*(h_1(x,u_n) - h_1(u))| + |x^*(h_2(y,u_n) - h_2(y,u))| \\
&\quad\quad + \iint_{[0,x]\cdot[0,y]} |x^*(f((t,s),u_n(t,s)) - f((t,s),u(t,s)))|\,dtds,
\end{aligned}
$$

which tends to zero as $n \to \infty$.

We are now in a position to show that the operator $F$ is a contraction with respect to some measure of weak noncompactness. Let $V \subset X_S$, $(x,y) \in \Delta$. We partition the intervals $[0,x]$, $[0,y]$ with points $0 = x_0 < x_1 < \ldots < x_m = x$, $0 = y_0 < y_1 < \ldots < y_n = y$, in the following manner: $x_i = \frac{i \cdot x}{m}, y_j = \frac{j \cdot y}{n}$ for $i = 0, 1, \ldots, m$, $j = 1, 2, \ldots, n$. Define $P_{ij} = \{(t,s) : x_{i-1} \le t \le x_i,\ y_{j-1} \le s \le y_j\}$ and $V(P_{ij}) = \{u(t,s) : u \in V,\ (t,s) \in P_{ij}\}$.

Let $A_v$ be the set where $v$ is bounded. Then $\mu(\Delta \setminus A_v) = 0$ and $\sup_A v(x,y) < \infty$. From the absolute convergence of the Pettis integral it follows that for each $\varepsilon > 0$ there exists $\delta$ such that for each $A \subset I$ with $\mu(A) < \frac{1}{2}\delta$, we have

$$\omega\Big(\iint_A f((t,s), V(t,s))\, dtds\Big) < \varepsilon.$$

Since $v_i$ is measurable, by the Luzin theorem for each $\delta > 0$ there exist closed sets $A_{v_i}$ such that $\mu(\Delta \setminus A_{v_i}) < \frac{1}{2}\delta$ and $v_i$ is continuous on $A_{v_i}$. Set $A_\delta = A_v \cap A_{v_1} \cap A_{v_2}$. Then $\mu(\Delta \setminus A_\delta) < \delta$ and there exist $(t_i, s_i) \in P_{ij} \cap A_\delta$ such that

$$v(t_i,s_i)\, v_1(t_i)\, v_2(s_i)\, \omega(V(t_i,s_i)) = \sup_{(t,s) \in P_{ij} \cap A_\delta} v(t,s)\, v_1(t)\, v_2(s)\, \omega(V(t,s)).$$

From the mean value theorem for the Pettis integral for each $w \in V$ we have

$$\begin{aligned}\iint_{([0,x]\cdot[0,y]) \cap A_\delta} f((t,s), w(t,s))\, dtds &= \iint_{([0,x]\cdot[0,y]) \cap A_\delta} f((t,s), w(t,s))\, dtds \\ &= \sum_{i=0}^{m} \sum_{j=0}^{n} \iint_{P_{ij} \cap A_\delta} f((t,s), w(t,s))\, dtds \\ &\subset \sum_{i=0}^{m} \sum_{j=0}^{n} (x_{i+1} - x_i)(y_{j+1} - y_j)\, \overline{conv}\{f(P_{ij} \cap A \cdot V(P_{ij} \cap A))\}.\end{aligned}$$

By the properties of the measure of weak noncompactness we have

$$\begin{aligned}\omega(F(V)(x,y)) &= \omega\Big(g_1(x) - g_1(0) + g_2(y) - h_1(x,V) - h_2(y,V) \\ &\qquad + \int_0^x\Big(\int_0^y f((t,s), V(t,s))\, dt\Big)ds\Big) \\ &\le \omega(h_1(x,V)) + \omega(h_2(y,V)) + \omega\Big(\iint_{[0,x]\cdot[0,y] \cap A_\delta} f((t,s), V(t,s))\, dtds\Big) + \varepsilon\end{aligned}$$

$$\begin{aligned}
&\leq \left(C^1 + C^2\right) \cdot \omega\left(V\right) \\
&\qquad + \omega\Big(\sum_{i=0}^{m}\sum_{j=0}^{n}(x_{i+1} - x_i)\left(y_{j+1} - y_j\right)\overline{conv} f\left(P_{ij} \cap A_\delta \cdot V\left(P_{ij} \cap A_\delta\right)\right)\Big) + \varepsilon \\
&\leq \left(C^1 + C^2\right) \cdot \omega\left(V\right) \\
&\qquad + \sum_{i=0}^{m}\sum_{j=0}^{n}(x_{i+1} - x_i)\left(y_{j+1} - y_j\right)\omega\left(\overline{conv} f\left(P_{ij} \cap A_\delta \cdot V\left(P_{ij} \cap A_\delta\right)\right)\right) + \varepsilon \\
&= \left(C^1 + C^2\right) \cdot \omega\left(V\right) \\
&\qquad + \sum_{i=0}^{m}\sum_{j=0}^{n}(x_{i+1} - x_i)\left(y_{j+1} - y_j\right)\omega\left(f\left(P_{ij} \cap A_\delta \cdot V\left(P_{ij} \cap A_\delta\right)\right)\right) + \varepsilon \\
&= \left(C^1 + C^2\right) \cdot \omega\left(V\right) \\
&\qquad + \sum_{i=0}^{m}\sum_{j=0}^{n}(x_{i+1} - x_i)\left(y_{j+1} - y_j\right)\omega\left(f\left(\{(t_i, s_j)\} \cdot V\left(t_i, s_j\right)\right)\right) + \varepsilon \\
&\leq \left(C^1 + C^2\right) \cdot \omega\left(V\right) \\
&\qquad + \sum_{i=0}^{m}\sum_{j=0}^{n}(x_{i+1} - x_i)\left(y_{j+1} - y_j\right)v\left(t_i, s_i\right)v_1\left(t_i\right)v_2\left(s_i\right)\omega\left(V\left(t_i, s_j\right)\right) + \varepsilon,
\end{aligned}$$

and so

$$\omega(F(V)(x,y)) \leq (C^1 + C^2) \cdot \omega(V) + \int_0^x\int_0^y v(t,s)v_1(t)v_2(s)\omega(V(t,s))dtds + \varepsilon.$$

Notice that the right side of the above inequality does not depend on the choice of $\delta$. Since $\varepsilon$ can be arbitrarily small, we have

$$\begin{aligned}
&\omega(F(V)(x,y)) \\
&\qquad \leq \left(C^1 + C^2\right) \cdot \omega\left(V\right) + \int_0^x\int_0^y \|v\|_\infty v_1\left(t\right) \cdot v_2\left(s\right)\omega\left(V\left(t,s\right)\right)dtds.
\end{aligned}$$

Define a new function

$$\varphi(V) = \sup_{(t,s)\in\Delta}\Big\{\omega(V(t,s))\exp\Big(-r\,\|v\|_\infty^{1/2}\Big(\int_0^t v_1(p)dp + \int_0^s v_2(p)dp\Big)\Big)\Big\},$$

where

$$r > \sqrt{\frac{1}{1-(C^1+C^2)}}.$$

It can be easily proved that $\varphi$ is an axiomatic measure of weak noncompactness (cf. [3]). Moreover, by Lemmas 6 and 5 we obtain

$$\begin{aligned}
\omega\left(F(V)(x,y)\right) &\leq \left(C^1+C^2\right)\cdot\omega\left(V\right)+\int_0^x\int_0^y \|v\|_\infty v_1\left(t\right)v_2\left(s\right)\omega\left(V\left(t,s\right)\right) \\
&\quad\cdot\exp\Big(-r\sqrt{\|v\|_\infty}\Big(\int_0^t v_1(p)dp+\int_0^s v_2(p)dp\Big)\Big) \\
&\quad\cdot\exp\Big(r\sqrt{\|v\|_\infty}\Big(\int_0^t v_1(p)dp+\int_0^s v_2(p)dp\Big)\Big)dtds \\
&\leq \left(C^1+C^2\right)\cdot\omega\left(V\right)+\varphi(V)\int_0^x\int_0^y \|v\|_\infty v_1\left(t\right)v_2\left(s\right) \\
&\quad\cdot\exp\Big(r\sqrt{\|v\|_\infty}\Big(\int_0^t v_1(p)dp+\int_0^s v_2(p)dp\Big)\Big)dtds \\
&\leq \left(C^1+C^2\right)\cdot\omega\left(V\right)+\varphi(V)\frac{1}{r^2}\int_0^x d\Big(\exp\Big(r\sqrt{\|v\|_\infty}\int_0^t v_1\left(p\right)dp\Big)\Big) \\
&\quad\cdot\int_0^y d\Big(\exp\Big(r\sqrt{\|v\|_\infty}\int_0^s v_2\left(p\right)dp\Big)\Big) \\
&= \left(C^1+C^2\right)\cdot\omega\left(V\right)+\varphi(V)\frac{1}{r^2}\exp\Big(r\sqrt{\|v\|_\infty}\Big(\int_0^x v_1\left(p\right)dp+\int_0^y v_2\left(p\right)dp\Big)\Big).
\end{aligned}$$

Thus

$$\begin{aligned}
&\omega\left(F(V)(t,s)\right)\cdot\exp\Big(-r\sqrt{\|v\|_\infty}\Big(\int_0^t v_1\left(p\right)dt_1+\int_0^s v_2\left(p\right)ds_1\Big)\Big) \\
&\leq \left(C^1+C^2\right)\cdot\omega(B)\cdot\exp\Big(-r\sqrt{\|v\|_\infty}\Big(\int_0^t v_1\left(p\right)dp+\int_0^s v_2(p)dp\Big)\Big)+\frac{1}{r^2}\cdot\varphi(B)
\end{aligned}$$

and

$$\varphi\left(F(B)\right)\leq\left(C^1+C^2+\frac{1}{r^2}\right)\varphi(B).$$

By the assumption (H3) we get

$$C^1+C^2+\frac{1}{r^2}<1.$$

Finally, from Theorem 9 it follows that the function $F$ has a fixed point.

The proof of the theorem is complete. ∎

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