DEMONSTRATIO MATHEMATICA
Vol. XLII No 3 2009

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SECOND ORDER EVOLUTION PROBLEM WITH DEPENDENT ON t AND NOT DENSELY DEFINED OPERATORS

Abstract. The purpose of this paper is to present some theorems on existence and uniqueness of solution for nonautonomous second order Cauchy problem with a dumping operator and with dependent on t not densely defined operators.

1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space and let $(A(t))_{t\in[0,T]}$, $(B(t))_{t\in[0,T]}$ be two families of linear not densely defined closed operators from X to X with domains dependent on t. We endow the space $\mathcal{C}(X)$ of closed linear operators $A : X \to X$ with the topology of generalized convergence (see [5], Ch. IV). The domain of a given operator $A : X \to Y$ is denoted by $\mathcal{D}(A)$. The space of bounded linear operators $A : X \to X$ is denoted by $\mathcal{B}(X)$, and $\mathrm{Aut}(X)$ is the subspace of $\mathcal{B}(X)$ of bijective linear bounded operators with bounded inverses.

Definition 1. Let $B : X \to X$ be a linear operator. An operator $A : X \to X$ is called B-bounded if $\mathcal{D}(B) \subset \mathcal{D}(A)$ and if there exist $a, b \in \mathbb{R}_+$ such that

$$\|Ax\| \leq a\|Bx\| + b\|x\| \quad \text{for } x \in \mathcal{D}(B).$$

We consider the following second order evolution problem

$$\frac{d^2u}{dt^2} = B(t)\frac{du}{dt} + A(t)u + f\left(t, u, \frac{du}{dt}\right), \quad t \in [0,T], \tag{1}$$

$$\begin{cases} u(0) = u_0, \\ \frac{du}{dt}(0) = u_1, \end{cases} \qquad u_0,\ u_1 \in X, \tag{2}$$

where $f : [0,T] \times X \times X \to X$ is a given function.

2000 *Mathematics Subject Classification*: 34G10, 34A12.

The problem of the form (1)–(2) arise in mathematical physics. The study of the cases in which operators A and B are independent of t and are densely defined can be found in [2]. In the paper [11] this problem has studied with reference to two different cases:

(a) the operators $A(t) = A$ and $B(t) = B$ are independent of t, $\mathcal{D}(B) \subseteq \mathcal{D}(A)$ and $\overline{\mathcal{D}(A)} \neq X$,
(b) $A(t)$ and $B(t)$ depend on t, the domain $\mathcal{D}(B(t)) = \mathcal{D}_B$ is independent of t, $\overline{\mathcal{D}_B} = X$ and $\mathcal{D}_B \subseteq \mathcal{D}(A(t))$ for all $t \in [0,T]$.

The present paper expands on the study in [11] to the case when:

1) operators A and B are dependent on t,
2) domains $\mathcal{D}(A(t))$, $\mathcal{D}(B(t))$ of operators $A(t)$ and $B(t)$, respectively are dependent on t and are not dense in space X.

In such a case it cannot be expected that the classical solution exists. The main goal of this paper is to present a construction of a new problem in an adequate space in which previously known theorems can be used.

Definition 2. ([2, Def. 3.1, p. 368]) A function u is said to be a classical solution of problem (1)–(2) if

(i) $u \in C^2([0,T],X)$,
(ii) $u(t) \in \mathcal{D}_t^A$ for $t \in [0,T]$ and the mapping $[0,T] \ni t \mapsto A(t)u(t) \in X$ is continuous,
(iii) $u'(t) \in \mathcal{D}_t^B$ for $t \in [0,T]$ and the mapping $[0,T] \ni t \mapsto B(t)u'(t) \in X$ is continuous,
(iv) u satisfies (1)–(2).

We note that if $\mathcal{D}_t^A$ and $\mathcal{D}_t^B$ depend on t problem (1)–(2) has usually no classical solutions. To define a generalized solution we will construct an extended problem in which domains are independent of t.

2. Construction of extended problem

Since now, for a given two families $A(t)$, $B(t) : X \to X$ of linear operators, we make the following assumptions $(\mathbf{Z}_1)$–$(\mathbf{Z}_6)$:

$(\mathbf{Z}_1)$ domains $\mathcal{D}(B(t)) = \mathcal{D}_t^B$ and $\mathcal{D}(A(t)) = \mathcal{D}_t^A$ are dependent on t and $\mathcal{D}_t^B \subseteq \mathcal{D}_t^A$ for all $t \in [0,T]$,
$(\mathbf{Z}_2)$ subspaces $Y_B := \overline{\mathcal{D}_t^B}$ and $Y_A := \overline{\mathcal{D}_t^A}$ are independent of $t \in [0,T]$,
$(\mathbf{Z}_3)$ the resolvent sets of $A(t)$ and $B(t)$ are independent of t, i.e.

$$\varrho(A(t)) = \varrho(A), \quad \varrho(B(t)) = \varrho(B)$$

and $0 \in \varrho(A) \cap \varrho(B)$,

($\mathbf{Z}_4$) operators $A^{-1}(t)B(0)$, $B^{-1}(0)A(t)$ corresponding to $t \in [0,T]$ are bounded and the mappings

(3) $$[0,T] \ni t \to \overline{B^{-1}(0)A(t)}x \in X \qquad \text{for } x \in Y_A,$$

(4) $$[0,T] \ni t \to B^{-1}(t)x \in X \quad \text{for } x \in X$$

are of class $\mathcal{C}^1$,

($\mathbf{Z}_5$) $(B(t))_{t\in[0,T]}$ is a stable family, i.e. there exists $M \geq 1$ and $\omega \in \mathbb{R}$ such that

(i) $(\omega, +\infty) \subset \varrho(B(t))$ for $t \in [0,T]$,

(ii) $$\left\| \prod_{j=1}^{k} \mathcal{R}(\lambda, B(t_j)) \right\| \leq \frac{M}{(\lambda - \omega)^k} \quad \text{for } \lambda > \omega \text{ and}$$

for any finite sequence $0 \leq t_1 \leq \ldots \leq t_k \leq T$,

($\mathbf{Z}_6$) for each $t, s \in [0,T]$ the operator $B^{-1}(t)B(s)$ is closable and the mapping

$$[0,T] \ni t \mapsto \overline{B^{-1}(t)B(s)} \in \mathcal{C}(Y_B), \quad \text{for } s \in [0,T]$$

is continuous.

Remark 1. By the Banach–Steinhaus theorem, it follows from ($\mathbf{Z}_4$) that operators $\overline{B^{-1}(0)A(t)}$, $B^{-1}(t)$ corresponding to $t \in [0,T]$ are uniformly bounded.

Remark 2. By ([12], Theorem 7 and [5], Ch. 4), it follows from ($\mathbf{Z}_3$) and ($\mathbf{Z}_6$) that

(a) $\overline{B^{-1}(t)B(s)} \in \operatorname{Aut}(Y_B)$, for $t, s \in [0,T]$,

(b) for each $s \in [0,T]$ the mappings

(5) $$[0,T] \ni t \mapsto \overline{B^{-1}(t)B(s)} \in \mathcal{B}(Y_B),$$

(6) $$[0,T] \ni t \mapsto \overline{B^{-1}(s)B(t)} \in \mathcal{B}(Y_B)$$

are continuous,

(c) the norms

(7) $$|\cdot|_t : X \ni x \to |x|_t := \left\|B^{-1}(t)x\right\| \leq M \left\|x\right\|$$

corresponding to $t \in [0,T]$ are equivalent (see [12])[1],

(d) there exists $K > 0$ such that

(8) $$|x|_t = \left\|B^{-1}(t)x\right\| = \left\|B^{-1}(t)B(0)B^{-1}(0)x\right\| \leq K\, |x|_0 .$$

[1] Existence of a constant M follows from the Banach–Steinhaus theorem, because of continuity of the mapping (4).

The completion X^B_{-1} of the space $(X, |\cdot|_0)$ to a Banach space is called the extrapolation space for $B(0)$. Since norms (7) are equivalent, the space X^B_{-1} is the extrapolation space for $B(t)$ for all $t \in [0,T]$.

Definition 3. ([7, p. 44]) A linear operator A is called a Hille-Yosida operator (of type (M,ω)) if there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$ and

$$\|R(\lambda, A))^n\| \leq \frac{M}{(\lambda-\omega)^n}, \quad \lambda > \omega, \ n = 1, 2, \ldots.$$

By the Hille-Yosida theorem, A generates a $\mathcal{C}_0$– semigroup if and only if it is a densely defined operator.

Remark 3. Since, by $(\mathbf{Z}_5)$, $B(0)$ is a Hille-Yosida operator, the space Y_B (defined in $(\mathbf{Z}_2)$) is dense in X^B_{-1} (cf. [7], Theorem 3.1.10).

Lemma 1. *For each $t \in [0,T]$ the mapping*

$$B(t) : X \supset \mathcal{D}^B_t \to X \subset X^B_{-1} \tag{9}$$

is an isomorphism of normed spaces.

Proof. Since, for each $t \in [0,T]$,

$$|B(t)x|_0 = \|B^{-1}(0)B(t)x\| \leq C\|x\| \quad \text{for } x \in \mathcal{D}^B_t,$$

the operator (9) is bounded. The inverse operator $B^{-1}(t) : X^B_{-1} \supset X \to X$ is bounded, because, by definition, $\|B^{-1}(0)x\| = |x|_0$ for $x \in \mathcal{D}^B_t \subset X$ and all norms (7) are equivalent. ∎

Now, by Lemma 1, $B(t)$ can be uniquely extended to Y_B and we can define

$$B_{-1}(t) : X^B_{-1} \supset Y_B \to X^B_{-1}$$

to be the extension of $B(t)$ onto Y_B considered as a subspace of X^B_{-1}. We can also uniquely extend $B^{-1}(t)$ to X^B_{-1} and define

$$B^{-1}_{-1}(t) : X^B_{-1} \to X$$

to be the extension of $B^{-1}(t)$ onto X^B_{-1}. The extension of the norm $|\cdot|_0$ to X^B_{-1} is given by

$$|x|_0 = \left\|B^{-1}_{-1}(0)x\right\| \quad \text{for } x \in X^B_{-1}. \tag{10}$$

Since operators $B_{-1}(t)B^{-1}_{-1}(t)$ and $B^{-1}_{-1}(t)B_{-1}(t)$ are bounded and

$$B_{-1}(t)B^{-1}_{-1}(t)x = x \quad \text{for } x \in X \subset X^B_{-1}$$

and

$$B^{-1}_{-1}(t)B_{-1}(t)x = x \quad \text{for } x \in \mathcal{D}^B_t \subset Y_B,$$

the operator $B_{-1}(t)$ is invertible and

$$(11)\qquad (B_{-1}(t))^{-1} = B_{-1}^{-1}(t) \quad \text{for } t \in [0,T].$$

Since

$$\left|B_{-1}^{-1}(t)x\right|_0 \leq M \left\|B_{-1}^{-1}(t)x\right\| = M\,|x|_t\,,$$

the operator $B_{-1}^{-1}(t) : X_{-1}^B \to X_{-1}^B$ is bounded, and so $B_{-1}(t)$ is closed. Thus, in this way we have obtained a family $(B_{-1}(t))_{t\in[0,T]}$ of closed densely defined linear operators with the same domain $Y_B \subset X_{-1}^B$ for all $t \in [0,T]$.

Note that, by $(\mathbf{Z}_4)$ and $(\mathbf{Z}_6)$ (see also Remark 2), for each $t \in [0,T]$ the operator

$$(12)\qquad A(t) : X \supset \mathcal{D}_t^A \to X_{-1}^B$$

is bounded. Indeed, for $x \in \mathcal{D}_t^A$ we have (by Banach–Steinhaus theorem)

$$(13)\qquad |A(t)x|_0 = \left\|B^{-1}(0)A(t)x\right\| \leq C\|x\|\,.$$

Hence the closure $\overline{A(t)}$ of $A(t)$ (considered as in (12)) is well-defined on $Y_A = \overline{\mathcal{D}_t^A}$. Thus we can define

$$A_{-1}(t) : X_{-1}^B \supset Y_A \to X_{-1}^B$$

to be $\overline{A(t)}$ with the domain Y_A considered as a subspace of X_{-1}^B.

Lemma 2. *The family $(A_{-1}(t))_{t\in[0,T]}$ is a family of densely defined closed linear operators with the same domain Y_A dense in X_{-1}^B.*

Proof. Since $Y_B \subset Y_A$ and Y_B is dense in X_{-1}^B (see Remark 3), Y_A is a dense subspace of X_{-1}^B.

By assumption $(\mathbf{Z}_4)$ and Lemma 1 the mappings

$$(14)\qquad B^{-1}(0)A(t) : X \supset \mathcal{D}_t^A \to \mathcal{D}_0^B \subset X,$$

$$(15)\qquad B(0) : X \supset \mathcal{D}_0^B \to X \subset X_{-1}^B$$

are isomorphisms of a normed spaces. Then

$$(16)\qquad A(t) = B(0)(B^{-1}(0)A(t)) : X \supset \mathcal{D}_t^A \to X \subset X_{-1}^B$$

as a composition of two isomorphisms is an isomorphism too. Since the extensions of bounded operators with dense domains are unique, $A_{-1}(t) : X \supset Y_A \to X_{-1}^B$ is also an isomorphism. Hence and by (7)

$$\left|A_{-1}^{-1}(t)x\right|_0 \leq M \left\|A_{-1}^{-1}(t)x\right\| \leq MC(t)\,|x|_0\,.$$

Thus the inverse to $A_{-1}(t)$ is bounded. Thus $A_{-1}(t)$ is closed. ∎

Lemma 3. *The operators $A_{-1}(t)$, $t \in [0,T]$, are uniformly $B_{-1}(t)$ bounded in $[0,T]$.*

Proof. Using (8) we have

$$|A(t)x|_0 = \left\|B^{-1}(0)A(t)x\right\| \leq C\,\|x\| \leq CK\,|B(t)x|_0 \quad \text{for } x \in \mathcal{D}_t^B.$$

Hence $|A_{-1}(t)x|_0 \le a\,|B_{-1}(t)x|_0$ for $x \in Y_B$, because operators (9) and (12) are bounded. ∎

Now problem (1)–(2) comes down to the following problem

$$\frac{d^2u}{dt^2} = B_{-1}(t)\frac{du}{dt} + A_{-1}(t)u + \tilde{f}\left(t, u, \frac{du}{dt}\right), \quad t \in [0,T], \tag{17}$$

$$\begin{cases} u(0) = u_0, \\ \dfrac{du}{dt}(0) = u_1, \end{cases} \qquad u_0, u_1 \in X_{-1}^B, \tag{18}$$

where $\tilde{f} : [0,T] \times X_{-1}^B \times X_{-1}^B \to X_{-1}^B$ is an extension of f and the families $(A_{-1}(t))_{t\in[0,T])}$ and $(B_{-1}(t))_{t\in[0,T])}$ have "better properties":

- (i) for each $t \in [0,T]$ the domain $\mathcal{D}(B_{-1}(t)) = Y_B$ is the same dense subspace of X_{-1}^B,
- (ii) for each $t \in [0,T]$ the domain $\mathcal{D}(B_{-1}(t)) = Y_B \subset Y_A$,
- (iii) for each $t \in [0,T]$ the operator $A_{-1}(t)$ is $B_{-1}(t)$ bounded in the space X_{-1}^B,
- (iv) for each $t \in [0,T]$ the operator $B_{-1}(t)$ is a generator of a C_0 semigroup (cf. [7], Theorem 3.1.11),
- (v) the family $(B_{-1}(t))_{t\in[0,T]}$ is a stable family of generators of C_0 semigroups (cf. [9], Theorem 5).

Definition 4. A function u is said to be a generalized solution of problem (1)–(2) if it is a classical solution of problem (17)–(18).

Lemma 4. *For any $x \in Y_A$, the mapping*

$$[0,T] \ni t \mapsto A_{-1}(t)x \in X_{-1}^B \tag{19}$$

is of class C^1.

Proof. Let $x \in Y_A$ and let $H : [0,T] \mapsto Y_A \subset X$ be the derivative of the mapping (3). Since $B_{-1}^{-1}(0)A_{-1}(t) = \overline{B^{-1}(0)A(t)}$ and

$$\left|\frac{A_{-1}(t+h)x - A_{-1}(t)x}{h} - B_{-1}(0)H(t)\right|_0 = \left\|\frac{B_{-1}^{-1}(0)A_{-1}(t+h)x - B_{-1}^{-1}(0)A_{-1}(t)x}{h} - H(t)\right\|,$$

$B_{-1}(0)H$ is the derivative of the mapping (19). Thus, the lemma is proved, because $B_{-1}(0)$ as an operator from Y_B to X_{-1}^B is bounded and because of the assumption $(\mathbf{Z}_4)$. ∎

Theorem 1. *If X is a reflexive space, $u_0 \in Y_A$, $u_1 \in Y$, assumptions $(\mathbf{Z}_1)$–$(\mathbf{Z}_6)$ hold and $f : [0,T] \times X^B_{-1} \times X^B_{-1} \ni (t,x,y) \mapsto X$ satisfies the Lipschitz condition i.e. there exist $L > 0$ such that*

$$\|f(t_1,x_1,y_1) - f(t_2,x_2,y_2)\| \le L(|t_1 - t_2| + |x_1 - x_2|_0 + |y_1 - y_2|_0) \quad \text{for } t_1,t_2 \in [0,T],\ x_1,x_2,y_1,y_2 \in X^B_{-1},$$

then there exists exactly one classical solution in X^B_{-1} of problem (17)–(18) and the solution is of class $C^1([0,T],X)$ which mean that problem (1)–(2) has exactly one generalized solution.

Proof. Since

$$|f(t_1,x_1,y_1) - f(t_2,x_2,y_2)|_0 = \left\|B^{-1}(0)(f(t_1,x_1,y_1) - f(t_2,x_2,y_2))\right\| \le C\,\|f(t_1,x_1,y_1) - f(t_2,x_2,y_2)\|,$$

f as a mapping from $[0,T] \times X^B_{-1} \times X^B_{-1}$ into X^B_{-1} satisfies the Lipschitz condition. Thus, by Theorem 4 from [11], there exists exactly one classical solution of problem (17)–(18). By definition, if u is a solution then

(i) $u \in C^2([0,T], X^B_{-1})$,

(ii) $u(t) \in \mathcal{D}(A_{-1}(t))$ for $t \in [0,T]$ and the mapping

(20) $$[0,T] \ni t \mapsto A_{-1}(t)u(t) \in X^B_{-1} \quad \text{is continuous,}$$

(iii) $u'(t) \in \mathcal{D}(B_{-1}(t))$ for $t \in [0,T]$ and the mapping

(21) $$[0,T] \ni t \mapsto B_{-1}(t)u'(t) \in X^B_{-1} \quad \text{is continuous,}$$

(iv) u satisfies conditions (17), (18).

But

(22) $$u(t) = u(0) + \int_0^t u'(s)ds.$$

Hence $u(0) \in Y$, $u'(t) \in Y$ and because of (21) the mapping

$$[0,T] \ni t \mapsto B_{-1}(0)u'(t) \in Y \text{ is continuous.}$$

Thus

$$B_{-1}(0)\int_0^t u'(s)ds = \int_0^t B_{-1}(0)u'(s)ds \in Y.$$

Since, by (22),

$$B_{-1}(0)u(t) = B_{-1}(0)u(0) + \int_0^t B_{-1}(0)u'(s)ds,$$

it follows that $(B_{-1}(0)u(t))' = B_{-1}(0)u'(t)$ and so

$$\left\| \frac{u(t+h)-u(t)}{h} - u'(t) \right\| = \left| B_{-1}(0) \left(\frac{u(t+h)-u(t)}{h} - u'(t) \right) \right|_0$$
$$= \left| \frac{B_{-1}(0)u(t+h) - B_{-1}(0)u(t)}{h} - B_{-1}(0)u'(t) \right|_0 \underset{h \to 0}{\longrightarrow} 0.$$

Since, by (21),

$$\left| B_{-1}(t)u'(t) - B_{-1}(t_0)u'(t_0) \right|_0 = \left\| B_{-1}^{-1}(0)[B_{-1}(t)u'(t) - B_{-1}(t_0)u'(t_0)] \right\|$$
$$= \left\| B_{-1}^{-1}(0)B_{-1}(t)[u'(t) - u'(t_0)] + B_{-1}^{-1}(0)[B_{-1}(t) - B_{-1}(t_0)]u'(t_0) \right\| \underset{h \to 0}{\longrightarrow} 0$$

and $\left\| B_{-1}^{-1}(0)B_{-1}(t) \right\|$ is bounded, it follows from ($\mathbf{Z}_4$) that

$$\left\| u'(t) - u'(t_0) \right\| \underset{t \to t_0}{\longrightarrow} 0 .$$

We have, thus, proved that $u \in C^1([0,T], X)$. ∎

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Received October 28, 2008; revised version December 14, 2008.