DEMONSTRATIO MATHEMATICA
Vol. XLII No 3 2009

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NEW PROPERTIES OF SOME FAMILIES OF HOLOMORPHIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES

Abstract. The paper concerns holomorphic functions in complete bounded n-circular domains of the space $\mathbb{C}^n$ and presents some properties of the above mentioned functions belonging to the families described by some geometrical or analytical conditions. This subject has been considered by many mathematicans, for example I.I. Bavrin, K. Dobrowolska, I. Dziubiński, S. Fukui, Z.J. Jakubowski, J. Kamiński, A. Marchlewska, Y. Michiwaki, J. A. Pfaltzgraff, R. Sitarski, T. J. Suffridge, J. Stankiewicz, I. Weinberg, A. Wrzesień, Ł. Żywień and the authors.

1. Introduction

A domain $G \subset \mathbb{C}^n$, $n \geq 2$, containing the origin is called complete n-circular, if $z\Lambda = (z_1\lambda_1, \dots, z_n\lambda_n) \in G$ for each $z = (z_1, \dots, z_n) \in G$ and every $\Lambda = (\lambda_1, \dots, \lambda_n) \in \overline{E^n}$, where E is the disc $\{\zeta \in \mathbb{C}\colon |\zeta| < 1\}$. Polidiscs

$$E^n(r_1, \dots, r_n) = \{z \in \mathbb{C}^n\colon |z_j| < r_j, j = 1, \dots, n\}$$

with center at the origin and radii $r_1 > 0, \dots, r_n > 0$ are good examples of such domains.

Another example is provided by the domains

$$A(n; \delta) = \Big\{z \in \mathbb{C}^n\colon \sum_{j=1}^{n} |z_j|^{\frac{1}{\delta}} < 1\Big\}, \quad \delta > 0.$$

The latter are generalizations of the cone $\{z \in \mathbb{C}^n\colon \sum_{j=1}^n |z_j| < 1\}$ and the Euclidean ball $\{z \in \mathbb{C}^n\colon \sum_{j=1}^n |z_j|^2 < 1\}$. Sometimes complete n-circular domains are called complete Reinhardt domains. In the paper we assume that G is a bounded complete n-circular domain.

Key words and phrases: n-circular domain, holomorphic function, Minkowski function, Bavrin's families, growth and distortion theorem.
2000 *Mathematics Subject Classification*: 32A30, 30C45.

Let us consider the Minkowski function $\mu_G\colon \mathbb{C}^n \to [0,\infty)$

$$\mu_G(z) = inf\{t > 0\colon \frac{1}{t}z \in G\}, \quad z \in \mathbb{C}^n.$$

We shall use the continuity of μ_G and the following facts as well:

(i) $G = \{z \in \mathbb{C}^n\colon \mu_G(z) < 1\}$,

(ii) $\partial G = \{z \in \mathbb{C}^n : \mu_G(z) = 1\}$.

Example 1. The Minkowski function for $E^n(r_1,\ldots,r_n)$, $r_1 > 0,\ldots,r_n > 0$ and $A(n;\delta)$, $\delta > 0$, can be expressed as

$$\mu_{E^n(r_1,\ldots,r_n)}(z) = max\left\{\frac{|z_1|}{r_1},\ldots,\frac{|z_n|}{r_n}\right\}, \quad z \in \mathbb{C}^n,$$

$$\mu_{A(n;\delta)}(z) = \left[\sum_{j=1}^{n}|z_j|^{\frac{1}{\delta}}\right]^{\delta}, \quad z \in \mathbb{C}^n.$$

Let $\mathcal{H}_G$ be the vector space of holomorphic functions $f\colon G \to \mathbb{C}$ and let $\mathcal{L} : \mathcal{H}_G \to \mathcal{H}_G$ be the Temljakov linear operator [22], which is defined by

$$\mathcal{L}f(z) = f(z) + Df(z)(z), \quad z \in G, \tag{1}$$

where $Df(z)$ is Frechet's derivative of f at the point z.
It is also known (see [22]) that the inverse $\mathcal{L}^{-1}$ of the Temljakov operator has the following form

$$(\mathcal{L}^{-1}f)(z) = \int_0^1 f(tz)dt, \quad z \in G. \tag{2}$$

I.I. Bavrin [1], [2] considered the subclasses $\mathcal{M}_G$ and $\mathcal{N}_G$ of the class $\mathcal{H}_G(1) = \{f \in \mathcal{H}_G : f(0) = 1\}$. We say that $f \in \mathcal{H}_G(1)$ belongs to $\mathcal{M}_G$ if $f(z) \neq 0$ for $z \in G$ and

$$Re\frac{\mathcal{L}f(z)}{f(z)} > 0, \quad z \in G. \tag{3}$$

Similarly, $f \in \mathcal{H}_G(1)$ belongs to $\mathcal{N}_G$ if $\mathcal{L}f(z) \neq 0$ for $z \in G$ and

$$Re\frac{\mathcal{L}\mathcal{L}f(z)}{\mathcal{L}f(z)} > 0, \quad z \in G. \tag{4}$$

The family $\mathcal{M}_G$ ($\mathcal{N}_G$) corresponds to the class $\mathcal{S}^*$ ($\mathcal{S}^c$) (see [6]) of normalized holomorphic univalent starlike (convex) functions $F : E \to \mathbb{C}$. Let us denote that the class $\mathcal{M}_G$ has been used in research some linear invariant families of locally biholomorphic mapping in $\mathbf{C}^n$ (see [17]).

In case $n = 2$ the classes $\mathcal{M}_G$, $\mathcal{N}_G$ have the following geometrical interpretation (see [1]). A function f belongs to $\mathcal{M}_G$ ($\mathcal{N}_G$) if and only if

(i) the function $z_1 f(z_1, z_2)$ is univalent starlike (convex) in the intersection of the domain G by every analitic plane $z_2 = \alpha z_1$, $\alpha \in \mathbb{C}$. In other words the function $z_1 f(z_1, \alpha z_1)$ of one variable is univalent starlike (convex) in the disc, which is the projection of the intersection $G \cap \{z_2 = \alpha z_1\}$ onto the plane $z_2 = 0$,

(ii) the function $z_2 f(0, z_2)$ is univalent starlike (convex) in the intersection $G \cap \{z_1 = 0\}$.

In his works I.I. Bavrin examined also the families $\mathcal{R}_G$ and $\mathcal{P}_G$ of functions in a way "close" to functions of the families $\mathcal{N}_G$ and $\mathcal{M}_G$, respectively. We say that $f \in \mathcal{H}_G(1)$ belongs to $\mathcal{R}_G$ if there exists a function $\varphi \in \mathcal{N}_G$ such that

$$Re \frac{\mathcal{L}f(z)}{\mathcal{L}\varphi(z)} > 0, \quad z \in G. \tag{5}$$

In turn we say that $f \in \mathcal{H}_G(1)$ belongs to $\mathcal{P}_G$ if there exists a function $\psi \in \mathcal{M}_G$ such that

$$Re \frac{f(z)}{\psi(z)} > 0, \quad z \in G. \tag{6}$$

The family $\mathcal{R}_G$ corresponds to the well-known class $\mathcal{S}^{cc}$ of close-to-convex functions in the unit disc E (see e.g. [4], [10], [12]). In turn, the family $\mathcal{P}_G$ corresponds to the class of close-to-starlike functions in the unit disc E (see e.g. [18]).

Remark 1. ([1]) If $f \in R_G$, then $\mathcal{L}f \in \mathcal{P}_G$ and conversely, if $f \in \mathcal{P}_G$, then $\mathcal{L}^{-1}f \in \mathcal{R}_G$, where $\mathcal{L}$ is the Temljakov operator defined by (1).

2. A new approach of the sharpness of estimations

We will deal with the problem of the precision of some estimates in the families $\mathcal{R}_G$ and $\mathcal{P}_G$. Let $rG = \{rz \colon z \in G\}$, $r \in [0, 1]$.

Let us notice that for a function $f \in \mathcal{P}_G$ the following estimation is valid

$$\frac{1-r}{(1+r)^3} \le |f(z)| \le \frac{1+r}{(1-r)^3}, \quad z \in \overline{rG}, \quad r \in [0,1),$$

which was given by Bavrin for the case $n = 2$. He also showed that this estimate is precise for $G = A(2; \delta)$ and for $G = E^2(r_1, r_2)$. Bavrin also stated that the same estimates hold for $n > 2$, but he provided no information about the precision of the estimates in another domains. We will present the estimates using the Minkowski function μ_G and we will show that for any complete n-circular domain $G \subset \mathbb{C}^n$ the above estimates are sharp. Cleary, it is sufficient to consider the case if $r \in (0,1)$. Let us start from the following observation

(7) $rG = \{z \in \mathbb{C}^n \colon \mu_G(z) < r\}, \ \partial(rG) = \{z \in \mathbb{C}^n \colon \mu_G(z) = r\}, \ r \in (0,1),$

where μ_G is the Minkowski function of G (both equalities follows from the properties (i) and (ii) of the Minkowski function μ_G). It turns out that a generalization of the above Bavrin result holds.

Proposition 1. *Let $n \geq 2$, $G \subset \mathbb{C}^n$ be a bounded complete n-circular domain and*

$$\Delta = \Delta(G) = \max_{z \in \overline{G}} \Big|\sum_{j=1}^{n} z_j\Big|. \tag{8}$$

If $f \in \mathcal{P}_G$, then

$$\frac{1-r}{(1+r)^3} \leq |f(z)| \leq \frac{1+r}{(1-r)^3}, \quad \mu_G(z) \leq r, \quad r \in [0,1). \tag{9}$$

The estimate is precise for every domain G and an extremal function has the form:

$$f(z) = \Delta^2 \frac{\Delta + e^{i\alpha}\sum_{j=1}^{n} z_j}{(\Delta - e^{i\alpha}\sum_{j=1}^{n} z_j)^3}, \quad z = (z_1, \ldots, z_n) \in G, \quad \alpha \in \mathbb{R}. \tag{10}$$

Proof. The inequality (9) follows from the above-mentioned results of Bavrin and from relations (7).

Now we will prove the second part of the theorem. Every function f defined by (10) belongs to the family $\mathcal{P}_G$, because the condition (6) is satisfied by f with the function ψ defined by

$$\psi(z) = \frac{\Delta^2}{(\Delta - e^{i\alpha}\sum_{j=1}^{n} z_j)^2}, \quad z = (z_1, \ldots, z_n) \in G,$$

which belongs to the family $\mathcal{M}_G$. We shall show that for every $r \in [0,1)$ there exists a point $\overset{\circ}{z}$, $\mu_G(\overset{\circ}{z}) = r$ such that function (10) with an appropriate α gives the equality in the upper part of the inequality (9).

It is obvious that $\Delta = max_{z \in \partial G} |\sum_{j=1}^{n} z_j|$, so there exists a point $\overset{*}{z} \in \partial G$ for which $|\sum_{j=1}^{n} \overset{*}{z}_j| = \Delta$. In virtue of the properties of the domain G and the Minkowski function μ_G there exists a point $\overset{\circ}{z} = (\overset{\circ}{z}_1, \ldots, \overset{\circ}{z}_n) \in G \setminus \{0\}$, $\mu_G(\overset{\circ}{z}) = r \in (0,1)$ for such that $\overset{*}{z} = (\mu_G(\overset{\circ}{z}))^{-1} \overset{\circ}{z}$. Hence, we have

$$\Big|\sum_{j=1}^{n} \overset{\circ}{z}_j\Big| = \Delta \mu_G(\overset{\circ}{z}). \tag{11}$$

For this point $\overset{\circ}{z}\in G\setminus\{0\}$ let us choose a function f of the form (10) with $\alpha=\alpha_0\in\mathbb{R}$ such that

$$e^{i\alpha_0}\sum_{j=1}^{n}\overset{\circ}{z}_j=\Big|\sum_{j=1}^{n}\overset{\circ}{z}_j\Big|. \tag{12}$$

Therefore

$$|f(\overset{\circ}{z})|=\frac{1+\mu_G(\overset{\circ}{z})}{(1-\mu_G(\overset{\circ}{z}))^3}=\frac{1+r}{(1-r)^3},$$

which makes the upper estimation (9) sharp for $z\in G$ such that $\mu_G(z)\le r$, i.e. for $z\in\overline{rG}$, $r\in(0,1)$. The equality in the upper bound (9) holds also for $r=0$.

In order to show the precision of the lower estimation in (9) it is sufficient to repeat the above reasoning with the parameter $\alpha=\alpha_0$ chosen in the following way

$$e^{i\alpha_0}\sum_{j=1}^{n}\overset{\circ}{z}_j=-\Big|\sum_{j=1}^{n}\overset{\circ}{z}_j\Big|. \ \blacksquare \tag{13}$$

Remark 2. Let $n=2$. If $G=A(2;\delta)$, then our extremal function (10) is identical with Bavrin's extremal function

$$f(z)=\frac{[2^{1-\delta}+e^{i\alpha}(z_1+z_2)]2^{2(1-\delta)}}{[2^{1-\delta}-e^{i\alpha}(z_1+z_2)]^3},\quad z=(z_1,z_2)\in G,\quad \alpha\in\mathbb{R}.$$

In order to check this fact it is sufficient to choose $\alpha=\alpha_0$ defined by (12) and (13) and observe that

$$\Delta(A(2;\delta))=2^{1-\delta}.$$

However, for $G=E^2(r_1,r_2)$ our extremal function (10) and the Bavrin's extremal function

$$f(z)=4r_1^2r_2^2\frac{2r_1r_2+e^{i\alpha}(r_2z_1+r_1z_2)}{[2r_1r_2-e^{i\alpha}(r_2z_1+r_1z_2)]^3},\quad z=(z_1,z_2)\in G,\quad \alpha\in\mathbb{R}$$

differ, because

$$\Delta(E^2(r_1,r_2))=r_1+r_2$$

and consequently $\Delta(E^2(r_1,r_2))\neq 2r_1r_2$, for $r_1\neq 1$ or $r_2\neq 1$.

Now, we consider the problem of sharpness of estimates of some quantities connected with the expansion of the function $f\in\mathcal{P}_G$ and $f\in\mathcal{R}_G$ into a series of homogeneous polynomials

$$f(z)=1+\sum_{k=1}^{\infty}Q_{f,k}(z),\quad z\in G. \tag{14}$$

In the case $n=2$ Bavrin ([1], §15) obtained the following estimates

$$B_k(G) \le \begin{cases} (k+1)^2 & \text{for } f \in \mathcal{P}_G, \\ k+1 & \text{for } f \in \mathcal{R}_G \end{cases} \tag{15}$$

for $k \in \mathbb{N}$, where

$$B_k(G) = \sup_{(z_1,z_2)\in G} |Q_{f,\,k}(z)| = \sup_{(z_1,z_2)\in G} \Big|\sum_{l=0}^{k} a_{k-l,l} z_1^{k-l} z_2^{l}\Big|.$$

He also proved that these estimates are sharp for domains $G = A(2;\delta)$ and $G = E^2(r_1, r_2)$. Moreover, he stated ([1], §19 [1]) that similar estimates hold for any $n > 2$, but he gave no information about the precision of the estimates in other domains.

We will show that these estimates are sharp for an arbitrary bounded complete n-circular domain $G \subset \mathbb{C}^n$, $n \ge 2$. Moreover, estimates (15) will be presented with the use of the Minkowski function μ_G.

It is known that homogeneous polynomials for $n \ge 2$ have the form

$$Q_{f,\,k}(z) = \sum_{\alpha_1+\cdots+\alpha_n=k} a_{\alpha_1\ldots\alpha_n} z_1^{\alpha_1}\ldots z_n^{\alpha_n}, \quad z=(z_1,\ldots,z_n)\in\mathbb{C}^n, \quad k\in\mathbb{N},$$

where for nonnegative integers α_j, $j=1,\ldots,n$ the coefficients $a_{\alpha_1\ldots\alpha_n}$ are complex numbers.

Now

$$B_k(G) = \sup_{z\in G}|Q_{f,\,k}(z)| = \sup_{z\in G}\Big|\sum_{\alpha_1+\cdots+\alpha_n=k} a_{\alpha_1\ldots\alpha_n} z_1^{\alpha_1}\ldots z_n^{\alpha_n}\Big|, \quad k\in\mathbb{N}.$$

Thus, due to the maximum modulus principle for holomorphic functions and properties of the Minkowski function we have

$$B_k(G) = \sup_{z\in\partial G}|Q_{f,\,k}(z)| = \sup_{\mu_G(z)=1}|Q_{f,\,k}(z)|, \quad k\in\mathbb{N}.$$

Let us denote

$$\mu_G(Q_{f,\,k}) = \sup_{w\in\mathbb{C}^n\setminus\{0\}} \frac{|Q_{f,\,k}(w)|}{(\mu_G(w))^k} = \sup_{\mu_G(v)=1}|Q_{f,\,k}(v)|, \quad k\in\mathbb{N}. \tag{16}$$

Then for $w\in\mathbb{C}^n$ we have [2]

$$|Q_{f,\,k}(w)| \le \mu_G(Q_{f,\,k})(\mu_G(w))^k. \tag{17}$$

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[1] By §19 we denote the Supplement in [1].

[2] Assuming moreover that G is convex, we have that μ_G is a norm $\|\cdot\|$ in $\mathbb{C}^n$ (see e.g. [19], Chapt. 8) and consequently, (17) takes the following form:
$|Q_{f,k}(w)| \le \|Q_{f,k}\|\cdot\|w\|^k$, $w\in\mathbb{C}^n$.

By the estimates (15) and the notation (16) for $k \in \mathbb{N}$, we obtain the following estimates for $f \in \mathcal{P}_G$ and $f \in \mathcal{R}_G$, respectively

$$\mu_G(Q_{f,k}) \leq (k+1)^2, \quad k \in \mathbb{N}, \tag{18}$$

$$\mu_G(Q_{f,k}) \leq k+1, \quad k \in \mathbb{N}. \tag{19}$$

Now, we can formulate the following proposition.

Proposition 2. *Let $G \subset \mathbb{C}^n$, $n \geq 2$ be a bounded complete n-circular domain and let $f \in \mathcal{H}_G(1)$ be a function of the form (14). Then the inequalities (18) and (19) hold for $f \in \mathcal{P}_G$ and $f \in \mathcal{R}_G$, respectively. Moreover, the estimates are sharp.*

Proof. It remains to prove the sharpness of the estimations (18) and (19). Let us consider, for instance, inequality (18). Firstly, observe that the function $f \in \mathcal{P}_G$ of the form (10) gives equality in (18). Indeed, this function on G develops into a series of homogeneous polynomials of the form (14), where

$$Q_{f,k}(z) = (k+1)^2\Big(\frac{e^{i\alpha}}{\Delta}\sum_{j=1}^{n} z_j\Big)^k, \quad z \in G, \quad k \in \mathbb{N}.$$

Moreover,

$$\mu_G(Q_{f,k}) = \sup_{\mu_G(z)=1} |Q_{f,k}(z)| = \sup_{z\in G}\left|(k+1)^2\left(\frac{e^{i\alpha}}{\Delta}\sum_{j=1}^{n} z_j\right)^k\right|$$
$$= (k+1)^2\frac{1}{\Delta^k}\Delta^k = (k+1)^2.$$

Similarly, one can show that the function $f \in \mathcal{R}_G$ of the form

$$f(z) = \frac{\Delta^2}{(\Delta - e^{i\alpha}\sum_{j=1}^{n} z_j)^2}, \quad z = (z_1, \ldots, z_n) \in G \tag{20}$$

gives equality in (19). ∎

We use the estimates (18) in the proof of the following theorem.

Theorem 1. *Let $n \geq 2$ and $G \subset \mathbb{C}^n$ be a bounded complete n-circular domain. For each function $f \in \mathcal{P}_G$ the following precise estimates hold*

$$|\mathcal{L}f(z)| \leq \frac{r^2+4r+1}{(1-r)^4}, \quad \mu_G(z) \leq r, \quad r \in [0,1), \tag{i}$$

$$\left|\frac{\mathcal{L}f(z)}{f(z)}\right| \leq \frac{r^2+4r+1}{1-r^2}, \quad \mu_G(z) \leq r, \quad r \in [0,1). \tag{ii}$$

Proof. Let us take $z \in G$ such that $\mu_G(z) = r \in [0,1)$, then from (17) and (18) we obtain

$$|\mathcal{L}f(z)| = \left|1 + \sum_{k=1}^{\infty}(k+1)Q_{f,\,k}(z)\right| \le 1 + \sum_{k=1}^{\infty}(k+1)^3 r^k = \frac{r^2+4r+1}{(1-r)^4}.$$

This yields claim (i).

Now we will prove the inequality (ii). By the definition of the family $\mathcal{P}_G$ there exist functions $g \in \mathcal{M}_G$ and

$$h \in \mathcal{C}_G = \{h \in \mathcal{H}_G(1) : \mathrm{Re}\, h(z) > 0, z \in G\}$$

such that $f(z) = g(z)h(z)$ for $z \in G$. Since $f \in \mathcal{P}_G$, $g \in \mathcal{M}_G$, $h \in \mathcal{C}_G$, we have $f(z)g(z)h(z) \neq 0$ for $z \in G$ (see definitions of $\mathcal{P}_G$, $\mathcal{M}_G$, $\mathcal{C}_G$) and

$$\frac{\mathcal{L}f(z)}{f(z)} = \frac{\mathcal{L}g(z)}{g(z)} + \frac{Dh(z)(z)}{h(z)}, \quad z \in G.$$

Let us find upper estimates of the modulus for both summands of this sum. For $g \in \mathcal{M}_G$ we have (see [1], §16 and §19)

$$\left|\frac{\mathcal{L}g(z)}{g(z)}\right| \le \frac{1+r}{1-r}, \quad z \in \overline{rG}, \quad r \in [0,1).$$

On the other hand I.I. Bavrin showed ([1], §13 and §19) that

$$|Dh(z)(z)| \le \frac{2r}{1-r^2}\,\mathrm{Re} h(z), \quad z \in \overline{rG}, \quad r \in [0,1). \tag{21}$$

Hence

$$\left|\frac{Dh(z)(z)}{h(z)}\right| \le \frac{2r}{1-r^2}\,\frac{\mathrm{Re} h(z)}{|h(z)|} \le \frac{2r}{1-r^2}, \quad z \in \overline{rG}, \quad r \in [0,1).$$

By combining the estimates of the summands of the above sum we obtain the estimate (ii) by (7).

Now we will show the sharpness of the estimate (ii). Let $f \in \mathcal{P}_G$ be defined by (10). Thus

$$\mathcal{L}f(z) = \Delta^2 \frac{\Delta^2 + 4\Delta e^{i\alpha}\sum_{j=1}^n z_j + (e^{i\alpha}\sum_{j=1}^n z_j)^2}{(\Delta - e^{i\alpha}\sum_{j=1}^n z_j)^4}, \quad z \in G, \tag{22}$$

and consequently,

$$\frac{\mathcal{L}f(z)}{f(z)} = \frac{\Delta^2 + 4\Delta e^{i\alpha}\sum_{j=1}^n z_j + (e^{i\alpha}\sum_{j=1}^n z_j)^2}{\Delta^2 - (e^{i\alpha}\sum_{j=1}^n z_j)^2}, \quad z \in G.$$

For every $r \in (0,1)$ let us take a point $\mathring{z} \in G\setminus\{0\}$, $\mu_G(\mathring{z}) = r$ which satisfies (11) and $\alpha = \alpha_0 \in \mathbb{R}$ defined by (12). Thus we obtain

$$\left|\frac{\mathcal{L}f(\mathring{z})}{f(\mathring{z})}\right| = \frac{1+4r+r^2}{1-r^2},$$

which proves the sharpness of the estimation (ii) for every bounded complete n-circular domain G.

Similarly, we can show that the function $f \in \mathcal{P}_G$ of the form (10) is an extremal function for the estimate (i). ∎

REMARK 3. There is no positive lower bound for the quantities $|\mathcal{L}f(z)|$, $\left|\frac{\mathcal{L}f(z)}{f(z)}\right|$ in the family $\mathcal{P}_G$.

Indeed, if f is a function defined by (10), then by introducing into the formula (22) both $\alpha = \alpha_0 \in \mathbb{R}$ defined by (13) and the point $\mathring{z} \in G$ with $\mu_G(\mathring{z}) = 2-\sqrt{3}$ satisfying the condition (11), we get $\mathcal{L}f(\mathring{z}) = 0$.

THEOREM 2. *Let $n \geq 2$ and $G \subset \mathbb{C}^n$ be a bounded complete n-circular domain. For each function $f \in \mathcal{R}_G$ the following precise estimates hold*

$$\frac{1-r}{1+r} \leq \left|\frac{f(z)}{\mathcal{L}f(z)}\right| \leq \frac{1+r}{1-r}, \quad \mu_G(z) \leq r,\ r \in [0,1). \tag{23}$$

The estimate is precise for every domain G and an extremal function has the form (8).

Proof. The inequalities in (23) hold for $r = 0$. Let us fix arbitrarily $z \in G$ such that $\mu_G(z) = r \in (0,1)$. In order to prove the above inequalities let us take a function $F : E \longrightarrow \mathbb{C}$ defined by $F(\zeta) = \zeta f(\zeta\frac{z}{\mu_G(z)})$, $\zeta \in E$. Thus $F'(\zeta) = \mathcal{L}f(\zeta\frac{z}{\mu_G(z)})$ and $F \in \mathcal{S}^{cc}$. In the class $\mathcal{S}^{cc}$ hold the following estimates

$$\frac{|\zeta|(1-|\zeta|)}{1+|\zeta|} \leq \left|\frac{F(\zeta)}{F'(\zeta)}\right| \leq \frac{|\zeta|(1+|\zeta|)}{1-|\zeta|}, \quad \zeta \in E.$$

It follows from the same estimates in the class $\mathcal{S}$ of all function f univalent in E, $f(0) = 0$, $f'(0) = 1$ (see [6]) by the facts that $\mathcal{S}^{cc} \subset \mathcal{S}$ and the extremal function in the class $\mathcal{S}$ also belongs to $\mathcal{S}^{cc}$.

From the above estimates we have

$$\frac{|\zeta|(1-|\zeta|)}{1+|\zeta|} \leq \left|\frac{\zeta f(\zeta\frac{z}{\mu_G(z)})}{\mathcal{L}f(\zeta\frac{z}{\mu_G(z)})}\right| \leq \frac{|\zeta|(1+|\zeta|)}{1-|\zeta|}, \quad \zeta \in E.$$

Consequently, putting $\zeta = \mu_G(z) = r$ and using the maximum modulus principle, we obtain (23).

Now it is sufficient to prove the sharpness of the estimates. Let us consider a function $f \in \mathcal{R}_G$ of the form (20), where Δ is defined by (8). For every $r \in (0,1)$ we choose $\overset{\circ}{z} \in G$, $\mu_G(\overset{\circ}{z}) = r$ satisfying (11) and $\alpha = \alpha_0 \in \mathbb{R}$ defined by (13). Therefore

$$\mathcal{L}f(\overset{\circ}{z}) = \Delta^2 \frac{(\Delta + e^{i\alpha_0} \sum_{j=1}^n \overset{\circ}{z}_j)}{(\Delta - e^{i\alpha_0} \sum_{j=1}^n \overset{\circ}{z}_j)^3}$$

and

$$\left| \frac{f(\overset{\circ}{z})}{\mathcal{L}f(\overset{\circ}{z})} \right| = \frac{\Delta + \Delta\mu_G(\overset{\circ}{z})}{\Delta - \Delta\mu_G(\overset{\circ}{z})} = \frac{1+r}{1-r},$$

which gives equality in the upper estimate (23).

In order to prove the sharpness of the lower estimate (23) we choose $\overset{\circ}{z}$ in the same way as above and $\alpha = \alpha_0 \in \mathbb{R}$ defined by (12). ■

3. A majorization problem

We will use Theorem 2 to solve an extremal problem concerning the majorization of functions belonging to the family $\mathcal{H}_G$. The above-mentioned problem relates to with the family $\mathcal{R}_G$.

Let $f, F \in \mathcal{H}_G$ and $r \in [0,1]$. If

$$|f(z)| \le |F(z)|, \quad z \in rG, \tag{24}$$

we say that the function F majorizes the function f in the set rG.

The second author (see [13]) has proved that if in a complete bounded two-circular domain $G \subset \mathbb{C}^2$ a function $F \in \mathcal{M}_G$ majorizes a function $f \in \mathcal{H}_G$, then $\mathcal{L}F$ majorizes $\mathcal{L}f$ in rG, where $r = r(\mathcal{M}_G) = 2 - \sqrt{3}$. Moreover, the number $r(\mathcal{M}_G)$ cannot be increased by taking G, to be the cone $A(2;1) = \{z \in \mathbb{C}^2 : |z_1| + |z_2| < 1\}$ in $\mathbb{C}^2$.

It turns out that there exists an analogous result for the superclass $\mathcal{R}_G$ of the class $\mathcal{M}_G$ and it is optimal in case of any complete bounded n-circular domain $G \subset \mathbb{C}^n$. This is the consequence of the following theorem.

Theorem 3. *Let $n \ge 2$ and $G \subset \mathbb{C}^n$ be a bounded complete n-circular domain. If a function $f \in \mathcal{H}_G$ is majorized in G by a function $F \in \mathcal{R}_G$, then*

$$|\mathcal{L}f(z)| \le T(r)|\mathcal{L}F(z)|, \quad \mu_G(z) = r \in [0,1), \tag{25}$$

where

$$T(r) = \begin{cases} 1 & \text{for } r \in [0, 2-\sqrt{3}] \\ \dfrac{4r^2 + (1-r)^4}{4r(1-r)^2} & \text{for } r \in [2-\sqrt{3}, 1) \end{cases}. \tag{26}$$

The function T cannot be replaced in (25) by any function with values $T(r)$ smaller than the values of T defined by (26).

Proof. Let $f \in \mathcal{H}_G$, $F \in \mathcal{R}_G$ and assume (24) holds with $r = 1$. Thus we have that

$$f(z) = \omega(z)F(z), \quad z \in G, \tag{27}$$

where $\omega \in \tilde{\mathcal{S}}_G = \mathcal{S}_G \cup \{1\}$ and $\mathcal{S}_G = \{\omega \in \mathcal{H}_G : \omega(G) \subset E\}$. Indeed, since $F(z)\mathcal{L}F(z) \neq 0$ for $F \in \mathcal{R}_G$, $z \in G$ (see [1]), we have in view of (24) with $r = 1$, that

$$\left|\frac{f(z)}{F(z)}\right| \leq 1, \quad z \in G.$$

Consequently the function $\omega(z) = \frac{f(z)}{F(z)}$, $z \in G$, is holomorphic in G and $|\omega(z)| < 1$ for $z \in G$ or $\omega(z) \equiv 1$ in G. Now, we can investigate the quotient $\frac{\mathcal{L}f(z)}{\mathcal{L}F(z)}$, $z \in G$. If $\mu_G(z) = r \in [0,1)$, then (27) and (23) give

$$\left|\frac{\mathcal{L}f(z)}{\mathcal{L}F(z)}\right| = \left|\frac{\mathcal{L}[\omega(z)F(z)]}{\mathcal{L}F(z)}\right| = \left|\frac{D\omega(z)(z)F(z)}{\mathcal{L}F(z)} + \omega(z)\right| \leq$$
$$\left|\frac{D\omega(z)(z)F(z)}{\mathcal{L}F(z)}\right| + |\omega(z)| \leq |D\omega(z)(z)|\frac{1+r}{1-r} + |\omega(z)|.$$

Let us also observe that for $\omega \in \tilde{\mathcal{S}}_G$ the following inequality holds

$$|D\omega(z)(z)| \leq \frac{r}{1-r^2}(1 - |\omega(z)|^2), \quad \mu_G(z) = r \in [0,1).$$

Obviously, for $\omega = 1$ this inequality is true. For $\omega \in S_G$ the inequality holds by a generalization of Bavrin's results [1] onto the case $\mathbb{C}^n$, $n \geq 2$ and by relations (7). As a result we have

$$\left|\frac{\mathcal{L}f(z)}{\mathcal{L}F(z)}\right| \leq \frac{-r}{(1-r)^2}\,|\omega(z)|^2 + |\omega(z)| + \frac{r}{(1-r)^2}, \quad \mu_G(z) = r \in [0,1).$$

Thus the right-hand part of the above inequality is a quadratic function of the variable x:

$$\frac{-r}{(1-r)^2}x^2 + x + \frac{r}{(1-r)^2}, \quad x = |\omega(z)|.$$

Its maximum is equal to

$$\frac{4r^2 + (1-r)^4}{4r(1-r)^2} \quad for \quad r \in [2-\sqrt{3}, 1)$$

and it is 1 for $r \in (0, 2-\sqrt{3})$. Hence we obtain (26).

In order to prove the second part of the theorem we take $r \in [2-\sqrt{3}, 1)$ and the point $\overset{\circ}{z} \in G$, $\mu_G(\overset{\circ}{z}) = r$ satisfying the condition (11) and the function

$$f(z) = F(z)\frac{\Delta\beta + e^{i\alpha}\sum_{j=1}^{n} z_j}{\Delta + \beta e^{i\alpha}\sum_{j=1}^{n} z_j}, \quad z = (z_1, \ldots, z_n) \in G,$$

where $\beta = (1-2r-r^2)(r(1+2r-r^2))^{-1}$, F is defined by the right-hand side of the formula (20) and Δ, α are determined by (8) and (12), respectively.

Thus $F \in \mathcal{R}_G$, f is majorized by F in G and for the point $\overset{\circ}{z}$ we have equality in (25). However, by putting $f = F$ for $r \in [0, 2-\sqrt{3}]$, where $F \in \mathcal{R}_G$, we have $\mathcal{L}f = \mathcal{L}F$ and equality in (25) holds for points $z \in G$ such that $\mu_G(z) = r \in [0, 2-\sqrt{3}]$. This completes the proof. ∎

The values of the function $\frac{4r^2+(1-r)^4}{4r(1-r)^2}$ are not greater than 1 for $r \in [0, 2-\sqrt{3}]$ and are greater than 1 for $r \in (2-\sqrt{3}, 1)$, so by Theorem 3 the following statement holds.

COROLLARY 1. *Let $n \geq 2$ and G be a bounded complete n-circular domain of $\mathbb{C}^n$. If a function $F \in \mathcal{R}_G$ majorizes a function $f \in \mathcal{H}_G$ in G, then the function $\mathcal{L}F$ majorizes the function $\mathcal{L}f$ in the domain $(2-\sqrt{3})G$. The number $2-\sqrt{3}$ cannot be replaced by any greater number.*

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Received February 4, 2008, revised version February 9, 2009.