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ON CLASSES OF MEROMORPHIC OR COMPLEX HARMONIC FUNCTIONS WITH A POLE AT THE INFINITY

Abstract. In this article we investigate some classes of meromorphic or complex harmonic functions with a pole, which are generated either by analytic conditions or by "coefficient inequalities". There are given theorems, which combine the geometrical properties of functions of the introduced classes. Some results broaden knowledge about the classes of functions, which were investigated in [15]. The main inspiration for the reaserch were the papers [4] and [11]. The part of results were presented in the XII-th International Mathematically-Informatical Conference in Chełm (2nd-5th July, 2006) [12].

1. On some classes of holomorphic functions in U

Let $U_r := \{z \in \mathbb{C} : |z| > r\}$, $r > 1$, $U_1 := U$. Let us consider functions H of the form

$$(1)\quad H(z;\xi) = \xi z + \sum_{n=1}^{\infty} a_n z^{-n}, \quad z \in U,\ \ \xi \in \mathbb{C}\setminus\{0\},\ \ a_n \in \mathbb{C},\ n = 1, 2, \ldots.$$

Definition 1. Let $\alpha \in \langle 0, 1\rangle$, $\operatorname{Re}\xi < 0$. We denote by $\mathcal{J}(\xi, \alpha)$ the class of functions H of the form (1) satisfying the condition

$$(2)\qquad \operatorname{Re}\Big\{\alpha \frac{H(z;\xi)}{z} + (\alpha - 1)H'_z(z;\xi) - 2\xi\alpha\Big\} > 0, \quad z \in U.$$

The class $\mathcal{J}(\xi, \alpha)$ for a fixed $\alpha \in \langle 0, 1\rangle$ and $\xi \in \mathbb{C}$, $Re\,\xi < 0$ is not empty. Indeed e.g. the function

$$I_\xi(z) = I(z;\xi) := \xi z,\ \operatorname{Re}\xi < 0,\ z \in U$$

is the function from the class $\mathcal{J}(\xi, \alpha)$.

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From (1) we obtain

$$\{\alpha\frac{H(z;\xi)}{z}+(\alpha-1)H'_z(z;\xi)-2\xi\alpha\}=$$
$$=-\xi+\frac{a_1}{z^2}+(2-\alpha)\frac{a_2}{z^3}+\cdots+(n+(1-n)\alpha)\frac{a_n}{z^{n+1}}+\ldots,$$

so, by (2), the condition $\operatorname{Re}\xi<0$ is essential.

Since $\operatorname{Re}\{\alpha\frac{H(z;\xi)}{z}+(\alpha-1)H'_z(z;\xi)-2\xi\alpha\}$ is the real harmonic function in $U\cup\{\infty\}$, thus $Re\,\xi=0$ is impossible. It is visible that in (2) there appears a convex combination of conditions:

$$\operatorname{Re}\{\frac{H(z;\xi)}{z}-2\xi\}>0,\quad \operatorname{Re}\{-H'_z(z;\xi)\}>0. \tag{3}$$

The second of them guarantees the locally univalence of function H in U and is equivalent to $Arg(-H'_z(z;\xi))\in(-\frac{\pi}{2},\frac{\pi}{2})$ for $z\in U$.

It is worth rememberig, that for functions holomorphic in the unit disc $\Delta:=\{z\in\mathbb{C}:|z|<1\}$, the conditions corresponding to (3) have been known for long time (see e.g. [2], [6], [9], [10], [13], [14]). The similar conditions concerning the functions with a pole can be found e.g. in [1], [3].

In the next investigations we need

Example 1. Let $\alpha\in\langle 0,1\rangle$, $\operatorname{Re}\xi<0$, $k\in\mathbb{N}$. Let us consider the functions

$$H_k(z;\xi)=\xi z+\frac{a_k}{z^k},\ a_k\in\mathbb{R},\ 0<|a_k|\le\frac{-\operatorname{Re}\xi}{\alpha+(1-\alpha)k},\quad z\in U. \tag{4}$$

The functions H_k are of the form (1) and

$$\operatorname{Re}\Big\{\alpha\frac{H_k(z;\xi)}{z}+(\alpha-1)H'_{kz}(z;\xi)-2\xi\alpha\Big\}>0,\quad z\in U,$$

thus $H_k\in\mathcal{J}(\xi,\alpha)$.

Let $\wp$ denotes the well-known class of Carathéodory of functions with a positive real part (e.g. [5], p. 40).

By $\wp^{\vee}$ let us denote the subclass of the class $\wp$ of functions p satisfying the condition $p_1=p'(0)=0$. Hence and from Definition 1 we obtain

Property 1. *Let $\alpha\in\langle 0,1\rangle$. If $H\in\mathcal{J}(\xi,\alpha)$, $\xi<0$, then function p of the form*

$$p(\xi,\alpha;\ \zeta)=-\frac{1}{\xi}\Big\{\alpha\frac{H(z;\xi)}{z}+(\alpha-1)H'_z(z;\xi)-2\xi\alpha\Big\}\Big|_{z=\frac{1}{\zeta}}, \tag{5}$$

where $\zeta\in\Delta$, is the function from the class $\wp^{\vee}$.

Conversely, if $p \in \wp^{\vee}$, $\xi < 0$, then a function H of the form (1), *which is the solution of the equation* (5), *belongs to the class* $\mathcal{J}(\xi, \alpha)$.

Definition 2. Let $\alpha \in \langle 0, 1 \rangle$, $\xi \in \mathbb{C} \setminus \{0\}$. By $\tilde{\mathcal{J}}(\xi, \alpha)$ let us denote the class of functions H of the form (1) satisfying the condition

$$\sum_{n=1}^{\infty} (\alpha + (1-\alpha)n)\, |a_n| \le |\xi|. \tag{6}$$

Remark 1. Let $\alpha \in \langle 0, 1 \rangle$.

1. If $\operatorname{Re} \xi < 0$, then we have:
   (a) $I_\xi \in \tilde{\mathcal{J}}(\xi, \alpha) \cap \mathcal{J}(\xi, \alpha)$;
   (b) the functions H_k of the form (4) are also the functions from the class $\tilde{\mathcal{J}}(\xi, \alpha)$;
2. If $H \in \tilde{\mathcal{J}}(\xi, \alpha)$, $\xi \in \mathbb{C} \setminus \{0\}$, then

$$|a_n| \le \frac{|\xi|}{\alpha + (1-\alpha)n}, \quad n = 1, 2, \ldots.$$

   The estimates are sharp. The functions H_k from the Example 1 are the extremal functions, with $a_k = \frac{\xi}{\alpha - (1-\alpha)k}$, $k = n$, $\xi \in \mathbb{C} \setminus \{0\}$.

Example 2. Let

$$\tilde{H}(z; \xi) = \xi(z + \frac{1}{z}), \quad z \in U, \ \xi \in \mathbb{C} \setminus \{0\}.$$

It is obvious that $\tilde{H} \in \tilde{\mathcal{J}}(\xi, \alpha)$ for all $\alpha \in \langle 0, 1 \rangle$. Moreover, let us notice that

$$\operatorname{Re}\left\{ \alpha \frac{\tilde{H}(z; \xi)}{z} + (\alpha - 1)\tilde{H}'_z(z; \xi) - 2\xi\alpha \right\} = \operatorname{Re}\left\{ \xi \left(\frac{1}{z^2} - 1 \right) \right\} > 0 \ \text{ in } U$$

only if $\xi < 0$. Therefore $\tilde{H} \in \mathcal{J}(\xi, \alpha)$ only if $\xi < 0$. Since the function $z \mapsto z + \frac{1}{z}$ maps conformally U onto the plane $\mathbb{C}$, which is cut along the segment $-2 \le w \le 2$, thus $\tilde{H}$ maps U onto the plane which is cut along the segment connecting the points $-2\xi, 2\xi$.

Next we have the theorems combining the functions from the class $\tilde{\mathcal{J}}(\xi, \alpha)$ with the fucntions from the class $\mathcal{J}(\xi, \alpha)$.

Theorem 1. *If $\xi < 0$, then $\tilde{\mathcal{J}}(\xi, \alpha) \subset \mathcal{J}(\xi, \alpha)$.*

Proof. We will prove that

$$\left| \alpha \frac{H(z; \xi)}{z} + (\alpha - 1)H'_z(z; \xi) - 2\xi\alpha - (-\xi) \right| < -\xi, \ z \in U.$$

For the sake of (1) and (6) for $z \in U$ we have

$$\left|\alpha\frac{H(z;\xi)}{z}+(\alpha-1)H'_z(z;\xi)-2\xi\alpha-(-\xi)\right|\le\sum_{n=1}^{\infty}(\alpha+n(1-\alpha))\left|\frac{a_n}{z^{n+1}}\right|$$
$$<|\xi|=-\xi.$$

Hence H satisfies the condition (2), thus $H\in\mathcal{J}(\xi,\alpha)$. ∎

The function $\tilde{H}$ from Example 2 shows that for freely fixed $\alpha\in\langle 0,1\rangle$ the assumption $\xi<0$ in Theorem 1 cannot be replaced with more general assumption $\operatorname{Re}\xi<0$.

Let $\xi<0$. Let us observe that no each function from the class $\mathcal{J}(\xi,\alpha)$ is a function of class $\tilde{\mathcal{J}}(\xi,\alpha)$, which is illustrated by the next example.

Example 3. Let us consider the function H^* of the form

$$H^*(z;\xi)=\xi\ z+\sum_{n=1}^{\infty}\frac{-2\xi}{\alpha+(1-\alpha)2n}z^{-2n},\quad z\in U,\ \xi<0,\ \alpha\in\langle 0,1\rangle.$$

Owing to Property 1 the function H^*, which is the solution of the equation (5) in the case of $p^*(\zeta)=\frac{1+\zeta^2}{1-\zeta^2}$, is the function from the class $\mathcal{J}(\xi,\alpha)$. For the function H^* the condition (6) does not hold, thus

$$\mathcal{J}(\xi,\alpha)\setminus\tilde{\mathcal{J}}(\xi,\alpha)\neq\emptyset,\ \xi<0.$$

It appears that functions belonging to some subclasses of the class $\mathcal{J}(\xi,\alpha)$ satisfy the condition (6).

Definition 3. Let $\alpha\in\langle 0,1\rangle$, $\operatorname{Re}\xi<0$. Let us denote by $\mathcal{J}^-(\xi,\alpha)$ the class of functions H from the class $\mathcal{J}(\xi,\alpha)$ of the form

$$H(z;\xi)=\xi\ z-\sum_{n=1}^{\infty}a_nz^{-n},\quad a_n\ge 0,\ n=1,2,\dots,\ z\in U. \tag{7}$$

It occurs

Theorem 2. *Each function H from the class $\mathcal{J}^-(\xi,\alpha)$ satisfies the condition* (6).

Proof. Since H is of the form (7) and satisfies (2) for any fixed $\alpha\in\langle 0,1\rangle$, $\operatorname{Re}\xi<0$, then for all $x\in(1,+\infty)$ we have

$$0<\operatorname{Re}\left\{\alpha\frac{H(x;\xi)}{x}+(\alpha-1)H'_x(x;\xi)-2\xi\alpha\right\}=-\operatorname{Re}\xi-\sum_{n=1}^{\infty}\frac{\alpha+(1-\alpha)n}{x^{n+1}}a_n.$$

Therefore,

$$0 \leq \sum_{n=1}^{\infty} \frac{\alpha + (1-\alpha)n}{x^{n+1}} a_n < -\mathrm{Re}\xi \leq |\xi|.$$

If we go to the limit with $x \to 1^+$, we obtain (6). ∎

From Theorems 1 and 2 it follows directly

COROLLARY 1. *Let $\xi < 0$. Function H belongs to the class $\mathcal{J}^-(\xi, \alpha)$ if and only if H is the function from the class $\tilde{\mathcal{J}}(\xi, \alpha)$.*

REMARK 2. From the Definition 2 and inequalities

$$1 \leq \alpha + (1-\alpha)n \leq n, \quad \alpha \in \langle 0, 1\rangle,\ n \in \mathbb{N}$$

we have

$$\tilde{\mathcal{J}}(\xi, 0) \subset \tilde{\mathcal{J}}(\xi, \alpha) \subset \tilde{\mathcal{J}}(\xi, 1).$$

Thus, owing to Corollary 1 we obtain

$$\mathcal{J}^-(\xi, 0) \subset \mathcal{J}^-(\xi, \alpha) \subset \mathcal{J}^-(\xi, 1).$$

The question about inclusions $\mathcal{J}(\xi, 0) \subset \mathcal{J}(\xi, \alpha) \subset \mathcal{J}(\xi, 1)$, $\xi < 0$ remains open.

DEFINITION 4. Let

$$H_i(z;\xi) = \xi_i\, z + \sum_{n=1}^{\infty} \frac{a_n^{(i)}}{z^n}, \quad z \in U,\ \xi_i \in \mathbb{C} \setminus \{0\},\ i = 1, 2.$$

The Hadamard convolution of the functions H_1, H_2 is the function $H_1 * H_2$ of the form

$$(H_1 * H_2)(z; \xi_1 \xi_2) := \xi_1\, \xi_2\, z + \sum_{n=1}^{\infty} \frac{a_n^{(1)} a_n^{(2)}}{z^n}, \quad z \in U.$$

This definition is based on the classic definition of convolution of the holomorphic functions, introduced by J. Hadamard [7]. Hence, owing to Definition 2 we have

PROPERTY 2. *If $H \in \tilde{\mathcal{J}}(\xi, \alpha)$ and Φ is a function of the form*

$$\Phi(z) = z + \sum_{n=1}^{\infty} \frac{b_n}{z^n}, \quad z \in U,$$

and $|b_n| \leq 1$, $n = 1, 2, \ldots$, then $H * \Phi \in \tilde{\mathcal{J}}(\xi, \alpha)$.

Directly from the above-mentioned property and from Theorem 2 we obtain

Property 3. *If* $H \in \mathcal{J}^-(\xi,\alpha)$, $\Psi(z) := z + \sum_{n=1}^{\infty} \frac{|b_n|}{z^n}$, $z \in U$, $|b_n| \leq 1$, $n = 1,2,\ldots$, *then* $H * \Psi \in \mathcal{J}^-(\xi,\alpha)$.

2. On some classes of complex harmonic functions in U

In the paper [15] we introduced and investigated the following classes of harmonic functions in U with a pole at the infinity.

Definition 5. Let $\alpha \in \langle 0,1\rangle$, $\xi,\eta \in \mathbb{C}$, $|\eta| < |\xi|$. Let us denote by $\tilde{\mathcal{J}}_H(\xi,\eta,\alpha)$ the class of functions F of the form

$$(8)\qquad F(z:\xi,\eta) = H(z;\xi) + \overline{G(z;\eta)},$$

$$H(z;\xi) = \xi z + \sum_{n=1}^{\infty} a_n z^{-n}, \quad G(z;\eta) = \eta z + \sum_{n=1}^{\infty} b_n z^{-n}, \quad z \in U,$$

which are complex harmonic in U and satisfy the condition

$$(9)\qquad \sum_{n=1}^{\infty} \big(\alpha + (1-\alpha)n\big)\big(|a_n| + |b_n|\big) \leq |\xi| - |\eta|.$$

In this part of the article we investigate their further properties. It appears that the coefficient condition (9), in the case of some values of parameters ξ η, is connected with some analytic condition.

Let us notice that if $F = H + \overline{G}$ is the function of the form (8) such that $\xi + \eta \neq 0$, then the function $W = H + G$ is of the form (1).

Definition 6. Let $\alpha \in \langle 0,1\rangle$, $\mathrm{Re}(\xi+\eta) < 0$. Let us denote by $\mathcal{J}_H(\xi,\eta,\alpha)$ the class of functions $F = H + \overline{G}$ of the form (8) such that $W = H + G$ is the function from the class $\mathcal{J}(\xi+\eta,\alpha)$, i.e.

$$\mathrm{Re}\left\{\alpha\frac{H(z;\xi)+G(z;\eta)}{z} + (\alpha-1)(H'_z(z;\xi)+G'_z(z;\eta)) - 2(\xi+\eta)\alpha\right\} > 0, \ z \in U.$$

Next, we have

Theorem 3. *Let* $\xi + \eta < 0$, $|\xi| > |\eta|$. *Then the inclusion*

$$\tilde{\mathcal{J}}_H(\xi,\eta,\alpha) \subset \mathcal{J}_H(\xi,\eta,\alpha)$$

is true.

Proof. Since $-(\xi+\eta) = |\xi+\eta| > |\xi| - |\eta| > 0$, thus if F satisfies (9), then it also satysfies the condition

$$(10)\qquad \sum_{n=1}^{\infty} (\alpha + (1-\alpha)n)(|a_n| + |b_n|) \leq -(\xi+\eta).$$

The function $W = H + G$ is of the form

$$W(z;\xi+\eta) = (\xi+\eta)z + \sum_{n=1}^{\infty}(a_n+b_n)z^{-n}, \quad z \in U,$$

and

$$\sum_{n=1}^{\infty}(\alpha+(1-\alpha)n)(|a_n+b_n|) \le \sum_{n=1}^{\infty}(\alpha+(1-\alpha)n)(|a_n|+|b_n|) \le |\xi+\eta|.$$

Hence, owing to Definition 2, it follows that $W \in \tilde{\mathcal{J}}(\xi+\eta,\alpha)$. Thus, from Theorem 1 and Definition 6, $F \in \mathcal{J}_H(\xi,\eta,\alpha)$. ■

Definition 7. Let $\alpha \in \langle 0,1\rangle$, $0 < \eta < -\xi$. Let us denote by $\mathcal{J}_H^-(\xi,\eta,\alpha)$ the subclass of the class $\mathcal{J}_H(\xi,\eta,\alpha)$, of functions of the form

$$\text{(11)} \quad F(z;\xi,\eta) = H(z) + \overline{G(z)},$$

$$H(z;\xi) = \xi z - \sum_{n=1}^{\infty} a_n z^{-n}, \quad G(z;\eta) = \eta z - \sum_{n=1}^{\infty} b_n z^{-n}, \quad z \in U,$$

$$a_n \ge 0,\ b_n \ge 0,\ n = 1,2,\dots.$$

Remark 3. If $0 < \eta < -\xi$, then the conditions (9) and (10) are equivalent.

Moreover, we have

Theorem 4. *Let $0 < \eta < -\xi$. The function F of the form (11) belongs to the class $\tilde{\mathcal{J}}_H(\xi,\eta,\alpha)$ if and only if it is the function from the class $\mathcal{J}_H^-(\xi,\eta,\alpha)$.*

Proof. Let $0 < \eta < -\xi$. By the Definition 7 and Theorem 3, it is enough to prove that each function from the class $\mathcal{J}_H^-(\xi,\eta,\alpha)$ satisfies the condition (9).

Since $F \in \mathcal{J}_H^-(\xi,\eta,\alpha)$, then the function $W = H + G$ is the map from the class $\mathcal{J}(\xi+\eta,\alpha)$ and

$$W(z;\xi+\eta) = (\xi+\eta)z - \sum_{n=1}^{\infty}(a_n+b_n)z^{-n},$$

$$z \in U,\ a_n + b_n \ge 0, n = 1,2,\dots, \xi+\eta < 0,$$

thus W is of the form (7). From Theorem 2 we obtain the thesis. ■

Example 4. Let $\alpha \in \langle 0,1\rangle$, $0 < \eta < -\xi$. Let us consider

$$F_1 = H_1 + \overline{G_1},$$

where

$$H_1(z;\xi) = \xi z - \frac{a_1}{z}, \quad G_1(z;\eta) = \eta z - \frac{b_1}{z}, \quad z \in U.$$

Then $F_1 \in \mathcal{J}_H^-(\xi,\eta,\alpha)$, if $a_1, b_1 \ge 0$ and $a_1 + b_1 \le -(\xi+\eta)$.

It is known [8] that if the harmonic function F of the form (8) satisfies the condition

$$\sum_{n=1}^{\infty} n(|a_n| + |b_n|) \leq |\xi| - |\eta|, \quad |\xi| > |\eta|,$$

then F is the function univalent and sense-preserving in U and each set $\mathbb{C} \setminus \overline{F(U_r)}$, $r > 1$ is starlike with respect to the origin, i.e. for all $r > 1$, $\theta \in \langle 0, 2\pi \rangle$, the inequality of the form

$$\frac{\partial}{\partial \theta}\left(argF(re^{i\theta})\right) > 0$$

holds.

In the paper [15] it was shown the theorem which gives the size of the optimal narrowness U_r of the domain U, where the condition (9) is sufficient and necessary for the function F of the form (11) to be univalent, sense-preserving and starlike with respect to the origin.

THEOREM A. *Let $\alpha \in \langle 0, 1 \rangle$ and let F be a function of the form* (11). *Let us set*

$$r_1(\alpha) = \sqrt[4]{\frac{3}{3 - 2\alpha}}, \quad r_2(\alpha) = \sqrt[5]{\frac{4}{4 - 3\alpha}}.$$

The condition (9) *is necessary and sufficient for univalence, sense-preservation and starlikness of F in U_r, for any $r \geq r^*$ where*

$$r^* = \begin{cases} 1 & \text{if } \alpha = 0 \\ r_1(\alpha) & \text{if } \alpha \in (0, \alpha_1\rangle \\ r_2(\alpha) & \text{if } \alpha \in (\alpha_1, 1\rangle \end{cases} \tag{12}$$

and α_1 is the least positive solution of the equation

$$3^5(4 - 3\alpha)^4 - 4^4(3 - 2\alpha)^5 = 0.$$

Directly from the above-mentioned theorem and from Theorem 4 we obtain

THEOREM 5. *Let F be of the form* (11), *$\alpha \in \langle 0, 1 \rangle, 0 < \eta < -\xi$. The function belongs to the class $\mathcal{J}_H^-(\xi, \eta, \alpha)$ if and only if it is univalent, sense-preserving and starlike with respect to the origin in U_r, for any $r \geq r^*$, where r^* is defined by* (12).

By the same method, we obtain

COROLLARY 2. *Let $\alpha \in \langle 0, 1 \rangle, \xi < 0$. The function H belongs to the class $\mathcal{J}(\xi, \alpha)$ if and only if it is univalent, sense-preserving and starlike with respect to the origin in U_r, for any $r \geq r^*$, where r^* is defined by* (12).

Directly from Theorems 3 and 4 we obtain the property connected with the Hadamard convolution.

Property 4. *Let* $\alpha \in \langle 0,1\rangle$, $0 < \eta < -\xi$, $\chi = \Phi + \overline{\Psi}$, *where* $\Phi(z) = z+\sum_{n=1}^{\infty} d_n z^{-n}$, $\Psi(z) = z+\sum_{n=1}^{\infty} e_n z^{-n}$, $z \in U$ *and* $0 \le d_n \le 1, 0 \le e_n \le 1$, $n = 1,2,\ldots$. *If* $F \in \mathcal{J}_H^-(\xi,\eta,\alpha)$, *then the convolution* $F * \chi = H * \Phi + \overline{G * \Psi}$ *is the function of the same class.*

3. On the convexity of functions from some classes generated by analytic conditions

Definition 8. Let $\alpha \in \langle 0,1\rangle$, $\operatorname{Re}\xi > 0$. We denote by $\mathcal{K}(\xi,\alpha)$ the class of functions H of the form (1) satisfying the condition

$$\operatorname{Re}\{(1-2\alpha)H'_z(z;\xi) + (1-\alpha)zH''_{zz}(z,\xi) + 2\xi\alpha\} > 0, \quad z \in U. \tag{13}$$

The class $\mathcal{K}(\xi,\alpha)$ is not empty, the function $I_{-\xi}$ belongs to $\mathcal{K}(\xi,\alpha)$.

According to (1) and (13), the condition $\operatorname{Re}\xi > 0$ is essential.

The classes $\mathcal{K}(\xi,\alpha)$ and $\mathcal{J}(\xi,\alpha)$ are closely related by the following theorem.

Theorem 6. *The function* H *belongs to the class* $\mathcal{K}(\xi,\alpha)$ *if and only if the function* Φ *of the form*

$$\Phi(z;\xi) = -zH'_z(z;\xi), \; z \in U,$$

belongs to the class $\mathcal{J}(-\xi,\alpha)$.

Definition 9. Let $\alpha \in \langle 0,1\rangle$, $\xi \in \mathbb{C}\setminus\{0\}$. Let us denote by $\tilde{\mathcal{K}}(\xi,\alpha)$ the class of functions H of the form (1) satisfying the condition

$$\sum_{n=1}^{\infty}(\alpha n + (1-\alpha)n^2)\,|a_n| \le |\xi|. \tag{14}$$

By Theorems 1 and 6 we get

Corollary 3. *Let* $\alpha \in \langle 0,1\rangle$, $\xi > 0$. *If the function* H *of the form* (1) *satisfies the coeffcient condition* (14), *then* $H \in \mathcal{K}(\xi,\alpha)$.

Remark 4. Owing to the function H^* from Example 2 and Theorem 6, we have that

$$\mathcal{K}(\xi,\alpha)\setminus\tilde{\mathcal{K}}(\xi,\alpha) \neq \emptyset.$$

Definition 10. Let $\alpha \in \langle 0,1\rangle$, $\operatorname{Re}\xi > 0$. Let us denote by $\mathcal{K}^-(\xi,\alpha)$ the class of functions H from the class $\mathcal{K}(\xi,\alpha)$ of the form (7).

We have

Corollary 4. *Let* $\operatorname{Re}\xi > 0$. *A function* H *belongs to the class* $\mathcal{K}^-(\xi,\alpha)$ *if and only if* H *is a function from the class* $\tilde{\mathcal{K}}(\xi,\alpha)$.

Similarly as in the Part 2, the classes of complex harmonic functions, corresponding to the above-mentioned classes of holomorphic functions, can be investigated.

Definition 11. Let $\alpha \in \langle 0,1\rangle$, $\xi,\eta \in \mathbb{C}$, $|\eta| < |\xi|$. Let us denote by $\tilde{\mathcal{K}}_H(\xi,\eta,\alpha)$ the class of complex harmonic in U functions F of the form (8), satisfying the condition

$$\sum_{n=1}^{\infty} \big(\alpha n + (1-\alpha)n^2\big)\big(|a_n| + |b_n|\big) \le |\xi| - |\eta|. \tag{15}$$

Let $\mathcal{K}_H(\xi,\eta,\alpha)$ denote the class of functions F of the form (8) such that $W = F + G \in \mathcal{K}(\xi+\eta,\alpha)$ for a fixed $\alpha \in \langle 0,1\rangle$ and $\operatorname{Re}(\xi+\eta) > 0$, i.e.

$$\begin{aligned}\operatorname{Re}\big\{&(1-2\alpha)(H'_z(z;\xi) + G'_z(z;\eta)) + \\ &+ (1-\alpha)z(H''_{zz}(z,\xi) + G''_{zz}(z,\eta)) + 2(\xi+\eta)\alpha\big\} > 0,\ z \in U.\end{aligned}$$

Moreover, by $\mathcal{K}^-_H(\xi,\eta,\alpha)$ we denote the subclass of the class $\mathcal{K}_H(\xi,\eta,\alpha)$ of functions of the form (11) for a fixed $\alpha \in \langle 0,1\rangle$, $0 < \xi < -\eta$.

Obviously, $\mathcal{K}(\xi,\alpha) \subset \mathcal{K}_H(\xi,\eta,\alpha)$.

Some properties of the classes $\tilde{\mathcal{K}}_H(\xi,\eta,\alpha)$, defined by the coefficient condition (15), were investigated in [15].

By Corollaries 3 and 4 and Definition 11, like in the Part 2, we obtain the following two corollaries.

Corollary 5. *Let $\xi+\eta > 0$, $|\xi| > |\eta|$. If $F \in \tilde{\mathcal{K}}_H(\xi,\eta,\alpha)$, then $F \in \mathcal{K}_H(\xi,\eta,\alpha)$.*

Corollary 6. *Let $0 < \xi < -\eta$. A function F of the form (11) belongs to the class $\tilde{\mathcal{K}}_H(\xi,\eta,\alpha)$ if and only if F is a function from the class $\mathcal{K}^-_H(\xi,\eta,\alpha)$.*

The following result is known.

Theorem B. [15] *Let $\alpha \in \langle 0,1\rangle$ and let F be a function of the form (11). The condition (15) is necessary and sufficient for univalence, sense-preservation and convexity of F in U_r, for any $r \ge r^*$, where r^* is defined by (12).*

Directly from the above-mentioned theorem and from Corollary 6 follows

Corollary 7. *Let F be a function of the form (11), $0 < \xi < -\eta$. The function F belongs to the class $\mathcal{K}^-_H(\xi,\eta,\alpha)$ if and only if it is univalent, sense-preserving and convex in U_r, for any $r \ge r^*$, where r^* is defined by (12).*

From Corollary 7 we obtain the corollary, analogous to Corollary 2, which gives the size of the optimal narrowness U_r of the domain U, where the condition (15) is necessary and sufficient for the holomorphic function of the form (7) to be univalent, sense-preserving and convex.

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