DEMONSTRATIO MATHEMATICA
Vol. XLII No 3 2009

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DERIVATIONS WITH ENGEL CONDITIONS ON MULTILINEAR POLYNOMIALS IN PRIME RINGS

Abstract. Let R be a prime ring with extended centroid C and characteristic different from 2, d a nonzero derivation of R, $f(x_1,\ldots,x_n)$ a nonzero multilinear polynomial over C such that $[d^2(f(x_1,\ldots,x_n)),d(f(x_1,\ldots,x_n))]_k = 0$ for all $x_1,\ldots,x_n$ in some nonzero right ideal ρ of R, where k is a fixed positive integer. If $d(\rho)\ \rho \neq 0$, then $\rho C = eRC$ for some idempotent e in the socle of RC and $f(x_1,\ldots,x_n)$ is central-valued on $eRCe$.

Throughout this paper R always denotes a prime ring with center $Z(R)$ and extended centroid C. For given $x,y \in R$, let $[x,y]_0 = x$, $[x,y]_1 = [x,y] = xy - yx$ and inductively for $k > 1$, $[x,y]_k = [[x,y]_{k-1},y]$.

A well-known result proved by Posner [24] states that R must be commutative if there exists a nonzero derivation d such that $[d(x),x] \in Z(R)$ for all $x \in R$. Many related generalizations of Posner's result have been obtained by a number of authors in literature. For details we refer to [4, 12, 13, 14, 15, 16, 17, 20, 25], where further references can be found. In [13], Lanski generalized the Posner's result, by replacing $x \in R$ with an element in a noncommutative Lie ideal L of R. More precisely, he proved that if $[d(x),x]_k = 0$ for all $x \in L$, where $k > 0$ is a fixed integer, then char $R = 2$ and R satisfies S_4, the standard identity of four variables. In [15], Lee and Lee considered a similar Engel condition, $[d(x),x]_k = 0$ in case $x \in \{f(x_1,\ldots,x_n) | x_1,\ldots,x_n \in I\}$, where I is a nonzero two-sided ideal of R and $f(x_1,\ldots,x_n)$ is a multilinear polynomial over C in R. They obtained the result that $f(x_1,\ldots,x_n)$ is central-valued on R except when char $R = 2$ and R satisfies S_4. In case $x \in \{f(x_1,\ldots,x_n) | x_1,\ldots,x_n \in \lambda\}$, where λ is a nonzero left ideal of R and $f(x_1,\ldots,x_n)$ is a multilinear polynomial over C in R, then Lee [17] proved that $\lambda C = RCe$ for some idempotent e in the socle of RC and $f(x_1,\ldots,x_n)$ is central-valued on $eRCe$ except when char

2000 *Mathematics Subject Classification*: 16W25, 16R50, 16N60.
Key words and phrases: prime ring, derivation, extended centroid, Martindale quotient ring.

$R = 2$ and $\dim_C eRCe = 4$. In the present paper, our aim is to study the identity in case $x \in \{d(f(x_1, \ldots, x_n)) | x_1, \ldots, x_n \in \rho\}$, where ρ is a nonzero right ideal of R.

Before beginning the proof of our main theorem, we first fix some notations concerning quotient rings. Denote by Q the two-sided Martindale's quotient ring of R and by C the center of Q, which is called the extended centroid of R. Note that Q is also a prime ring with C a field. We will make a frequent use of the following notation: $f(x_1, \ldots, x_n) = x_1 x_2 \ldots x_n + \sum_{I \neq \sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \ldots x_{\sigma(n)}$ for some $\alpha_\sigma \in C$ where S_n is the permutation group over n elements and we denote by $f^d(x_1, \ldots, x_n)$ the polynomial obtained from $f(x_1, \ldots, x_n)$ by replacing each coefficient α_σ with $d(\alpha_\sigma)$. Thus we write

$$d(f(x_1, \ldots, x_n)) = f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, d(x_i), \ldots, x_n)$$

and

$$\begin{aligned} d^2(f(x_1, \ldots, x_n)) &= d(f^d(x_1, \ldots, x_n)) + d(\sum_i f(x_1, \ldots, d(x_i), \ldots, x_n)) \\ &= f^{d^2}(x_1, \ldots, x_n) + \sum_i f^d(x_1, \ldots, d(x_i), \ldots, x_n) \\ &\quad + \sum_i f^d(x_1, \ldots, d(x_i), \ldots, x_n) \\ &\quad + \sum_{i \neq j} f(x_1, \ldots, d(x_i), \ldots, d(x_j), \ldots, x_n) \\ &\quad + \sum_i f(x_1, \ldots, d^2(x_i), \ldots, x_n) \\ &= f^{d^2}(x_1, \ldots, x_n) + 2 \sum_i f^d(x_1, \ldots, d(x_i), \ldots, x_n) \\ &\quad + 2 \sum_{i<j} f(x_1, \ldots, d(x_i), \ldots, d(x_j), \ldots, x_n) \\ &\quad + \sum_i f(x_1, \ldots, d^2(x_i), \ldots, x_n). \end{aligned}$$

1. The case for $\rho = R$

We first consider the matrix ring case:

Lemma 1.1. *Let F be a field of characteristic $\neq 2$ and $R = M_k(F)$, the $k \times k$ matrix algebra over the field F. Suppose that $a \in R$ and $f(x_1, \ldots, x_n)$*

is a multilinear polynomial over F such that

$$[[a,[a,f(x_1,\ldots,x_n)]],[a,f(x_1,\ldots,x_n)]]_m=0$$

for all $x_1,\ldots,x_n\in R$, where m is a fixed positive integer. Then either $a\in F\cdot I_k$ or $f(x_1,\ldots,x_n)$ is central-valued on R.

Proof. If $k=1$, the result holds trivially. So assume that $k\geq 2$. We assume further that char $F\neq 2$ and proceed to show that $a\in F\cdot I_k$ if $f(x_1,\ldots,x_n)$ is not central-valued on R. Suppose that $f(x_1,\ldots,x_n)$ is not central-valued on R.

Since $f(x_1,\ldots,x_n)$ is not central on R, by [22, Lemma 2, Proof of Lemma 3] there exists a sequence of matrices $r=(r_1,\ldots,r_n)$ in R such that $f(r_1,\ldots,r_n)=\gamma e_{ij}$ with $0\neq\gamma\in F$ and $i\neq j$. Since the set $f(R)=\{f(x_1,\ldots,x_n),x_i\in R\}$ is invariant under the action of all inner automorphisms of R, $f(r)=\gamma e_{ij}$ holds for any $i\neq j$. Thus

$$\begin{aligned}0&=[[a,[a,f(r_1,\ldots,r_n)]],[a,f(r_1,\ldots,r_n)]]_m\\&=[[a,[a,\gamma e_{ij}]],[a,\gamma e_{ij}]]_m\\&=\sum_{s=0}^{m}(-1)^s\binom{m}{s}[a,\gamma e_{ij}]^s[a,[a,\gamma e_{ij}]][a,\gamma e_{ij}]^{m-s}.\end{aligned}$$

Left and right multiplying by e_{ij} we obtain

$$\begin{aligned}0&=e_{ij}\sum_{s=0}^{m}(-1)^s\binom{m}{s}(\gamma ae_{ij})^s(a^2\gamma e_{ij}-2a\gamma e_{ij}a+\gamma e_{ij}a^2)(-\gamma e_{ij}a)^{m-s}e_{ij}\\&=e_{ij}\sum_{s=0}^{m}(-1)^m\binom{m}{s}(\gamma ae_{ij})^s(-2a\gamma e_{ij}a)(\gamma e_{ij}a)^{m-s}e_{ij}\\&=(-1)^{m+1}2\gamma^{m+1}\sum_{s=0}^{m}\binom{m}{s}e_{ij}(ae_{ij})^{m+2}\\&=(-1)^{m+1}2^{m+1}\gamma^{m+1}a_{ji}^{m+2}e_{ij}.\end{aligned}$$

This implies that $a_{ji}=0$ for any $i\neq j$. Thus a is a diagonal matrix. Now for any F-automorphism θ of R, a^θ enjoy the same property as a does, namely,

$$[[a^\theta,[a^\theta,f(x_1,\ldots,x_n)]],[a^\theta,f(x_1,\ldots,x_n)]]_m=0$$

for all $x_1,\ldots,x_n\in R$. Hence, a^θ must be diagonal. Write, $a=\sum_{i=0}^{k}a_{ii}e_{ii}$; then for $s\neq t$, we have

$$(1+e_{ts})a(1-e_{ts})=\sum_{i=0}^{k}a_{ii}e_{ii}+(a_{ss}-a_{tt})e_{ts}$$

diagonal. Hence, $a_{ss}=a_{tt}$ and so a is a scalar matrix that is $a\in F\cdot I_k$.

Lemma 1.2. *Let R be a prime ring of characteristic different from 2 and $f(x_1,\dots,x_n)$ a multilinear polynomial over C. If for any $i=1,\dots,n$,*

$$[f(r_1,\dots,y_i,\dots,r_n),f(r_1,\dots,r_n)]_k=0$$

for all $y_i,r_1,\dots,r_n\in R$, then the polynomial $f(x_1,\dots,x_n)$ is central-valued on R.

Proof. Let a be a noncentral element of R. Then replacing y_i with $[a,r_i]$, we have that

$$\Big[\sum_{i=0}^{n} f(r_1,\dots,[a,r_i],\dots,r_n),f(r_1,\dots,r_n)\Big]_k=0$$

which gives,

$$[a,f(r_1,\dots,r_n)]_{k+1}=0$$

for all $r_1,\dots,r_n\in R$ implying $f(r_1,\dots,r_n)$ is central-valued on R [15, Theorem].

Theorem 1.3. *Let R be a prime ring of characteristic different from 2 and $f(x_1,\dots,x_n)$ a multilinear polynomial over C. Suppose that d is a nonzero derivation of R such that*

$$[d^2(f(x_1,\dots,x_n)),d(f(x_1,\dots,x_n))]_k=0$$

for all $x_1,\dots,x_n\in R$, where k is a fixed positive integer. Then $f(x_1,\dots,x_n)$ is central-valued on R.

Proof. Let $f(x_1,\dots,x_n)$ be noncentral-valued on R. Assume first that d is Q-inner, i.e., $d(x)=[a,x]$ for all $x\in R$ and for some $a\in Q$. Since d is nonzero, $a\notin C$. Thus R satisfies the generalized polynomial identity

$$g(x_1,\dots,x_n)=[[a,[a,f(x_1,\dots,x_n)]],[a,f(x_1,\dots,x_n)]]_k.$$

Since $f(x_1,\dots,x_n)$ is noncentral-valued on R and $a\notin C$, this is a nontrivial GPI. By Chuang [5] this GPI is also satisfied by Q. In case C is infinite, we have $g(r_1,\dots,r_n)=0$ for all $r_1,\dots,r_n\in Q\otimes_C\overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both Q and $Q\otimes_C\overline{C}$ are centrally closed [7, Theorem 2.5 and 3.5], we may replace R by Q or $Q\otimes_C\overline{C}$ according as C is finite or infinite. Thus we may assume that R is centrally closed over C which is either finite or algebraically closed and $g(r_1,\dots,r_n)=0$ for all $r_1,\dots,r_n\in R$. By Martindale's theorem [23], R is a primitive ring having nonzero socle H with C as the associated division ring. In light of Jacobson's theorem [10, p. 75], R is isomorphic to a dense ring of linear transformations on some vector space V over C. Assume first that V is finite dimensional over C. Then the density of R on V implies that $R\cong M_k(C)$ with $k=\dim_C V$. By Lemma 1.1, we have $f(x_1,\dots,x_n)$ is central-valued on R.

Assume next that V is infinite dimensional over C. Since V is infinite dimensional over C then as in Lemma 2 in [26], the set $f(R)$ is dense on R and so from

$$[[a,[a,f(r_1,\dots,r_n)]],[a,f(r_1,\dots,r_n)]]_k = 0$$

for all $r_1,\dots,r_n \in R$, we have

$$[[a,[a,r]],[a,r]]_k = 0 \tag{1}$$

for all $r \in R$. Let e be an idempotent element of H. Replacing r with $er(1-e)$ in (1), we obtain

$$\begin{aligned}0 &= [[a,[a,er(1-e)]],[a,er(1-e)]]_k \\ &= \sum_{j=0}^{k}(-1)^j\binom{k}{j}[a,er(1-e)]^j \\ &\qquad (a^2er(1-e)-2aer(1-e)a+er(1-e)a^2)[a,er(1-e)]^{k-j}.\end{aligned}$$

Left multiplying by $(1-e)$ and right multiplying by er, we obtain

$$\begin{aligned}0 &= \sum_{j=0}^{k}(-1)^j\binom{k}{j}(1-e)(aer(1-e))^j(-2aer(1-e)a)(-er(1-e)a)^{k-j}er \\ &= (-1)^{k+1}2((1-e)aer)^{k+2}\sum_{j=0}^{k}\binom{k}{j} \\ &= (-1)^{k+1}2^{k+1}((1-e)aer)^{k+2}.\end{aligned}$$

By [8], it follows that $(1-e)aer = 0$ for any $r \in R$, implying $(1-e)ae = 0$. Similarly, replacing r with $(1-e)re$ in (1), we shall get $ea(1-e) = 0$. Thus for any idempotent $e \in H$, we have $(1-e)ae = 0 = ea(1-e)$ that is $[a,e] = 0$. Therefore, $[a,E] = 0$, where E is the additive subgroup generated by all idempotents of H. Since E is non central Lie ideal of H, this implies $a \in C$ (see [3, Lemma 2]), a contradiction.

Assume next that d is not Q-inner. Then R satisfies the differential identity

$$\begin{aligned}&[f^{d^2}(r_1,\dots,r_n) + 2\sum_i f^d(r_1,\dots,d(r_i),\dots,r_n) \\ &\qquad\qquad + 2\sum_{i<j} f(r_1,\dots,d(r_i),\dots,d(r_j),\dots,r_n) \\ &+ \sum_i f(r_1,\dots,d^2(r_i),\dots,r_n), f^d(r_1,\dots,r_n) + \sum_i f(r_1,\dots,d(r_i),\dots,r_n)]_k.\end{aligned}$$

By Kharchenko's [11] theorem, R satisfies the polynomial identity

$$\Big[f^{d^2}(r_1,\ldots,r_n)+2\sum_i f^d(r_1,\ldots,x_i,\ldots,r_n) + 2\sum_{i<j} f(r_1,\ldots,x_i,\ldots,x_j,\ldots,r_n) + \sum_i f(r_1,\ldots,y_i,\ldots,r_n), f^d(r_1,\ldots,r_n)+\sum_i f(r_1,\ldots,x_i,\ldots,r_n)\Big]_k.$$

In particular, R satisfies blended component

$$[f(r_1,\ldots,y_i,\ldots,r_n), f(r_1,\ldots,x_i,\ldots,r_n)]_k$$

in the indeterminates $r_1,\ldots,r_{i-1},r_{i+1},\ldots,r_n,x_i,y_i$ which implies that $f(r_1,\ldots,r_n)$ is central-valued on R by Lemma 1.2.

2. The case for ρ.

We need the following lemmas.

Lemma 2.1. *Let ρ be a nonzero right ideal of R and d a derivation of R. Then the following conditions are equivalent:*
(i) *d is an inner derivation induced by some $b\in Q$ such that $b\rho=0$;*
(ii) $d(\rho)\rho=0$.

For its proof, we refer to [9].

Lemma 2.2. *Let R be a prime ring. If $[d^2(f(x_1,\ldots,x_n)), d(f(x_1,\ldots,x_n))]_k = 0$ for all $x_1,\ldots,x_n\in\rho$, where k is a fixed positive integer, then either R satisfies a nontrivial generalized polynomial identity or $d(\rho)\rho=0$.*

Proof. Suppose, on the contrary, that R does not satisfy any nontrivial generalized polynomial identity and then we derive that $d(\rho)\rho=0$. We may assume that R is noncommutative, otherwise R satisfies trivially a nontrivial GPI. Now we consider the following two cases:

<u>Case I.</u> Suppose that d is Q-inner derivation induced by an element $a\in Q$. Then for any $x\in\rho$,

$$[[a,[a,f(xX_1,\ldots,xX_n)]],[a,f(xX_1,\ldots,xX_n)]]_k$$

is a GPI for R, so it is the zero element in $Q*_C C\{X_1,X_2,\ldots,X_n\}$. Denote $l_Q(\rho)$ the left annihilator of ρ in Q. Suppose first that $\{1,a,a^2\}$ are linearly C-independent modulo $l_Q(\rho)$, that is $(\alpha a^2+\beta a+\gamma)\rho=0$ if and only if $\alpha=\beta=\gamma=0$. Since R is not a GPI-ring, a fortiori it cannot be a PI-ring. Thus, by [19, Lemma 3] there exists $x_0\in\rho$ such that $\{a^2x_0, ax_0, x_0\}$ are linearly C-independent. In this case, we have that

$$(2)\qquad \sum_{j=0}^{k}(-1)^j\binom{k}{j}(af(x_0X_1,\dots,x_0X_n)-f(x_0X_1,\dots,x_0X_n)a)^j$$
$$(a^2f(x_0X_1,\dots,x_0X_n)-2af(x_0X_1,\dots,x_0X_n)a+f(x_0X_1,\dots,x_0X_n)a^2)$$
$$(af(x_0X_1,\dots,x_0X_n)-f(x_0X_1,\dots,x_0X_n)a)^{k-j}=0.$$

In this expansion $a^2f(x_0X_1,\dots,x_0X_n)(af(x_0X_1,\dots,x_0X_n))^k$ appears nontrivially, a contradiction.

Therefore, $\{1,a,a^2\}$ are linearly C-dependent modulo $l_R(\rho)$, that is there exist $\alpha,\beta,\gamma\in C$, not all zero, such that $(\alpha a^2+\beta a+\gamma)\rho=0$. Suppose that $\alpha=0$. Then $\beta\neq 0$, otherwise $\gamma=0$. Thus by $(\beta a+\gamma)\rho=0$, we have that $(a+\beta^{-1}\gamma)\rho=0$. Since a and $a+\beta^{-1}\gamma$ induce the same inner derivation, we may replace a by $a+\beta^{-1}\gamma$ in the basic hypothesis. Therefore, in any case we may suppose $a\rho=0$ and then by Lemma 2.1, $d(\rho)\rho=0$.

Next suppose that $\alpha\neq 0$. In this case there exist $\lambda,\mu\in C$ such that $a^2x_0=\lambda ax_0+\mu x_0$ for all $x_0\in\rho$. Choose $x_0\in\rho$ such that ax_0 and x_0 are linearly C-independent, otherwise we have again $a\rho=0$ and hence by Lemma 2.1, $d(\rho)\rho=0$. Thus right multiplying by $f(x_0X_1,\dots,x_0X_n)$ in (2) and then replacing a^2x_0 with $\lambda ax_0+\mu x_0$, we get, R satisfies

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}(af(x_0X_1,\dots,x_0X_n)-f(x_0X_1,\dots,x_0X_n)a)^j$$
$$((\lambda a+\mu)f(x_0X_1,\dots,x_0X_n)-2af(x_0X_1,\dots,x_0X_n)a$$
$$+f(x_0X_1,\dots,x_0X_n)(\lambda a+\mu))$$
$$(af(x_0X_1,\dots,x_0X_n)-f(x_0X_1,\dots,x_0X_n)a)^{k-j}f(x_0X_1,\dots,x_0X_n).$$

In this sum the terms

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}(af(x_0X_1,\dots,x_0X_n))^j$$
$$(-2af(x_0X_1,\dots,x_0X_n)a)(-f(x_0X_1,\dots,x_0X_n)a)^{k-j}f(x_0X_1,\dots,x_0X_n)$$
$$=(-1)^{k+1}2^{k+1}(af(x_0X_1,\dots,x_0X_n))^{k+2}$$

appears nontrivially, a contradiction, because ax_0 and x_0 are linearly C-independent.

<u>CASE II.</u> Next suppose that d is an outer derivation. If for all $x\in\rho$, $d(x)\in xC$, then $[d(x),x]=0$ which implies that R is commutative (see [2]),

a contradiction. Therefore there exists $x \in \rho$ such that $d(x) \notin xC$ i.e., x and $d(x)$ are linearly C-independent. By our assumption we have that R satisfies

$$[f^{d^2}(xX_1,\dots,xX_n)+2\sum_i f^d(xX_1,\dots,d(x)X_i+xd(X_i),\dots,xX_n)$$
$$+2\sum_{i<j} f(xX_1,\dots,d(x)X_i+xd(X_i),\dots,d(x)X_j+xd(X_j),\dots,xX_n)$$
$$+\sum_i f(xX_1,\dots,d^2(x)X_i+2d(x)d(X_i)$$
$$+xd^2(X_i),\dots,xX_n), f^d(xX_1,\dots,xX_n)$$
$$+\sum_i f(xX_1,\dots,d(x)X_i+xd(X_i),\dots,xX_n)]_k.$$

By Kharchenko's theorem [11],

$$(3)\quad [f^{d^2}(xX_1,\dots,xX_n)+2\sum_i f^d(xX_1,\dots,d(x)X_i+xr_i,\dots,xX_n)$$
$$+2\sum_{i<j} f(xX_1,\dots,d(x)X_i+xr_i,\dots,d(x)X_j+xr_j,\dots,xX_n)$$
$$+\sum_i f(xX_1,\dots,d^2(x)X_i+2d(x)r_i+xs_i,\dots,xX_n), f^d(xX_1,\dots,xX_n)$$
$$+\sum_i f(xX_1,\dots,d(x)X_i+xr_i,\dots,xX_n)]_k=0$$

for all $X_1,\dots,X_n,r_1,\dots,r_n,s_1,\dots,s_n \in R$. In particular, for $X_1=r_2=\dots=r_n=s_1=0$,

$$[2f^d(xr_1,\dots,xX_n)+2\sum_{j\geq 2} f(xr_1,\dots,d(x)X_j,\dots,xX_n)$$
$$+f(2d(x)r_1,\dots,xX_n), f(xr_1,\dots,xX_n)]_k=0$$

which is a nontrivial GPI for R, because x and $d(x)$ are linearly C-independent, a contradiction.

Theorem 2.3. *Let R be a prime ring with extended centroid C and characteristic different from 2, d a nonzero derivation of R, $f(x_1,\dots,x_n)$ a nonzero multilinear polynomial over C such that $[d^2(f(x_1,\dots,x_n)), d(f(x_1,\dots,x_n))]_k = 0$ for all $x_1,\dots,x_n$ in some nonzero right ideal ρ of R, where k is a fixed positive integer. If $d(\rho)\rho \neq 0$, then $\rho C = eRC$ for some idempotent e in the socle of RC and $f(x_1,\dots,x_n)$ is central-valued on $eRCe$.*

Proof. Assume first that $[f(\rho),\rho]\rho = 0$, that is $[f(x_1,\dots,x_n),x_{n+1}]x_{n+2} = 0$ for all $x_1,\dots,x_{n+2} \in \rho$. Then by [18, Proposition], $\rho C = eRC$ for some idempotent $e \in soc(RC)$. Since $[f(\rho),\rho]\rho = 0$, we have $[f(\rho R),\rho R]\rho R = 0$ and hence by [5], $[f(\rho Q),\rho Q]\rho Q = 0$. In particular, $[f(\rho C),\rho C]\rho C = 0$. Since $\rho C = eRC$, we have $[f(eRC),eRC]eRC = 0$, or equivalently $[f(eRCe),eRCe] = 0$ which shows that $f(x_1,\dots,x_n)$ is central-valued on $eRCe$.

Next assume that $[f(\rho),\rho]\rho \neq 0$ that is $[f(x_1,\dots,x_n),x_{n+1}]x_{n+2}$ is not an identity for ρ. By Lemma 2.2, R is a prime GPI-ring and so is Q (see [1] and [5]). Since Q is centrally closed over C, it follows from [23] that Q is a primitive ring with $H = Soc(Q) \neq 0$. Then $[f(\rho H),\rho H]\rho H \neq 0$. For otherwise, $[f(\rho Q),\rho Q]\rho Q = 0$ by [1] and [5], a contradiction. Choose $a_1,\dots,a_{n+2} \in \rho H$ such that $[f(a_1,\dots,a_n),a_{n+1}]a_{n+2} \neq 0$. Let $a \in \rho H$. Since H is a regular ring, there exists $e^2 = e \in H$ such that $eH = aH + a_1H + \cdots + a_{n+2}H$. Then $e \in \rho H$ and $a = ea$, $a_i = ea_i$ for $i = 1,\dots,n+2$. Thus, we have $f(eHe) = f(eH)e \neq 0$. By our assumption and by [21, Theorem 2], we may also assume that $[d^2(f(x_1,\dots,x_n)),d(f(x_1,\dots,x_n))]_k$ is an identity for ρQ. In particular, $[d^2(f(x_1,\dots,x_n)),d(f(x_1,\dots,x_n))]_k$ is an identity for ρH and so for eH. It follows that, for all $r_1,\dots,r_n \in H$, $0 = [d^2(f(er_1,\dots,er_n)),d(f(er_1,\dots,er_n))]_k$. We may write $f(x_1,\dots,x_n) = t(x_1,\dots,x_{n-1})x_n + h(x_1,\dots,x_n)$, where x_n never appears as last variable in any monomials of h. Let $r \in H$. Then replacing r_n with $r(1-e)$ we have

$$(4)\quad 0 = [d^2(t(er_1,\dots,er_{n-1})er(1-e)),d(t(er_1,\dots,er_{n-1})er(1-e))]_k.$$

Now we know the fact that $d(x(1-e))e = -x(1-e)d(e)$, $(1-e)d(ex) = (1-e)d(e)ex$ and thus

$$\begin{aligned}(1-e)d^2(ex(1-e))e &= (1-e)d\{d(e)ex(1-e) + ed(ex(1-e))\}e \\ &= (1-e)d(e)d(ex(1-e))e + (1-e)d(e)d(ex(1-e))e \\ &= -2(1-e)d(e)ex(1-e)d(e).\end{aligned}$$

Thus left multiplying by $(1-e)$ and right multiplying by e, we get from (4) that

$$\begin{aligned}0 &= (1-e)[d^2(t(er_1,\dots,er_{n-1})er(1-e)),d(t(er_1,\dots,er_{n-1})er(1-e))]_k e \\ &= (1-e)\sum_{j=0}^{k}(-1)^j\binom{k}{j}\Big\{d(t(er_1,\dots,er_{n-1})er(1-e))\Big\}^j \cdot \\ &\qquad \Big\{d^2(t(er_1,\dots,er_{n-1})er(1-e))\Big\}\Big\{d(t(er_1,\dots,er_{n-1})er(1-e))\Big\}^{k-j} e\end{aligned}$$

$$= (1-e)\sum_{j=0}^{k}(-1)^j\binom{k}{j}\Big\{d(e)t(er_1,\ldots,er_{n-1})er(1-e)\Big\}^j.$$
$$\Big\{d^2(t(er_1,\ldots,er_{n-1})er(1-e))\Big\}\Big\{-t(er_1,\ldots,er_{n-1})er(1-e)d(e)\Big\}^{k-j}$$
$$= (1-e)\sum_{j=0}^{k}(-1)^j\binom{k}{j}\Big\{d(e)t(er_1,\ldots,er_{n-1})er(1-e)\Big\}^j.$$
$$\Big\{-2d(e)t(er_1,\ldots,er_{n-1})er(1-e)d(e)\Big\}$$
$$\Big\{-t(er_1,\ldots,er_{n-1})er(1-e)d(e)\Big\}^{k-j}$$
$$= (-1)^{k+1}2\{(1-e)d(e)t(er_1,\ldots,er_{n-1})er\}^{k+1}(1-e)d(e)\sum_{j=0}^{k}\binom{k}{j}$$
$$= (-1)^{k+1}2^{k+1}\{(1-e)d(e)t(er_1,\ldots,er_{n-1})er\}^{k+1}(1-e)d(e).$$

Since char $R \neq 2$, this gives

$$0 = \{(1-e)d(e)t(er_1,\ldots,er_{n-1})er\}^{k+2}$$

for all $r \in H$. By [8], $(1-e)d(e)t(er_1,\ldots,er_{n-1})eH = 0$ which implies $(1-e)d(e)t(er_1e,\ldots,er_{n-1}e) = 0$ for all $r_1,\ldots,r_{n-1} \in H$. Since eHe is a simple Artinian ring and $t(eHe) \neq 0$ is invariant under the action of all inner automorphisms of eHe, by [6, Lemma 2], $(1-e)d(e) = 0$ and so $d(e) = ed(e) \in eH$. Thus $d(eH) \subseteq d(e)H + ed(H) \subseteq eH \subseteq \rho H$ and $d(a) = d(ea) \in d(eH) \subseteq \rho H$. This means that $d(\rho H) \subseteq \rho H$. It is easily seen that $d(l_H(\rho H)) \subseteq l_H(\rho H)$ holds and so d naturally induces a derivation δ on the prime ring $\overline{\rho H} = \frac{\rho H}{\rho H \cap l_H(\rho H)}$ defined by $\delta(\overline{x}) = \overline{d(x)}$ for $x \in \rho H$, where $l_H(\rho H)$ denotes the left annihilator of ρH in H. Thus by assumption we have $[\delta^2(f(x_1,\ldots,x_n)), \delta(f(x_1,\ldots,x_n))]_k$ is a differential identity for $\overline{\rho H}$. By Theorem 1.3, either $\delta(\overline{\rho H}) = 0$ or $f(x_1,\ldots,x_n)$ is central-valued on $\overline{\rho H}$. If $\delta(\overline{\rho H}) = 0$ that is $d(\rho H)\rho H = 0$, then $0 = d(\rho\rho H)\rho H = d(\rho)\rho H\rho H$ implying $d(\rho)\rho = 0$, a contradiction. If $f(x_1,\ldots,x_n)$ is central-valued on $\overline{\rho H}$, then $[f(x_1,\ldots,x_n), x_{n+1}]x_{n+2}$ is an identity for ρH, again a contradiction. Thus the proof of the theorem is complete.

Acknowledgement. The author wishes to thank the referee for his valuable comments and suggestions.

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Received June 6, 2008; revised version November 29, 2008.