DEMONSTRATIO MATHEMATICA
Vol. XLII No 3 2009

Andrzej Walendziak, Magdalena Wojciechowska

SEMISIMPLE AND SEMILOCAL PSEUDO BL-ALGEBRAS

Abstract. The concepts of semisimple and semilocal pseudo BL-algebras are investigated. Many facts corresponding with them are considerated. Moreover, we give a negative answer to the question from [Di Nola, Georgescu and Iorgulescu (Multiplae Valued Logic 8: 715-750, 2002), Problem 1.33].

1. Introduction

BL-algebras were introduced by Hájek [10] in 1998. The class of BL-algebras contains the MV-algebras introduced by Chang ([1]). Georgescu and Iorgulescu ([6]) introduced pseudo MV-algebras which are a non-commutative generalization of MV-algebras. In 2000, there were introduced pseudo BL-algebras as a natural generalization of BL-algebras and of pseudo MV-algebras. Georgescu and Iorgulescu ([8]) made the connection between pseudo BL-algebras and pseudo BCK-algebras. Kühr ([14]) proved that pseudo BL-algebras are equivalent to certain bounded DRℓ-monoids. Iorgulescu ([13]) showed that the category of pseudo Iséki algebras is equivalent to the category of pseudo BL-algebras. Pseudo BL-algebras correspond to a pseudo-basic fuzzy logic (see [11] and [12]). The paper [2] contains definition and basic properties of pseudo BL-algebras.

In [9], there are characterized and defined some classes of pseudo BL-algebras: local, good, perfect, peculiar and bipartite pseudo BL-algebras. In this paper there are given characterizations of other classes of pseudo BL-algebras: semisimple and semilocal pseudo BL-algebras. In particular, we show that the class of semisimple pseudo BL-algebras is not a quasivariety (and therefore it is not a variety). From this we obtain that representable pseudo BL-algebras are not semisimple in general. Thus Problem 1.33 of [3] is solved.

2000 *Mathematics Subject Classification*: 03G25, 06F05.
Key words and phrases: pseudo BL-algebra, semisimple (semilocal, maximal) pseudo BL-algebra.

2. Preliminaries

Definition 2.1. Let $(A; \vee, \wedge, \odot, \to, \rightsquigarrow, 0, 1)$ be an algebra of type $(2,2,2,2,2,0,0)$. A is called *a pseudo BL-algebra* if it satisfies the following axioms, for any $x, y, z \in A$:

(C1) $(A; \vee, \wedge, 0, 1)$ is a bounded lattice,
(C2) $(A; \odot, 1)$ is a monoid,
(C3) $x \odot y \leq z \Leftrightarrow x \leq y \to z \Leftrightarrow y \leq x \rightsquigarrow z$,
(C4) $x \wedge y = (x \to y) \odot x = x \odot (x \rightsquigarrow y)$,
(C5) $(x \to y) \vee (y \to x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$.

In this sequel, we shall agree that the operations $\vee$, $\wedge$, $\odot$ have priority towards the operations $\to$, $\rightsquigarrow$. A pseudo BL-algebra $(A; \vee, \wedge, \odot, \to, \rightsquigarrow, 0, 1)$ is *nontrivial* if and only if $0 \neq 1$. For any pseudo BL-algebra $(A; \vee, \wedge, \odot, \to, \rightsquigarrow, 0, 1)$, the reduct $\mathcal{L}(A) = (A; \vee, \wedge, 0, 1)$ is a bounded distributive lattice. A pseudo BL-chain is a pseudo BL-algebra such that its lattice order is linear.

Throughout this paper A will denote a pseudo BL-algebra. For any $x \in A$ and $n = 0, 1, \ldots$, we put $x^0 = 1$ and $x^{n+1} = x^n \odot x$.

Proposition 2.2. ([2]) *The following properties hold in A (for any $x, y, z \in A$):*

(a) $x \leq y \Leftrightarrow x \to y = 1$;
(b) $y \leq x \to y$, $y \leq x \rightsquigarrow y$;
(c) $x \odot y \leq x$, $x \odot y \leq y$;
(d) $0 \odot x = x \odot 0 = 0$;
(e) $x \vee z \to y \vee z \geq x \to y$;
(f) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$;
(g) $(x \vee y) \odot z = (x \odot z) \vee (y \odot z)$.

For any $x \in A$, we define $x^- = x \to 0$ and $x^\sim = x \rightsquigarrow 0$.

Proposition 2.3. ([2]) *The following properties hold in A (for any $x, y \in A$):*

(a) $y \leq x^- \Leftrightarrow y \odot x = 0$;
(b) $y \leq x^\sim \Leftrightarrow x \odot y = 0$;
(c) $x \leq y$ *implies* $y^- \leq x^-$ *and* $y^\sim \leq x^\sim$;
(d) $x \leq (x^\sim)^-$, $x \leq (x^-)^\sim$;
(e) $x \odot x^\sim = x^- \odot x = 0$.

DEFINITION 2.4. A nonempty set F is called a *filter* of A if the following conditions hold:

(F1) if $x, y \in F$, then $x \odot y \in F$;
(F2) if $x \in F$, $y \in A$ and $x \leq y$, then $y \in F$.

Under this definition, $\{1\}$ and A are the simple examples of filters. A filter F of A is *proper* if $F \neq A$. We denote by $\mathrm{Fil}(A)$ the set of all filters of A.

PROPOSITION 2.5. ([2]) *If* $F \in \mathrm{Fil}(A)$, *then*:

(a) $1 \in F$;
(b) *if* $x, y \in F$, *then* $x \wedge y \in F$;
(c) *if* $x \in F$, $y \in A$, *then* $y \to x \in F, y \rightsquigarrow x \in F$.

For every subset $X \subseteq A$, the smallest filter of A which contains X, i.e., the intersection of all filters $F \supseteq X$, is called *generated* by X, and is denoted by $[X)$.

REMARK 2.6. ([2])

(a) If X is a filter, then $[X) = X$.
(b) If $X \subseteq A$, then $[X) = \{y \in A : x_1 \odot x_2 \odot \cdots \odot x_n \leq y$ for some $n \geq 1$ and $x_1, x_2, \ldots, x_n \in X\}$.
(c) If $X = \{x\}$, then we shall write $[x)$ instead of $[\{x\})$ and $[x) = \{y \in A : x^n \leq y$ for some $n \geq 1\}$.

DEFINITION 2.7. Let F be a proper filter of A.

(a) F is called *prime* if for all $x, y \in A$, $x \vee y \in F$ implies $x \in F$ or $y \in F$.
(b) F is called *maximal* (or *ultrafilter*) if whenever H is a filter such that $F \subseteq H \subseteq A$, then either $H = F$ or $H = A$.

PROPOSITION 2.8. ([2]) *Any ultrafilter of* A *is a prime filter of* A.

PROPOSITION 2.9. ([2]) *Any proper filter of* A *can be extended to an ultrafilter.*

We denote by $\mathrm{Max}(A)$ the set of all ultrafilters of A. Write $\mathcal{M}(A) = \bigcap\{F : F \in \mathrm{Max}(A)\}$.

DEFINITION 2.10. A filter H of A is called *normal* if for every $x, y \in A$,

$$x \to y \in H \Leftrightarrow x \rightsquigarrow y \in H.$$

We denote by $\mathrm{Max_n}(A)$ the set of normal ultrafilters of A. Suppose that A possesses at least one ultrafilter which is normal. We define $\mathcal{M}_{\mathrm{n}}(A) = \bigcap\{F : F \in \mathrm{Max_n}(A)\}$. If $\mathrm{Max_n}(A) = \emptyset$, we set $\mathcal{M}_{\mathrm{n}}(A) = A$.

From [5] (p. 499) we get

Proposition 2.11. *If A is a pseudo BL-chain, then* $\mathrm{Max}(A) = \mathrm{Max_n}(A) = \{F\}$, *where* $F = \{x \in A : x^n > 0$ *for any* $n \in \mathbb{N}\}$.

Proposition 2.12. ([3]) *If H is a proper normal filter of A, then H is an ultrafilter of A if and only if for any $x \in A$,*

$$x \notin H \Leftrightarrow (x^n)^- \in H \text{ for some } n \in \mathbb{N}.$$

Example 2.13. ([16]) Let $a, b, c, d \in \mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers. We put by definition

$$(a, b) \leq (c, d) \Leftrightarrow a < c \text{ or } (a = c \text{ and } b \leq d).$$

For any $x, y \in \mathbb{R} \times \mathbb{R}$, we define operations $\vee$ and $\wedge$ as follows: $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$. Let $A = \{(\frac{1}{2}, b) \in \mathbb{R}^2 : b \geq 0\} \cup \{(a, b) \in \mathbb{R}^2 : \frac{1}{2} < a < 1, b \in \mathbb{R}\} \cup \{(1, b) \in \mathbb{R}^2 : b \leq 0\}$. For any $(a, b), (c, d) \in A$, we put:

$$(a, b) \odot (c, d) = \left(\frac{1}{2}, 0\right) \vee (ac, bc + d),$$

$$(a, b) \to (c, d) = \left(\frac{1}{2}, 0\right) \vee \left[\left(\frac{c}{a}, \frac{d - b}{a}\right) \wedge (1, 0)\right],$$

$$(a, b) \rightsquigarrow (c, d) = \left(\frac{1}{2}, 0\right) \vee \left[\left(\frac{c}{a}, \frac{ad - bc}{a}\right) \wedge (1, 0)\right].$$

Then $(A; \vee, \wedge, \odot, \to, \rightsquigarrow, (\frac{1}{2}, 0), (1, 0))$ is a pseudo BL-algebra. Let $H = \{(1, b) : b \leq 0\}$. We show that it is a normal ultrafilter of A. Obviously, H is a filter. Suppose that $(a, b), (c, d) \in A$. Then

$$\begin{aligned}(a, b) \to (c, d) \in H &\Leftrightarrow \left(\frac{1}{2}, 0\right) \vee \left[\left(\frac{c}{a}, \frac{d - b}{a}\right) \wedge (1, 0)\right] \in H \\ &\Leftrightarrow \frac{c}{a} \geq 1 \Leftrightarrow \left(\frac{1}{2}, 0\right) \vee \left[\left(\frac{c}{a}, \frac{ad - bc}{a}\right) \wedge (1, 0)\right] \in H \\ &\Leftrightarrow (a, b) \rightsquigarrow (c, d) \in H.\end{aligned}$$

By definition, H is normal. We now apply Proposition 2.12 to show that H is maximal. Let $x = (a, b) \notin H$. Then $\frac{1}{2} \leq a < 1$, and we have $x^n = (\frac{1}{2}, 0)$ for some $n \in \mathbb{N}$. Hence $(x^n)^- = (\frac{1}{2}, 0)^- = (\frac{1}{2}, 0) \to (\frac{1}{2}, 0) = (1, 0) \in H$. Assume now $x \in H$, that is, $x = (1, b)$, $b \leq 0$. Then $x^n = (1, nb) \in H$ for all $n \in \mathbb{N}$, and therefore $(x^n)^- = (1, nb)^- = (\frac{1}{2}, -nb) \notin H$. It is proved that H is an ultrafilter.

For a filter H and $x \in A$, we denote:

$$x \odot H = \{x \odot h : h \in H\} \text{ and } H \odot x = \{h \odot x : h \in H\}.$$

PROPOSITION 2.14. ([3]) *Let H be a filter of A. The following conditions are equivalent:*

(a) *H is normal;*
(b) *for each $x \in A$, $x \odot H = H \odot x$.*

As a consequence of Remark 2.6 and Proposition 2.14 we have

PROPOSITION 2.15. *Let H_1 and H_2 be normal filters of A. Then*

$$[H_1 \cup H_2) = \{x \in A : h_1 \odot h_2 \leq x \text{ for some } h_1 \in H_1 \text{ and } h_2 \in H_2\}.$$

Following [3], for any normal filter H of A, we define a congruence $\equiv_H$ on A by

$$x \equiv_H y \Leftrightarrow (x \to y) \odot (y \to x) \in H.$$

We also have $x \equiv_H y \Leftrightarrow (x \rightsquigarrow y) \odot (y \rightsquigarrow x) \in H$. Applying Proposition 2.2 (c) we get

(1) $$x \equiv_H y \Leftrightarrow x \to y,\ y \to x \in H \Leftrightarrow x \rightsquigarrow y,\ y \rightsquigarrow x \in H.$$

In [3] it is proved that the map $H \to \equiv_H$ is an isomorphism between the lattice of normal filters and the lattice of congruences of A. We denote by x/H the congruence class of an element $x \in A$, that is, $x/H = x/\equiv_H$. On the set $A/H = \{x/H : x \in A\}$ we define the natural operations induced from those of A. The resulting quotient algebra $(A/H; \vee, \wedge, \odot, \to, \rightsquigarrow, 0/H, 1/H)$ becomes a pseudo BL-algebra, called the *quotient algebra of A by the normal filter H*. The map $\varphi : A \to A/H$, defined by $\varphi(x) = x/H$ for all $x \in A$, is a homomorphism from A onto the quotient pseudo BL-algebra A/H.

PROPOSITION 2.16. ([9]) *Let H be a normal filter of A and let $x \in A$. Then:*

(a) $x/H = 1/H \Leftrightarrow x \in H$;
(b) $x/H = 0/H \Leftrightarrow x^\sim \in H \Leftrightarrow x^- \in H$.

If $\varphi : A \to B$ is a homomorphism of pseudo BL-algebras, then the kernel of φ is the set $\mathrm{Ker}(\varphi) = \{x \in A : \varphi(x) = 1\}$. The following propositions are easily obtained:

PROPOSITION 2.17. *Let $\varphi : A \to B$ be a homomorphism of pseudo BL-algebras. Then:*

(a) $\mathrm{Ker}(\varphi)$ *is a normal filter of A;*
(b) $A/\mathrm{Ker}(\varphi) \cong B$.

Proposition 2.18. *Let H be a normal filter of A. Then there is a bijection between the filters of A containing H and the filters of A/H.*

Proposition 2.19. *Let $H_1, \ldots, H_m$ be normal filters of A such that $[H_i \cup H_j) = A$ for $i, j = 1, \ldots, m$ and $i \neq j$. Let $x_1, \ldots, x_m \in A$. Then there is* $x \in A$ such that $x \equiv_{H_i} x_i$ *for $i = 1, \ldots, m$.*

Proof. First, let $m = 2$. Since $[H_1 \cup H_2) = A$, by Proposition 2.15 there exist $h_{12} \in H_1$ and $h_{21} \in H_2$ such that $h_{12} \odot h_{21} = 0$. From Proposition 2.3 (a) we have $h_{12} \leq h_{21}^{-}$. Then $h_{21}^{-} \in H_1$, and hence $h_{21} \equiv_{H_1} 0$ by Proposition 2.16. Since $h_{12} \leq h_{21}^{-}$, applying Proposition 2.3 (c) we get $(h_{21}^{-})^{\sim} \leq h_{12}^{\sim}$. From Proposition 2.3 (d) we obtain $h_{21} \leq (h_{21}^{-})^{\sim}$. Therefore, $h_{21} \leq h_{12}^{\sim}$, and consequently, $h_{12}^{\sim} \in H_2$. Proposition 2.16 now shows that $h_{12} \equiv_{H_2} 0$.

Pick $x = (h_{12} \odot x_1) \vee (h_{21} \odot x_2)$, where $x_1, x_2 \in A$. Note that using Proposition 2.2 (d) we have

$$\begin{aligned} x/H_1 &= (h_{12}/H_1 \odot x_1/H_1) \vee (h_{21}/H_1 \odot x_2/H_1) \\ &= (1/H_1 \odot x_1/H_1) \vee (0/H_1 \odot x_2/H_1) \\ &= x_1/H_1. \end{aligned}$$

Thus $x \equiv_{H_1} x_1$. Similarly, $x \equiv_{H_2} x_2$. Now let m be arbitrary. For $i, j = 1, \ldots, m$ and $i \neq j$, there exist $h_{ij} \in H_i$ and $h_{ji} \in H_j$ such that $h_{ij} \odot h_{ji} = 0$. Considering $x = \bigvee_{i=1}^{m} (h_{i1} \odot \cdots \odot h_{i,i-1} \odot h_{i,i+1} \odot \cdots \odot h_{i,m} \odot x_i)$ and reasoning as above we see that $x \equiv_{H_i} x_i$ for $i = 1, \ldots, m$. ∎

Let I be a nonempty set and $(A_i : i \in I)$ be the indexed system of pseudo BL-algebras. The direct product $\prod(A_i : i \in I)$ is defined in the usual way. We will denote by $\pi_i, i \in I$, the i-th projection function.

Proposition 2.20. *Let $A_1, \ldots, A_k$ be pseudo BL-algebras and let $A = A_1 \times \cdots \times A_k$. Then*

$$\mathrm{Fil}(A) = \mathrm{Fil}(A_1) \times \cdots \times \mathrm{Fil}(A_k).$$

Proof. $F_i \in \mathrm{Fil}(A_i)$ for $i = 1, \ldots, k$, then $F_1 \times \cdots \times F_k$ is a filter of A. Conversely, if F is a filter of A, then for $i = 1, \ldots, k$, $F_i = \pi_i(F)$ is a filter of A_i and $F = F_1 \times \cdots \times F_k$. From this we conclude that the assertion follows. ∎

Proposition 2.21. ([3]) *Let H be a proper normal filter of A. Then A/H is a pseudo BL-chain if and only if H is a prime filter of A.*

An algebra A is *simple* if A has exactly two congruences: $0_A = \{(x, x) : x \in A\}$ and $1_A = A^2$. Clearly, a pseudo BL-algebra A is simple if it has a unique proper normal filter. Observe that a nontrivial pseudo BL-chain

A is simple if and only if $\mathrm{Fil}(A) = \{\{1\}, A\}$. Indeed, let A be simple and $F \neq \{1\}$ be a proper filter of B. By Proposition 2.9, F can be extended to an ultrafilter M. From Proposition 2.11 we see that M is normal. This contradicts the fact that A is simple. Then $\mathrm{Fil}(A) = \{\{1\}, A\}$. The converse is obvious.

Proposition 2.21 and Proposition 3.2 of [4] together yield.

Proposition 2.22. *A normal filter H of A is maximal if and only if A/H is a simple pseudo BL-chain.*

Let $B(A)$ be the Boolean algebra of all complemented elements in the distributive lattice $\mathcal{L}(A) = (A; \vee, \wedge, 0, 1)$.

Proposition 2.23. ([9]) *If $e \in B(A)$, then $[e) = \{x \in A : e \leq x\}$ and $([e); \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, e, 1)$ is a pseudo BL-algebra.*

Proposition 2.24. ([9]) *If $e \in B(A)$ and $x \in A$, then:*

(a) $e \odot x = e \wedge x$;
(b) $e \vee e^- = 1$ *and* $e \wedge e^- = 0$;
(c) $e^- = e^\sim$ *is the complement of* e.

Proposition 2.25. ([9]) *If $A = A_1 \times A_2$, then there is $e \in B(A)$ such that $A_1 \cong [e)$ and $A_2 \cong [e^-)$.*

Proposition 2.26. *If $x \in A$ and $e \in B(A)$, then $(x \vee e) \odot (x \vee e^-) = x$.*

Proof. Applying Propositions 2.2 (f, g) and 2.24 we have

$$\begin{aligned}(x \vee e) \odot (x \vee e^-) &= [(x \vee e) \odot x] \vee [(x \vee e) \odot e^-] \\ &= [(x \odot x) \vee (e \odot x)] \vee [(x \odot e^-) \vee (e \odot e^-)] \\ &= (x \odot x) \vee (x \wedge e) \vee (x \wedge e^-) \\ &= (x \odot x) \vee [x \wedge (e \vee e^-)] = (x \odot x) \vee x = x. \end{aligned}$$ ∎

3. Semisimple pseudo BL-algebras

Definition 3.1. A pseudo BL-algebra A is *semisimple* if the intersection of all maximal congruences of A is the congruence 0_A.

Since, in a pseudo BL-algebra A, the congruences are in bijective correspondence with the normal filters, it follows that A is *semisimple* if and only if $\mathcal{M}_{\mathrm{n}}(A) = \{1\}$. Obviously, every simple pseudo BL-algebra is semisimple.

Theorem 3.2. *Let A be a pseudo BL-algebra. The following are equivalent:*

(a) *A is semisimple;*
(b) *there is a family $\{H_i : i \in I\}$ of normal ultrafilters of A with $\bigcap\{H_i : i \in I\} = \{1\}$;*
(c) *A is a subdirect product of simple pseudo BL-chains.*

Proof. (a) $\Rightarrow$ (b): Follows from definition.

(b) $\Rightarrow$ (c): Let $\{H_i : i \in I\}$ be a family of normal ultrafilters of A such that $\bigcap\{H_i : i \in I\} = \{1\}$. Write $A_i = A/H_i$ for $i \in I$. From Proposition 2.22 we deduce that A_i are simple pseudo BL-chains. Now, define $\varphi : A \to \prod(A_i : i \in I)$ by

$$\varphi(x) = (x/H_i : i \in I) \text{ for all } x \in A.$$

Evidently, φ is a homomorphism. Let $\varphi(x) = \varphi(y)$. Then $x/H_i = y/H_i$ for all $i \in I$. By (1), $x \to y$, $y \to x \in \bigcap\{H_i : i \in I\} = \{1\}$. Therefore, $x \to y = y \to x = 1$. From Proposition 2.2 (a) it follows that $x = y$. Consequently, φ is injective. It is easy to see that $\pi_i \circ \varphi$ maps A onto A_i. Thus A is a subdirect product of the simple pseudo BL-chains A_i, $i \in I$.

(c) $\Rightarrow$ (a): Let $\psi : A \to \prod(A_i : i \in I)$ be an injective homomorphism, where A_i are simple BL-chains, and let $\pi_i \circ \psi : A \to A_i$ be surjective. Set $\mathrm{Ker}(\pi_i \circ \psi) = H_i$ for $i \in I$. From Proposition 2.17 we conclude that H_i is a normal filter of A and $A/H_i \cong A_i$. In consequence, A/H_i is simple. By Proposition 2.22, H_i is maximal. Let $x \in \bigcap\{H_i : i \in I\}$. Then $\pi_i(\psi(x)) = 1$ for all $i \in I$, and hence $\psi(x) = 1$. Since ψ is injective we obtain $x = 1$. Therefore, $\bigcap\{H_i : i \in I\} = \{1\}$. Consequently, A is semisimple. ∎

Proposition 3.3. *Any subalgebra of semisimple pseudo BL-algebra is semisimple.*

Proof. Let A be a semisimple pseudo BL-algebra and let B be a subalgebra of A. By Theorem 3.2, there is a family $\{H_i : i \in I\}$ of normal ultrafilters of A such that $\bigcap\{H_i : i \in I\} = \{1\}$. Observe that $H_i \cap B \in \mathrm{Max_n}(B)$ for each $i \in I$. By definition, $H_i \cap B$ is a normal proper filter of B and from Proposition 2.12 we see that it is maximal. Moreover,

$$\bigcap\{H_i \cap B : i \in I\} = \left(\bigcap\{H_i : i \in I\}\right) \cap B = \{1\} \cap B = \{1\}.$$

Now, applying Theorem 3.2 we conclude that B is a semisimple pseudo BL-algebra. ∎

Proposition 3.4. *Let A_1 and A_2 be semisimple pseudo BL-algebras. Then the direct product $A = A_1 \times A_2$ is also semisimple.*

Proof. By Theorem 3.2 there exist families $\{H_i : i \in I_1\} \subseteq \mathrm{Max_n}(A_1)$ and $\{F_i : i \in I_2\} \subseteq \mathrm{Max_n}(A_2)$ such that $\bigcap\{H_i : i \in I_1\} = \{1\}$ and $\bigcap\{F_i : i \in I_2\} = \{1\}$. Let

$$U_i = \begin{cases} H_i \times A_2 & \text{if } i \in I_1, \\ A_1 \times F_i & \text{if } i \in I_2. \end{cases}$$

We set $I = I_1 \cup I_2$. It is clear that U_i $(i \in I)$ are normal ultrafilters of A and $\bigcap\{U_i : i \in I\} = \{1\}$. Consequently, A is a semisimple pseudo BL-algebra. ∎

In a similar way, we get the following more general result.

THEOREM 3.5. *Any direct product of semisimple pseudo BL-algebras is a semisimple pseudo BL-algebra.*

From Proposition 3.3 and Theorem 3.5 we have

COROLLARY 3.6. *The class of all semisimple pseudo BL-algebras is closed under the formation of subalgebras and direct products.*

PROPOSITION 3.7. *The class of all semisimple pseudo BL-algebras is not closed under the formation of ultraproducts (and hence it is not a quasivariety).*

Proof. Let $[0,1]$ be the unit interval of real numbers $\mathbb{R}$. For any $x, y \in \mathbb{R}$, define $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$. For $x, y \in [0,1]$ we put

$$x \odot y = (x + y - 1) \vee 0 \text{ and } x \to y = (y - x + 1) \wedge 1.$$

Then $A = ([0,1]; \vee, \wedge, \odot, \to, \to, 0, 1)$ is a (pseudo) BL-chain. Proposition 2.11 shows that $\mathcal{M}_\mathrm{n}(A) = \{x \in A : x^n > 0 \text{ for all } n \in \mathbb{N}\}$. It is easy to see that

$$x^n = [n(x - 1) + 1] \vee 0$$

for $x \in [0,1]$ and $n \in \mathbb{N}$. We have

$$x^n > 0 \Leftrightarrow n(x - 1) + 1 > 0 \Leftrightarrow n(1 - x) < 1.$$

Hence, if $x^n > 0$ for all $n \in \mathbb{N}$, then $x = 1$. Therefore, $\mathcal{M}_\mathrm{n}(A) = \{1\}$ and consequently, A is semisimple.

Let $\mathcal{F}$ be an ultrafilter over $\mathbb{N}$ containing all cofinite subsets of $\mathbb{N}$. Let B be the ultrapower of A determined by $\mathcal{F}$, in symbols, $B = A^{\mathbb{N}}/\mathcal{F}$. By the fundamental ultraproduct theorem, B is a (pseudo) BL-algebra. Let $b = (b_k : k \in \mathbb{N})$, where $b_k = 1 - \frac{1}{k}$ for $k \in \mathbb{N}$. We prove that

$$(b^n)^-/\mathcal{F} \leq b/\mathcal{F} \text{ for all } n \in \mathbb{N}. \tag{2}$$

Fix $n \in \mathbb{N}$ and let $k > n$. We have $b_k^n = [n(b_k - 1) + 1] \vee 0 = 1 - \frac{n}{k}$, and hence $(b_k^n)^- = 1 - b_k^n = \frac{n}{k} \leq 1 - \frac{1}{k} = b_k$. From this we obtain (2). Observe that $b/\mathcal{F} \in \mathrm{Max_n}(B)$. On the contrary, suppose that $b/\mathcal{F} \notin H$ for some normal ultrafilter H of B. By Proposition 2.12, there is $m \in \mathbb{N}$ such that $[(b/\mathcal{F})^m]^- \in H$. From (2) it follows that $[(b/\mathcal{F})^m]^- = (b^m)^-/\mathcal{F} \leq b/\mathcal{F}$. Therefore, $b/\mathcal{F} \in H$. This contradiction shows that $b/\mathcal{F} \in \mathrm{Max_n}(B)$. Since $b/\mathcal{F} \neq 1/\mathcal{F}$, $\mathrm{Max_n}(B) \neq \{1/\mathcal{F}\}$. Thus B is not semisimple. ∎

We shall say that a pseudo BL-algebra is *representable* if it can be represented as a subdirect product of pseudo BL-chains. Kühr [14] proved that A is a representable pseudo BL-algebra if and only if there exists a family $\{P_i : i \in I\}$ of normal prime filters of A such that $\bigcap\{P_i : i \in I\} = \{1\}$. Consequently, if A is semisimple, then A is representable. The converse implication is not true in general, that is, the question of [3] (Problem 1.33) has a negative answer. Indeed, the class of representable BL-algebras is a variety (see Theorem 3.4 of [14]) but the class of semisimple BL-algebras is not a variety.

4. Semilocal pseudo BL-algebras

Definition 4.1. A pseudo BL-algebra is called *semilocal* if it has only finitely many normal ultrafilters.

Theorem 4.2. *Let A be a pseudo BL-algebra. The following are equivalent:*

(a) *A is semilocal;*

(b) *$A/\mathcal{M}_n(A)$ is isomorphic to a direct product of finitely many simple pseudo BL-chains;*

(c) *$A/\mathcal{M}_n(A)$ has finitely many filters.*

Proof. For now on throughout our proof, we will let U stand for $\mathcal{M}_n(A)$.

(a) $\Rightarrow$ (b): Assume that A is semilocal. If $\mathrm{Max_n}(A) = \emptyset$, then $A/U = A/A$ is a one-element pseudo BL-algebra and so it is the direct product of empty family of algebras. Now, let $\{H_1, \ldots, H_k\}$ be the set of all normal ultrafilters of A. Then $U = H_1 \cap \cdots \cap H_k$. By Proposition 2.22, A/H_i are simple pseudo BL-chains. We define the map $\varphi : A/U \to A/H_1 \times \cdots \times A/H_k$ by $\varphi(x/U) = (x/H_1, \ldots, x/H_k)$. Then φ is clearly a homomorphism. We show that φ is an isomorphism. Let $(x_1/H_1, \ldots, x_k/H_k) \in A/H_1 \times \cdots \times A/H_k$. Since $[H_i \cup H_j) = A$ for $i, j = 1, \ldots, k$ and $i \neq j$, we conclude (by Proposition 2.19) that there exists $x \in A$ such that $x/H_i = x_i/H_i$ for $i = 1, \ldots, k$. Hence $(x_1/H_1, \ldots, x_k/H_k) = (x/H_1, \ldots, x/H_k) = \varphi(x/U)$. Consequently, φ is surjective. Now, it suffices to show that φ is injective. Suppose that $\varphi(x/U) = \varphi(y/U)$ for $x, y \in A$. Hence $x/H_i = y/H_i$ for each

$i = 1, \ldots, k$. Then $x \to y \in H_i$ and $y \to x \in H_i$ for $i = 1, \ldots, k$, that is, $x \to y \in U$ and $y \to x \in U$. Therefore, $x/U = y/U$. It is proved that φ is an isomorphism.

(b) $\Rightarrow$ (c): Let $A/U \cong A_1 \times \cdots \times A_k$, where A_i are simple pseudo BL-chains for $i = 1, \ldots, k$. Proposition 2.20 gives $|\mathrm{Fil}(A/U)| = |\mathrm{Fil}(A_1) \times \cdots \times \mathrm{Fil}(A_k)|$. Since $\mathrm{Fil}(A_i)$ has two elements for every $i = 1, \ldots, k$, we have $|\mathrm{Fil}(A/U| = 2^k$. Thus A/U has finitely many filters.

(c) $\Rightarrow$ (a): To obtain a contradiction, suppose that A has infinitely many normal ultrafilters F_n, $n \in \mathbb{N}$. Obviously, all F_n/U are filters of A/U. Observe that

$$F/U = F'/U \Rightarrow F = F' \tag{3}$$

for all $F, F' \in \mathrm{Max_n}(A)$. Let $F/U = F'/U$ and let $x \in F$. Then $x/U \in F'/U$ and hence $x/U = y/U$ for some $y \in F'$. By (1), $y \to x \in U \subseteq F'$. Consequently, $x \wedge y = (y \to x) \odot y \in F'$. Therefore, $x \in F'$. This clearly forces $F \subseteq F'$. Similarly, $F' \subseteq F$, and we obtain $F = F'$. Thus (3) holds. From (3) it follows that A/U has infinitely many filters F_n/U, $n \in \mathbb{N}$, which is impossible. ∎

DEFINITION 4.3. Let $\{a_i : i \in I\}$ be a family of elements of a pseudo BL-algebra A and $\{H_i : i \in I\}$ be a family of normal filters of A. We say that the family $\{(a_i, H_i) : i \in I\}$ has a property (P) if for any finite subset J of I, there is $x_J \in A$ with $x_J \equiv_{H_i} a_i$ for any $i \in J$.

DEFINITION 4.4. A is called *maximal* if for any family $\{(a_i, H_i) : i \in I\}$ with property (P) there exists $x \in A$ such that $x \equiv_{H_i} a_i$ for any $i \in I$.

REMARK 4.5. If A has finitely many normal filters, then A is maximal. Hence any simple pseudo BL-algebra is maximal.

LEMMA 4.6. *A finite direct product of maximal pseudo BL-algebras is a maximal pseudo BL-algebra.*

Proof. We only need to prove that if A_1 and A_2 are maximal, then $A = A_1 \times A_2$ is also maximal. By Proposition 2.25, $A_1 \simeq [e)$ and $A_2 \simeq [e^-)$ with $e \in B(A)$. Let H be a normal filter of A. From Proposition 2.24 we conclude that $[e)$ is a normal filter of A. Therefore, $H \cap [e)$ is also a normal filter of A. Let $x, y \in A$ and $x \equiv_H y$. We show that $x \vee e \equiv_{H \cap [e)} y \vee e$. Since $x \equiv_H y$, we have $x \to y$, $y \to x \in H$. It suffices to prove that $x \vee e \to y \vee e$, $y \vee e \to x \vee e \in H \cap [e)$. By Proposition 2.2 (b), $x \vee e \to y \vee e \geq y \vee e \in [e)$, that is, $x \vee e \to y \vee e \in [e)$. From Proposition 2.2 (e) we obtain $x \vee e \to y \vee e \geq x \to y \in H$. Therefore, $x \vee e \to y \vee e \in H$. So $x \vee e \to y \vee e \in H \cap [e)$ and similarly, $y \vee e \to x \vee e \in H \cap [e)$. Thus $x \vee e \equiv_{H \cap [e)} y \vee e$. Likewise, we can prove that $x \vee e^- \equiv_{H \cap [e^-)} y \vee e^-$.

Now let $\{(a_i, H_i) : i \in I\}$ be a family in A with the property (P). Then the families $\{(a_i \vee e, H_i \cap [e)) : i \in I\}$ and $\{(a_i \vee e^-, H_i \cap [e^-)) : i \in I\}$ verify the property (P) in maximal pseudo BL-algebras $[e)$ and $[e^-)$, respectively. Let $y \in [e)$ and $z \in [e^-)$ such that $y \equiv_{H_i \cap [e)} a_i \vee e$ and $z \equiv_{H_i \cap [e^-)} a_i \vee e^-$ for any $i \in I$. Hence $y \odot z \equiv_{F_i} (a_i \vee e) \odot (a_i \vee e^-)$, and from Proposition 2.26 we conclude that $y \odot z \equiv_{H_i} a_i$. ∎

THEOREM 4.7. *If A is a maximal pseudo BL-algebra, then it is semilocal.*

Proof. Let $\mathcal{G} = \{(x_H, H) : x_H \in A, H \in \mathrm{Max_n}(A)\}$. Observe that the family $\mathcal{G}$ has the property (P). Indeed, let $\{H_1, \ldots, H_m\} \subseteq \mathrm{Max_n}(A)$. Since $[H_i \cup H_j) = A$ for $i \neq j$, we conclude from Proposition 2.19 that there exists $x^* \in A$ such that $x^* \equiv_{H_i} x_{H_i}$ for $i = 1, \ldots, m$. Thus $\mathcal{G}$ satisfies (P).

Let $F = \{x \in A : \{H \in \mathrm{Max_n}(A) : x \notin H\}$ is finite$\}$. It is easily seen that F is a normal filter of A. Let us consider the family

$$\mathcal{H} = \{(1, F)\} \cup \{(0, H) : H \in \mathrm{Max_n}(A)\}.$$

We will show that $\mathcal{H}$ has the property (P). Take a subfamily

$$\{(1, F), (0, H_1), \ldots, (0, H_m)\}$$

of $\mathcal{H}$. It is obvious that

$$\bigcap \{H : H \in \mathrm{Max_n}(A) - \{H_1, \ldots, H_m\}\} \subseteq F. \tag{4}$$

Since $\mathcal{G}$ satisfies (P), the family

$$\{(0, H_1), \ldots, (0, H_m)\} \cup \{(1, H) : H \in \mathrm{Max_n}(A) - \{H_1, \ldots, H_m\}\}$$

also satisfies (P). By assumption, A is maximal, and hence there is $x \in A$ such that $x/H_i = 0/H_i$ for all $i = 1, \ldots, m$ and $x/H = 1/H$ for all $H \in \mathrm{Max_n}(A) - \{H_1, \ldots, H_m\}$. Proposition 2.16 shows that $x \in H$ for all $H \neq H_1, \ldots, H_m$. We conclude from (4) that $x \in F$, what implies that $x/F = 1/F$. Therefore, $\mathcal{H}$ has the property (P).
By hypothesis, there exists $y \in A$ such that $y/F = 1/F$ and $y/H = 0/H$ for all $H \in \mathrm{Max_n}(A)$. From this we deduce that $y \in F$ and $y^\sim \in H$ for any $H \in \mathrm{Max_n}(A)$. Applying Proposition 2.3 (e) we see that $y \notin H$ for all $H \in \mathrm{Max_n}(A)$. It follows that $\mathrm{Max_n}(A) = \{H \in \mathrm{Max_n}(A) : y \notin H\}$. Since $y \in F$, we conclude that $\mathrm{Max_n}(A)$ is finite. Hence A is semilocal. ∎

THEOREM 4.8. *For a pseudo BL-algebra A, the following are equivalent:*
(a) *A is semisimple and maximal;*
(b) *A is semisimple and semilocal;*
(c) *A is isomorphic to a direct product of finitely many simple pseudo BL-chains;*
(d) $|\mathrm{Max_n}(A)| < \aleph_0$ *and* $\mathcal{M}_\mathrm{n}(A) = \{1\}$.

Proof. (a) $\Rightarrow$ (b): Let A be semisimple and maximal. By Theorem 4.7 we have (b).

(b) $\Rightarrow$ (c): Follows from Theorem 4.2.

(c) $\Rightarrow$ (d): Let $A \cong B = A_1 \times \cdots \times A_k$, where A_i are simple pseudo BL-chains. It is clear that F is an ultrafilter of B if and only if there is an $i \in \{1, \ldots, k\}$ such that $F = A_1 \times \cdots \times A_{i-1} \times F_i \times A_{i+1} \times \cdots \times A_k$, where $F_i = \{1\}$ is the unique ultrafilter of A_i. Hence (d) holds.

(d) $\Rightarrow$ (a): By definition, A is semisimple. From Remark 4.5 we see that A is maximal. ∎

From Theorem 4.8 we have

COROLLARY 4.9. *Let A be a semisimple pseudo BL-algebra. Then A is maximal if and only if A is semilocal.*

References

[1] C. C. Chang, *Algebraic analysis of many valued logics*, Trans. Amer. Math. Soc. 88 (1958), 467-490.

[2] A. Di Nola, G. Georgescu, A. Iorgulescu, *Pseudo-BL algebras I*, Multiple-Valued Logic 8 (2002), 673–714.

[3] A. Di Nola, G. Georgescu, A. Iorgulescu, *Pseudo-BL algebras II*, Multiple-Valued Logic 8 (2002), 715–750.

[4] A. Dvurečenskij, J. Rachůnek, *Probabilistic averaging in bounded Rℓ-monoids*, Semigroup Forum 72 (2006), 190–206.

[5] A. Dvurečenskij, *Every linear pseudo BL-algebra admits a state*, Soft Computing 11 (2007), 495–501.

[6] G. Georgescu, A. Iorgulescu, *Pseudo MV-algebras: a noncommutative extension of MV-algebras*, The Proceedings of the Fourth International Symposium on Economic Informatics, Bucharest, Romania, May 1999, 961–968.

[7] G. Georgescu, A. Iorgulescu, *Pseudo BL-algebras: a noncommutative extension of BL-algebras*, Abstracts of the Fifth International Conference FSTA 2000, Slovakia 2000, 90–92.

[8] G. Georgescu, A. Iorgulescu, *Pseudo BCK-algebras: an extension of BCK-algebras.* Proceedings of DMTCS'01: Combinatorics, Computability and Logic, Springer, London 2001, 97–114.

[9] G. Georgescu, L. L. Leustean, *Some classes of pseudo-BL algebras.* J. Austral. Math. Soc. 73 (2002), 127–153.

[10] P. Hájek, *Metamathematics of Fuzzy Logic,* Kluwer, Amsterdam, 1998.

[11] P. Hájek, *Fuzzy logics with noncommutative conjuctions*, J. Logic Comput. 13 (2003), 469–479.

[12] P. Hájek, *Observations on non-commutative fuzzy logic,* Soft Computing 8 (2003), 38–43.

[13] A. Iorgulescu, *Pseudo Iséki algebras. Connections with pseudo BL-algebras,* Journal of Multiple-Valued Logic and Soft Computing 11 (2005), 263–308.

[14] J. Kühr, *Pseudo BL-algebras and DRℓ-monoids*, Math. Bohem. 128 (2003), 199–208.

[15] J. Rachůnek, *A non-commutative generalizations of MV-algebras*, Math. Slovaca 52 (2002), 255–273.
[16] A. Walendziak, M. Wojciechowska, *Another axiomatization of pseudo BL-algebras*, to appear.

Andrzej Walendziak
WARSAW INFORMATION TECHNOLOGY
Newelska 6
01–447 WARSZAWA, POLAND
E-mail: walent@interia.pl

Magdalena Wojciechowska
INSTITUTE OF MATHEMATICS AND PHYSICS
UNIVERSITY OF PODLASIE
3 Maja 54
08–110 SIEDLCE, POLAND
E-mail: magdawojciechowska6@wp.pl

Received August 30, 2008; revised version January 30, 2009.