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APPROXIMATION OF MARKET VALUATIONS ON THE SET OF RISK MEASURES

Abstract. In this work we introduce the problem of choice of a risk measure providing best approximation of risk estimates derived from market valuations. We begin with a brief overview of connections between pricing and risk measurement issues which reveal importance of the problem we consider and lead to the mathematical formulation. In the main result under fairly general assumptions we establish the existence of the solution. In the second part we define a problem of finding a risk measure optimal with respect to the capital requirements. We impose additional assumptions, all of which have strong practical justification and in this particular setting we show that a solution exists and is a spectral measure of risk. As an example of application we show that there is some optimal spectral measure of risk for speculative position created in a market model with CIR short rate dynamics.

Introduction

In 2006 Basel Committee published a set of rules briefly called Revised Basel II which describes standards of financial security which must be observed by participants of the financial markets. This code states e.g. that any risky position created by a bank, requires some amount of capital to be held in case of a loss on the position. This is called a capital requirement. The amount to be held obviously depends on the level of risk of the particular position. Key issue is that Basel II allows financial institutions to implement their own systems of risk measurement. Therefore the systems can be suited to the institutions' risk attitudes and profiles of investment.

Calculation of capital requirements is one of the most important problems in modern finance. Another one is pricing financial products in the presence of market incompleteness. Practical difficulties arise when classical martingale valuation methods are employed. Alternative ways of pricing suggested

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by several authors, e.g. Cherny and Hodges [3], Jaschke and Kuchler [7] revealed strong relationships between risk measurement and pricing problems. In fact prices can be derived from risk estimates (called risk numbers) as well as risk measures (functions which assign risk to contingent claims) can be defined by price systems (valuation functionals on contingent claims).

Motivated by this correspondence and Basel II code, we formulate two problems which are of interest for pricing purposes and are related to holding capital requirements. We consider a one period financial market model on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume there are M risky positions $\{X1, \dots, X_M\} = \mathcal{L} \subseteq L_\infty(\Omega, \mathcal{F}, \mathbb{P})$ which can be entered in the market. There is some fixed set $\mathcal{A}$ of the risk measures (to be defined later). We also assume we have an exogenously given vector of risk estimates (risk numbers) $r = (r_1, \dots, r_M) \in \mathbb{R}^M$ corresponding to the positions $X_1, \dots, X_M \in \mathcal{L}$. If we consider a situation in which risk numbers r are somehow derived from the prices observed in the market, we are in a position in which it is reasonable to assume that r is a result of some estimation and hence it may not correspond to actual risk numbers produced by any risk measure $\rho \in \mathcal{A}$. This suggests a very natural

Question 1. Which measure of risk $\rho \in \mathcal{A}$ produces risk numbers for $X \in \mathcal{L}$ which in some sense provide the best approximation to the given risk numbers $r \in \mathbb{R}^M$?

The second problem under consideration is directly related to Basel II. In many financial institutions, e.g. investment banks capital requirements are regarded as a burden. Therefore in this paper we are interested in minimizing capital requirements so that more money is available for speculative purposes.

So far the problem has not been considered in the literature. As we shall see, it can be viewed as a dual-criterion optimization problem in which we look for a risk measurement methodology which is consistent with the security standards imposed by market regulators and at the same time minimizes average capital requirements. We consider the most popular measures of risk, e.g. *VaR*, *TCE*, *WCE*, *ES* and spectral measures of risk (all of which are defined later). We show that the mathematical formulation is a special case of the problem mentioned in Question 1. Hence the existence of the solution would be an easy consequence of the results obtained in the first part of this work. However, in the second part we impose stronger assumptions on the distributions of the payoffs $X \in \mathcal{L}$ under which we are able to tell more about the solution, i.e. we prove that it is a spectral measure of risk. This result is our contribution to the ongoing debate on proper risk measurement methodologies.

We begin with some notation and definitions of risk measures which are considered in this paper. In Section 1 we formulate the problem according to the Question 1 and provide assumptions. We work with bounded random variables X such that $\mathbb{P}(X = \operatorname{ess\,inf} X) = 0$ and consider the most popular risk measures used both in practice and theory, i.e.: $VaR_{\bar{\alpha}}$, $TCE_{\bar{\alpha}}$, $WCE_{\bar{\alpha}}$, $ES_{\bar{\alpha}}$ and spectral measures of risk $\mathcal{M}_{\phi,c}$. In Section 2 we establish the existence of the solution of a problem under our fairly general assumptions. In Section 3 we consider a problem of the existence of a risk measure optimal with respect to the capital requirements as a very special case of the general problem. Under stronger assumptions on the distributions of the payoffs we show that the solution is a spectral measure of risk. Numerical example in CIR model is also provided.

Conventions and notation

Consider some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In this paper any random variable $X \in L_\infty(\Omega, \mathcal{F}, \mathbb{P})$ is interpreted as a risky payoff (not as a loss). F_X denotes distribution function of X and $q_\alpha(X)$ denotes the lower α-quantile of X. As in [5] we have the following

Definition 1. Let $\mathcal{L} \subseteq L_\infty(\Omega, \mathcal{F}, \mathbb{P})$ be a set of some risky payoffs. Any mapping $\rho : \mathcal{L} \to \mathbb{R}$ is called a measure of risk if it is monotone, i.e. $X \leq Y \Rightarrow \rho(X) \geq \rho(Y)$, $X, Y \in \mathcal{L}$ and translation invariant, i.e. $\rho(X + c) = \rho(X) - c$, $X \in \mathcal{L}$, $c \in \mathbb{R}$.

Later we shall consider the most popular measures of risk, namely: Value-at-Risk (VaR), Tail Conditional Expectation (TCE), Worst Conditional Expectation (WCE), Expected Shortfall (ES) and a class of statistics called spectral measures of risk which were introduced in [2]. For convenience we provide the definition.

Definition 2. For $\alpha \in (0, 1]$ we have

1. $VaR_\alpha(X) = -\sup\{x \mid F_X(x) < \alpha\} \stackrel{def}{=} -q_\alpha(X)$,
2. $TCE_\alpha(X) = -E(X|X \leq q_\alpha(X))$
3. $WCE_\alpha(X) = -\inf\{E_A^P(X) |\, A \in \mathcal{F} : P(A) > \alpha\}$, where $P_A(\cdot) = P(\cdot|A)$
4. $ES_\alpha(X) = -\frac{1}{\alpha}\int_0^\alpha F_X^{\leftarrow}(p)dp$
5. $ES_0 = -\inf\{x|F_X(x) > 0\}$,
6. $\mathcal{M}_{\phi,c}(X) = cES_0(x) - (1-c)\int_0^1 \phi(p)F_X^{\leftarrow}(p)dp$, where $F_X^{\leftarrow}(p) = -VaR_p(X)$.

1. General formulation of the problem

We begin with a definition of the risk approximation error

Definition 3. Let $\mathcal{L} = \{X_k : k = 1, \ldots, M\}$ be a set of some risky payoffs, $r \in R^M$, $\Psi : \mathbb{R} \to \mathbb{R}$ a continuous function and by μ denote some probability measure on $2^{\{1,\ldots,M\}}$ such that $\forall k \leq M \quad \tilde{\mu}_M(\{k\}) > 0$. μ is called measure of expectations. For a risk measure $\rho \in \mathcal{A}$, an average Ψ-risk approximation error of a vector of risk numbers r with respect to the measure μ is a number

$$ARAE_r(\rho, \Psi, \mu) = \sum_{k=1}^{M} \Psi(\rho(X_k) - r_k)\mu(\{k\}). \tag{1.1}$$

Measure μ reflects expectations of an investor regarding positions he is going to enter. μ can be naturally interpreted as a frequency-based probability. Argument based on Kolmogorov's Strong Law of Large Numbers shows that the goal of minimizing a risk approximation error per position can be achieved by minimizing $ARAE_r$.

On the other hand if we put $r = 0$ and interpret Ψ as a function which translates risk numbers $\rho(X_k)$ into capital charges $\Psi(\rho(X_k))$ similar argument suggests that $ARAE_0$ is an average capital requirement per position when methodology ρ is employed and charges expressed in money are provided by Ψ. We shall return to this interpretation in Section 3 with $\Psi(x) = x$.

The above considerations motivate the following

Definition 4. Let $\mathcal{L} = \{X_k : k = 1, \ldots, M\}$ be a set of some risky payoffs and let Ψ, r, μ be as in Definition 3. Measure of risk measure providing best approximation of risk numbers r in the class $\mathcal{A}$ with respect to the function Ψ, under measure of expectations μ is a solution of the problem

$$min_{\rho \in \mathcal{A}} ARAE_r(\rho, \Psi, \mu). \tag{1.2}$$

Problem. First establish the existence of the solution of (1.2) under some moderate assumptions on the distributions of payoffs. In the next section apply obtained result to find a risk measure optimal with respect to the capital requirements and show that under stronger assumptions the solution is a spectral measure of risk.

2. Existence of the solution

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We begin with a few important definitions.

Definition 5. Spectral class $\mathbb{H}$ is a set of functions $\phi \in L^1([0,1])$, which satisfy the following: 1. $\phi(p) \geq 0, \quad p \in [0,1]$, 2. $\int_0^1 \phi(p)dp = 1$, 3. ϕ is nonincreasing.

Definition 6. Let $\bar{\alpha} \in (0,1)$. A set of $\bar{\alpha}$-feasible measures of risk is given by

$$\mathcal{A}_{\bar{\alpha}} = \{VaR_{\bar{\alpha}},\, TCE_{\bar{\alpha}},\, WCE_{\bar{\alpha}},\, ES_{\bar{\alpha}},\, \mathcal{M}_{\phi,c},\ c \in [0,1],\, \phi \in \mathbb{H}\}. \tag{2.1}$$

We shall work under the assumption that distributions of the payoffs satisfy

Condition 1. $\mathbb{P}(X = \operatorname{ess\,inf} X) = 0$,
which is not very demanding from the practical point of view.

We have a standard result

Proposition 1. *$X \in L_\infty(\Omega, \mathcal{F}, \mathbb{P})$ satisfies Condition 1 if and only if there exists $\delta > 0$ such that the function $p \to F_X^{\leftarrow}(p)$ is continuous and strictly increasing on $[0, \delta)$.*

Now we are ready to prove the main theorem.

Theorem 1. *Let $\mathcal{L} = \{X_k : X_k$ satisfies Condition 1$\}$, μ be measure of expectations on $2^{\{1,\dots,M\}}$ and let $\mathcal{A}_{\bar{\alpha}}$ be a set of $\bar{\alpha}$-feasible risk measures for some $\bar{\alpha} \in (0,1)$. Then with $\mathcal{A} = \mathcal{A}_{\bar{\alpha}}$ there exist $\bar{\rho} \in \mathcal{A}$ which is a solution of* (1.2).

Proof. We show that Problem (1.2) is equivalent to minimizing continuous function on a compact subset of some Euclidean space.

Let

$$\mathcal{S}_{\mathcal{A}} = \{v \in \mathbb{R}^M : \exists \rho \in \mathcal{A},\, v_k = \rho(X_k),\, k = 1, \dots, M\}$$

and

$$\tilde{\mathcal{S}} = \{v \in \mathbb{R}^M : \exists \mathcal{M}_{\phi,c} : \phi \in \mathbb{H},\, c \in [0,1],\ v_k = \mathcal{M}_{\phi,c}(X_k),\, k = 1, \dots, M\}.$$

Obviously $\tilde{\mathcal{S}} \subseteq \mathcal{S}_{\mathcal{A}}$. Denote $\mathcal{S} = \{M_{\phi,c} : \phi \in \mathbb{H},\, c \in [0,1]\}$. Thus we have a correspondence $A : \mathcal{A} \to \mathcal{S}_{\mathcal{A}}$ given by

$$A(\rho) = (\rho(X_1), \dots, \rho(X_k)) := v^\rho \in \mathbb{R}^M.$$

For $M_{\phi,c} = \rho \in \mathcal{S}$ we shall especially write

$$A(\mathcal{M}_{\phi,c}) = (\mathcal{M}_{\phi,c}(X_1), \dots, \mathcal{M}_{\phi,c}(X_k)) \overset{not.}{=} v^{\phi,c} \in \mathbb{R}^M. \tag{2.2}$$

Let $f(v) = \sum_{k=1}^M \Psi(v_k - r_k)\, \mu(\{k\})$ for $v \in \mathbb{R}^M$. Consider a problem

$$\min_{v \in \mathcal{S}_{\mathcal{A}}} f(v). \tag{2.3}$$

Because $f(v^\rho) = ARAE_r(\rho, \Psi, \mu)$ it is obvious that v^ρ is a solution to (2.3) if and only if ρ is a solution to (1.2). Hence it suffices to show that there exists a solution to (2.3). f is continuous (in Euclidean topology). We shall show that $\mathcal{S}_{\mathcal{A}}$ is compact. Because $\bar{\alpha}$ is fixed it is enough to show that $\tilde{\mathcal{S}}$ is compact.

Boundedness follows almost immediately. Indeed. Denote

$$d = \max_k |\operatorname{ess\,inf} X_k|.$$

Take arbitrary $v^{\phi,c} \in \tilde{\mathcal{S}}$. From (2.2) and the definition of $\mathcal{M}_{\phi,c}$ we obtain

$$\|v^{\phi,c}\|^2 = \sum_{k=1}^{M} (v_k^{\phi,c})^2 = \sum_{k=1}^{M} \Big(cES_0(X_k) - (1-c)\int_0^1 \phi(p) F_{X_k}^{\leftarrow}(p)dp \Big)^2.$$

For every k $ES_0(X) \le d$, $-F_X^{\leftarrow}(p) = VaR_p(X) \le d$. Hence

$$\|v^{\phi,c}\|^2 \le \sum_{k=1}^{M} \Big(cd + (1-c)d\int_0^1 \phi(p)dp \Big)^2 = Md^2 < \infty,$$

which shows boundedness of $\tilde{\mathcal{S}} \subseteq \mathbb{R}^M$.

Closedness. It follows from Proposition 2.2 and Theorem 4.1 in [2] we have that $\forall \mathcal{M} \in \mathcal{S}$ there exists a probability measure ν on $\mathcal{B}([0,1])$ such that

$$\forall X \in L_\infty \qquad \mathcal{M}(X) = \int_0^1 ES_\alpha(X)\nu(d\alpha). \tag{2.4}$$

To prove closedness we shall refer to (2.4) and our assumptions. Let $(v^n)_{n\in\mathbb{N}} \subseteq \tilde{\mathcal{S}}$ be any sequence such that $v^n \to \bar{v} \in \mathbb{R}^M$. To show that $\bar{v} \in \tilde{\mathcal{S}}$, we have to find $\phi \in \mathbb{H}$, $c \in [0,1]$ such that for $k = 1, \ldots, M$ we have

$$\mathcal{M}_{\phi,c}(X_k) = \bar{v}_k.$$

Using (2.2) we see that sequence (v^n) corresponds to some sequence of risk measures $(\mathcal{M}^n)$. For every k we have:

$$\mathcal{M}^n(X_k) = v_k^n \xrightarrow{n} \bar{v}_k := N(X_k). \tag{2.5}$$

To prove that $\bar{v} \in \tilde{\mathcal{S}}$, it suffices to show that $N \in \mathcal{S}$, i.e. there exists a probability measure ν_N such that $\forall X \in L_\infty$ $N(X) = \int_0^1 ES_\alpha(X)\nu_N(d\alpha)$. Observe that:

1. From (2.4) it follows that

$$\forall n\ \exists \nu_n \quad \forall X \in L_\infty\ \mathcal{M}^n(X) = \int_0^1 ES_\alpha(X)\nu_n(d\alpha).$$

2. For $k = 1, \ldots, M$, function $\alpha \to ES_\alpha(X_k)$ is continuous on $[0,1]$. Indeed: continuity on $(0,1]$ follows straightforwardly from Definition 2. We only need to verify continuity at 0. In our case $ES_0(X_k) = -\operatorname{ess\,inf} X_k \stackrel{df.}{=} d_k$. Let $\alpha_n \searrow 0$. It follows from our assumptions and Proposition 1 that for $k = 1, \ldots, M$ the function $\alpha \to VaR_\alpha(X_k)$ is continuous on $[0, \delta)$ and $VaR_0(X_k) = d_k$. Hence $\forall \varepsilon > 0\, \exists m_0$ such that $\forall m > m_0$ we have

$d_k \geq ES_{\alpha_m}(X_k) = \frac{1}{\alpha_m}\int_0^{\alpha_m} VaR_p(X_k)dp \geq d_k - \varepsilon$. But ε is arbitrary which shows that $ES_\alpha(X_k) \to d_k$ when $\alpha \searrow 0$. This yields continuity of $ES_\alpha(X_k)$ for every $1 \leq k \leq M$.

3. For every n, $\nu_n([0,1]) = 1$, which means the family of probability measures $\{\nu_n\}_{n\in\mathbb{N}}$ is tight. Hence Prohorov Theorem (Theorem 5, Chapter 8, [6]) shows that there exists a subsequence (ν_{n_l}) weakly convergent to some probability measure $\hat{\nu}$. From the definition of weak convergence and 2. we conclude that for $k = 1, \ldots, M$

$$\mathcal{M}^{n_l}(X_k) = \int_0^1 ES_\alpha(X_k)\nu_{n_l}(d\alpha) \xrightarrow{l\to\infty} \int_0^1 ES_\alpha(X_k)\hat{\nu}(d\alpha) = \hat{N}(X_k), \tag{2.6}$$

where $\hat{N} \in \mathcal{S}$ straight from the definition.

Now from (2.5), (2.6) and uniqueness of a limit, it follows that $N = \hat{N} \in \mathcal{S}$. The proof of closedness of $\tilde{\mathcal{S}}$ is finished.

Now, since $\mathcal{S}_\mathcal{A}$ is compact and f is continuous, in (2.3) we conclude that minimum value is admitted for some v^ρ. Corresponding ρ is a solution of (1.2). ∎

3. Application – risk measure optimal with respect to capital requirements

Now, we define the problem of choice of a risk measure optimal with respect to the capital requirements according to the motivations provided in Section 1.

Definition 7. Let $\mathcal{L} = \{X_k : k = 1, \ldots, M\}$ be a set of some risky payoffs. Measure of risk optimal in the class $\mathcal{A}$, under measure of expectations μ is a solution of the problem

$$min_{\rho\in\mathcal{A}} \sum_{k=1}^{M} \rho(X_k)\mu(\{k\}). \tag{3.1}$$

3.1. Results. First we impose stronger assumption on the distribution of the payoffs. The assumptions which have strong practical justification enable us to prove Lemma 1 which is our key argument in the discussion of properties of *VaR* and coherent measures of risk.

We shall consider payoffs X, for which distribution function F_X satisfy:

Condition 2.

- F_X is continuous on $(-\infty, 0)$ and strictly increasing on $(-d, 0)$, for some $0 < d < 1$;
- $F_X(x) = 1$ for $x \geq 0$;
- $\lim_{x\to -d^+} F_X(x) = 0$, $F_X(x) > 0$ for $x > -d$;

- $\lim_{x\to 0^-} F_X(x) = 1 - \mathbb{P}(\{X = 0\}) < 1$.

Payoffs for which Condition 2 is satisfied appear in economies where bonds are tradable securities. A suitable numerical example built in the framework of CIR model is provided in the Paragraph 3.2.

We have standard fact

PROPOSITION 2. *Let X be a random variable such that F_X satisfies Condition 2. Then $VaR_p(X)$ has the following properties: 1. $p \to VaR_p(X)$ is continuous, 2. $VaR_0(X) = d$, $VaR_p(X) < d$ for $p > 0$, 3. $p \in [1 - \mathbb{P}(\{X = 0\}), 1] \Leftrightarrow VaR_p(X) = 0$, 4. $p \to VaR_p(X)$ is nonincreasing.*

We shall use

Assumption 1.

1. In (3.1) we set $\mathcal{L} = \tilde{\mathcal{L}}_M = \{X_k : for\ F_{X_k}$ Condition 2 holds, $k = 1, \ldots, M\}$,
2. μ is measure of expectations on $2^{\{1,\ldots,M\}}$.

We have another assumption, which we justify below.

Assumption 2. In (3.1) we take: $\mathcal{A} = \mathcal{A}_{\bar{\alpha}}$, where $\mathcal{A}_{\bar{\alpha}}$ is a set of $\bar{\alpha}$-feasible measures of risk for some fixed $\bar{\alpha} \in (0, 1)$.

The fact that $\bar{\alpha}$ is fixed models the situation where security standards for risk measurement methodologies are provided by the market regulator. Lack of such standards could pose a serious danger to the whole financial system.

We have chosen $\mathcal{A}$ in order to consider the most popular measures of risk which posses the property that $\rho(X)$ can be straightforwardly interpreted as a capital requirement for a position X.

REMARK 1. It is crucial that $\bar{\alpha}$ is fixed. If we left the choice of $\bar{\alpha}$ to the investor, then under our assumptions $VaR_1(X) = 0$, which yields the minimal risk. However this means that a non-positive payoff requires no capital charge which is not acceptable from the economic point of view.

Denote $p_k = \mathbb{P}(X_k = 0)$, $k = 1, \ldots, M$. We have

Assumption 3. For $\bar{\alpha} \in (0, 1)$ which is considered in Assumption 2, we require that:

$$\forall k = 1, \ldots, M \quad 1 - \bar{\alpha} > p_k > \frac{\bar{\alpha}(1 - VaR_{\bar{\alpha}}(X_k))}{VaR_{\bar{\alpha}}(X_k)}.$$

Number p_k can be interpreted as the probability that our expectations are correct. For the problem to have economic sense, one should think that $\bar{\alpha}$ is 0.05 or smaller. Then the inequalities above can be interpreted as follows: investor takes some risk (probability that he is right does not exceed $1 - \bar{\alpha}$), however he does not speculate aggressively as $p_k > \frac{\bar{\alpha}(1-VaR_{\bar{\alpha}}(X_k))}{VaR_{\bar{\alpha}}(X_k)}$. We claim that in most cases those equations are not very restrictive as it is illustrated in the example presented at the end of the paper.

First we find a spectral measure of risk $\mathcal{M}_{\bar{\phi},0}$ which produces lower risk numbers than $VaR_{\bar{\alpha}}$ pointwise on $\tilde{\mathcal{L}}_M$.

Lemma 1. *Under Assumptions* 1, 2 *and* 3 *there exists* $\bar{\phi}$ *such that*

$$\forall X \in \tilde{\mathcal{L}}_M \qquad \mathcal{M}_{\bar{\phi},0}(X) \leq VaR_{\bar{\alpha}}(X) \tag{3.2}$$

Proof. From the assumptions and in particular from the finiteness of $\tilde{\mathcal{L}}_M$ it follows that there exists $0 < \delta < 1$ such that

$$\forall k \quad p_k > \frac{(\delta + \bar{\alpha}(1-\delta))(1 - VaR_{\bar{\alpha}}(X_k))}{(1-\delta)VaR_{\bar{\alpha}}(X_k)}. \tag{3.3}$$

We can always pick δ small enough, so that $\delta < VaR_{\bar{\alpha}}(X_k)$ for every $k = 1, \ldots, M$. Indeed. From Assumption 3, $\bar{\alpha} < 1 - p_k$ for every k, hence by 3. in Proposition 2 we see that for every k we have $VaR_{\bar{\alpha}}(X_k) > 0$.

Let $b = 1-\delta$, $a = \frac{1-(1-\delta)(1-\bar{\alpha})}{\bar{\alpha}}$. Define $\bar{\phi}(p) = a\mathbb{I}_{[0,\bar{\alpha}]} + b\mathbb{I}_{(\bar{\alpha},1]}$. By verifying Conditions 1, 2, 3 of Definition 5, one easily shows that $\bar{\phi} \in \mathbb{H}$. Now choose any $X = X_k \in \tilde{\mathcal{L}}_M$, for which we show inequality in (3.2). Because X is fixed, for simplicity we shall write: $VaR_p := VaR_p(X)$.

First we show the inequality

$$b\int_{\bar{\alpha}}^{1}(1 - VaR_p)dp \geq 1 - VaR_{\bar{\alpha}}. \tag{3.4}$$

From Proposition 2 we see that for $p \geq 1 - p_k$, we have $VaR_p = 0$, hence $1 - VaR_p = 1$. Furthermore under Assumption 3 we have $1 - p_k > \bar{\alpha}$. It follows that

$$b\int_{\bar{\alpha}}^{1}(1 - VaR_p)dp = b\int_{\bar{\alpha}}^{1-p_k}(1 - VaR_p)dp + bp_k.$$

Now using Proposition 2, we obtain

$$b\int_{\bar{\alpha}}^{1-p_k}(1 - VaR_p)dp \geq b(1 - VaR_{\bar{\alpha}})(1 - p_k - \bar{\alpha}).$$

Using this and (3.3), we arrive at:

$$\begin{aligned} b\int_{\bar{\alpha}}^{1}(1 - VaR_p)dp &\geq b\big(p_k + (1 - VaR_{\bar{\alpha}})(1 - p_k - \bar{\alpha})\big) > \\ &> (1 - VaR_{\bar{\alpha}})\big(\delta + \bar{\alpha}(1-\delta) + (1-\bar{\alpha})(1-\delta)\big) \\ &= 1 - VaR_{\bar{\alpha}}. \end{aligned}$$

Observe that

$$\mathcal{M}_{\bar{\phi},0}(X) = -\int_0^1 \bar{\phi}(p) \underbrace{F_X^{\leftarrow}(p)}_{-VaR_p(X)} dp = \int_0^1 \bar{\phi}(p) VaR_p dp.$$

From Proposition 2: $VaR_p \in (VaR_{\bar{\alpha}}, d) \subseteq (VaR_{\bar{\alpha}}, 1)$ for $p \in (0, \bar{\alpha})$, and hence

$$\begin{aligned}\mathcal{M}_{\bar{\phi},0}(X) &= \int_0^1 \bar{\phi}(p) VaR_p dp = a \int_0^{\bar{\alpha}} VaR_p dp + b \int_{\bar{\alpha}}^1 VaR_p dp \leq \\ &\leq a\bar{\alpha} + b(1-\bar{\alpha}) + b \int_{\bar{\alpha}}^1 ((VaR_p - 1)) dp \overset{(*)}{=} 1 - b \int_{\bar{\alpha}}^1 (1 - VaR_p) dp \leq \\ &\overset{(**)}{\leq} 1 - (1 - VaR_{\bar{\alpha}}) = VaR_{\bar{\alpha}}(X),\end{aligned}$$

where (*) follows from the fact that $\bar{\phi} \in \mathbb{H}$ and (**) follows from (3.4). Thus we obtained (3.2). It is clear that $\mathcal{M}_{\bar{\phi},0}$ is a spectral measure of risk. ■

Now, we are ready to prove the final theorem.

Theorem 2. *Under Assumptions* 1, 2 *and* 3, *there exist* $\hat{\phi} \in \mathbb{H}, \hat{c} \in [0,1]$ *such that* $\mathcal{M}_{\hat{\phi},\hat{c}}$ *is a solution of* (3.1).

Proof. Consider a general Problem (1.2). If we set $r = 0$, $\Psi(x) = x$ and take μ as a measure of expectations, we see that Problem (3.1) is just the special case of (1.2). Since under Assumption 1 the hypotheses of the Theorem 1 hold we conclude that the solution exists. From Corollary 5.2 in [1] it follows that if solution of (3.1) exists, it is either $VaR_{\bar{\alpha}}$ or some spectral measure of risk. Now Lemma 1 shows that minimum value is necessarily attained in the set of the spectral measures of risk. ■

3.2. Example. We shall describe a market model for which Assumptions 1, 2 i 3 are satisfied.

Let $\bar{\alpha} = 0.05$. We consider a set of measures of risk $\mathcal{A} = \mathcal{A}_{\bar{\alpha}}$. Thus Assumption 2 holds.

Construction of the set $\mathcal{L}$ and choice of measure μ. Consider *CIR* model ([4]) on $(\Omega, \mathcal{F}, \mathbb{P}^*)$ with filtration $(\mathcal{F}_t)$, in which dynamics of the short rate is given by $dr_t = (b - ar_t)dt + \sigma\sqrt{r_t}dW_t^1$, where $a, b, \sigma > 0$ and W^1 is $\mathbb{P}^*$-Wiener process. Consider another financial instrument S with dynamics $dS_t = S_t(r_t dt + \nu(t)dW_t^2)$, where ν is a continuous function and W^2 is $\mathbb{P}^*$-Wiener process independent of W^1. It is clear that model extended with this instrument is arbitrage-free.

Fix numbers $0 = t < T_* < T \leq T_\infty$, which have the interpretation: t-present date, T_*-investor's horizon, T-maturity of a bond which investor uses to construct speculative positions, T_∞-market horizon. Furthermore let $\eta : [0, T_*] \to \mathbb{R}$ be some continuous function. Denote $S^*_{T_*} = \frac{S_{T_*}}{B_{T_*}}$ and let $A = \{\omega \in \Omega \ : \ S^*_{T_*}(\omega) \le \eta(T_*)\}$. Define position

$$X_1 = P(T_*, T)(\mathbb{I}_A - 1). \tag{3.5}$$

Setting $\mathcal{L} = \{X_1\}$ and taking measure μ as Dirac delta at $\{1\}$, we are in a situation, in which Assumption 1 holds if F_{X_1} satisfies Condition 2.

Argument which shows that F_{X_1} satisfies Condition 2. First we find distribution function of X_1. From general theory (e.g. [9]) we know that at time t, price of a bond with maturity T is given by $P(t,T) = e^{m(t,T)-n(t,T)r_t}$, where m and n are some functions. One easily verifies that $-1 \le X_1 \le 0$, $\mathbb{P}^* - a.s.$ Hence for $t \le -1$, $F_{X_1}(t) = 0$ and for $t \ge 0$, $F_{X_1}(t) = 1$. We have to investigate the case $-1 < t < 0$. Using the fact that W^1 and W^2 are independent Wiener processes, we easily obtain:

$$\begin{aligned} F_{X_1}(t) &= \mathbb{P}^*(X_1 \le t) = \mathbb{P}^*(A' \cap \{P(T_*, T) \ge -t\}) \\ &= (1 - p_1) F_{r_{T_*}}\left(\frac{m(T_*, T) - \ln(-t)}{n(T_*, T)}\right), \end{aligned}$$

where $p_1 = \mathbb{P}^*(A) = \mathbb{P}^*(\{X_1 = 0\})$.

From on we shall assume that $b = \frac{1}{4}\sigma^2$. In [10] Rogers showed that $r_t = Y_t^2$, where (Y_t) is Ornstein-Uhlenbeck process with mean function $\bar{m}$ and variance function V (compare [8], Chapt. 5, Example 6.8).

It follows that r_{T_*} has a continuous distribution with positive density on $[0, \infty)$. Because $m(T_*, T) < 0$ and $n(T_*, T) > 0$, X_1 has continuous distribution on $[-e^{m(T_*,T)}, 0)$. Observe that $\lim_{t \to 0^-} F_{X_1}(t) = 1 - p_1$. We see that distribution function of X_1 satisfies Condition 2 with $d = e^{m(T_*,T)}$ regardless of the values of parameters a, σ. This means that our choice of $\mathcal{L}$ and μ makes our model consistent with Assumption 1.

Specification of the model so that Assumption 3 holds. Consider the model described above in which $a = 0.5$, $\sigma = 1$, $T = 2$, $T_* = 1$, $\bar{m}(0) = 0.5$, $V(0) = 1$, $S_0 = 1$, $\nu(s) \equiv 1.5$, $\eta(s) \equiv 1$, $s \in [0, T_*]$.

Remark 2. With values of the parameters specified above, we have the situation with natural economic interpretation: X_1 is the instrument which is hedged if investor does not take satisfying profit on the position created by the instrument S.

Numeric computations reveal that: $p_1 = \mathbb{P}^*(X_1 = 0) \approx 0.7734$, in particular $p_1 \in (0.77, 0.78)$, $1 - \bar{\alpha} = 0.95$, $VaR_{\bar{\alpha}}(X_1) \in (0,75, 0.755)$. Hence $\frac{\bar{\alpha}(1 - VaR_{\bar{\alpha}}(X_1))}{VaR_{\bar{\alpha}}(X_1)} < \frac{0.05(1-0.75)}{0.75} = \frac{1}{60} < \frac{77}{100} < 1 - \bar{\alpha}$. This shows that Assumption 3 is satisfied.

References

[1] C. Acerbi, D. Tasche, *On the coherence of the expected shortfall*, Journal of Banking and Finance 26 (2002), 1487–1503.
[2] C. Acerbi, *Spectral measures of risk: a coherent representation of subjective risk aversion*, Journal of Banking & Finance 26 (2002), 1505–1518.
[3] A. Cherny, S. D. Hodges, *The theory of no good-deal pricing in financial markets*, FORC preprint, Imperial College Working Paper, (1999).
[4] J. C. Cox, J. E. Ingersoll, S. A. Ross, *A theory of the term structure of interest rates*, Econometrica 53 (1985), 385–407.
[5] J. Jakubowski, *Modelowanie Rynków Finansowych*, SCRIPT, 2006 (in Polish).
[6] J. Jakubowski, R. Sztencel, *Wstęp do Teorii Prawdopodobieństwa*, SCRIPT, 2001 (in Polish).
[7] S. Jaschke, U. Kuchler, *Coherent risk measures and good-deal bounds*, Finance and Stochastics 5 (2001), 181-200.
[8] I. Karatzas, S. E. Shreve, *Brownian Motion and Stochastic Calculus*, Springer, 2005.
[9] M. Musiela, M. Rutkowski, *Martingale Methods in Financial Modelling*, Springer-Verlag, 1997.
[10] L. C. G. Rogers, *Which model for term-structure of interest rates should one use?*, IMA Vol. Math. Appl. 65 (1995), 93–116.
[11] G. Szego, *Risk Measures for the 21 Century*, John Wiley & Sons, 2004.

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