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ASCOLI'S THEOREMS

Abstract. We present some elegant and pretty general versions of Ascoli theorem for non-Hausdorff spaces. The proofs are simple and elementary (e.g. we do not use the notion of joint continuity) while the results seem to be a very good starting point for more sophisticated theorems. Due to Lemma 10 (it extends [1], Th. 2.2) each of our theorems concerning even continuity is sufficiently general to contain the respective "equicontinuous" one (see Remark 18). Kelley in [2] (as regards the Ascoli theorems) was oriented mainly on continuous mappings while the present paper shows that it is natural first to get "continuous on compacta" results and then the "continuous" versions become their simple consequences.

Let $\mathcal{S}$ be the family of all finite subsets of a set X, and for X being a topological space let $\mathcal{K}$ be the family of all compact subsets of X.

We adopt Y^X to be the family of all mappings on X to Y. If X, Y are topological spaces then $\mathcal{C} \subset Y^X$ is the family of all continuous mappings; for each $\mathcal{A} \subset 2^X$, $\mathcal{C}_\mathcal{A}$ is the family of all mappings that are continuous on the members of $\mathcal{A}$. In particular $\mathcal{C}_\mathcal{K} \subset Y^X$ is the family of all mappings continuous on compacta.

For $(Y, \mathcal{V})$ being a uniform space and $\mathcal{A} \subset 2^X$ let $\mathcal{V}|\mathcal{A}$ be the uniformity (in Y^X) of uniform convergence on members of $\mathcal{A}$ ([2], p. 228). The respective uniform topology in Y^X is denoted by $\mathcal{J}_\mathcal{A}$. In place of $\mathcal{J}_{\{X\}}$ we write $\mathcal{J}$. In particular we have $\mathcal{J}_\mathcal{S} = \mathcal{P}$ ($\mathcal{P}$ - the topology of pointwise convergence); each member V of $\mathcal{V}$ can be treated as a multivalued mapping with $V(y) = \{z \in Y : (y, z) \in V\}$, $y \in Y$.

Remark 1. If each member of $\mathcal{A}$ is contained in a member of $\mathcal{B}$ then we have $\mathcal{J}_\mathcal{A} \subset \mathcal{J}_\mathcal{B}$ and consequently if $\mathcal{A}$ covers X then $\mathcal{P} = \mathcal{J}_\mathcal{S} \subset \mathcal{J}_\mathcal{A}$ holds.

Let us recall ([2], p. 234) that for a topological space X and a uniform space $(Y, \mathcal{V})$ a family $\mathcal{F} \subset Y^X$ is *equicontinuous on a set* $B \subset X$ if the

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following is satisfied

(1) for each $x \in B$, $V \in \mathcal{V}$ there is a neighbourhood U of x such that for all $f \in \mathcal{F}$ we have $f(B \cap U) \subset V(f(x))$.

If $B = X$ then $\mathcal{F}$ is equicontinuous.

Condition (1) means that x is an inner point in B of $B \cap \bigcap\{f^{-1}(V(f(x))) : f \in \mathcal{F}\}$, $x \in B$, $V \in \mathcal{V}$.

Lemma 2. *Let X be a topological space, $\mathcal{A} \subset 2^X$ and let $(Y, \mathcal{V})$ be a uniform space. If $\mathcal{F} \subset \mathcal{C}_{\mathcal{A}}$ is precompact in $(Y^X, \mathcal{V}|\mathcal{A})$ then $\mathcal{F}$ is equicontinuous on members of $\mathcal{A}$.*

Proof. Let us assume that $V, W \in \mathcal{V}$ are symmetric and $W \circ W \circ W \subset V$. Let $B \in \mathcal{A}$ be arbitrary. From the precompactness of $\mathcal{F}$ it follows that there exists a finite set $\mathcal{G} \subset \mathcal{F}$ such that each $f \in \mathcal{F}$ satisfies $f(x) \in W(g(x))$, $x \in B$ for the respective $g \in \mathcal{G}$. From the continuity of $g \in \mathcal{G}$ on B it follows that for each $x \in B$ there exists a neighbourhood U of x such that $g(B \cap U) \subset (W \circ g)(x)$. We may require such U to be common for all $g \in \mathcal{G}$ this last set being finite. For $f \in W \circ g$ (equivalently $g \in W \circ f$) on B we have $f(z) \in (W \circ g)(z)$, $z \in B$ and hence for $z \in B \cap U$ we obtain $f(z) \in (W \circ g)(z) \subset (W \circ g)(B \cap U) \subset (W \circ W \circ g)(x) \subset (W \circ W \circ W \circ f)(x) \subset (V \circ f)(x)$, i.e. $f(B \cap U) \subset (V \circ f)(x)$. This last inclusion means equicontinuity of $\mathcal{F}$ on B, x being arbitrary.

Theorem 3. (cf. [2], Th. 15 p. 232) *Let X be a topological space and let $(Y, \mathcal{V})$ be a uniform space. If $\mathcal{F} \subset Y^X$ is equicontinuous on compacta then $\mathcal{J}_{\mathcal{K}} = \mathcal{P}$ on $\mathcal{F}$.*

Proof. In view of Remark 1 it is sufficient to show that $\mathcal{J}_{\mathcal{K}} \subset \mathcal{P}$ for $\mathcal{F}$. Let us assume that $V, W \in \mathcal{V}$ are symmetric, $V \circ V \circ V \subset W$ and let $B \subset X$ be a compact set. Let us consider a neighbourhood $W \circ g$ of a $g \in \mathcal{F}$ in topology $\mathcal{J}_{\mathcal{A}}$ in $\mathcal{F}$. We will prove that it contains a neighbourhood of g in $\mathcal{P}$. Let $\mathcal{U} = \{B \cap U_x\}_{x \in B}$ be an open cover of B such that $f(B \cap U_x) \subset V(f(x))$, $x \in B$, $f \in \mathcal{F}$ (see (1)). There exists a finite set $Z \subset B$ such that $\{U_z\}_{z \in Z}$ covers B. Then for $\mathcal{H} = \{f \in \mathcal{F}\colon f(z) \in (V \circ g)(z), z \in Z\} \in \mathcal{P}$ we have $f(B \cap U_z) \subset (V \circ f)(z) \subset (V \circ V \circ g)(z)$ and by (1) $g(B \cap U_z) \subset (V \circ g)(z)$, i.e. $g(z) \in (V \circ g)(y)$, $y \in B \cap U_z$. Hence we obtain $f(y) \in (V \circ V \circ V \circ g)(y)$, $y \in B \cap U_z$. In consequence we have $\mathcal{H} \subset W \circ g$ on B, $z \in Z$ being arbitrary.

From Lemma 2 and Theorem 3 we obtain

Theorem 4. *Let X be a topological space and $(Y, \mathcal{V})$ a uniform space. A family $\mathcal{F} \subset \mathcal{C}_{\mathcal{K}}$ is precompact (compact) in $(Y^X, \mathcal{V}|\mathcal{K})$ iff*

(i) *$\mathcal{F}$ is equicontinuous on compacta*

(ii) *$\mathcal{F}$ is precompact in $(Y^X, \mathcal{V}|\mathcal{S})$ (compact in $\mathcal{P}$).*

Let us recall that $\mathcal{J}_{\mathcal{S}} = \mathcal{P}$.

A topological space X is called a k_3-space ([3], p. 195) if $\mathcal{C}_{\mathcal{K}} \subset Y^X$ equals to $\mathcal{C} \subset Y^X$ for every regular space Y. In particular every k-space (and consequently locally compact space or space satisfying the first countability axiom) is a k_3-space. There exist k_3-spaces which are not k-spaces ([3], Th. 5.6(i)).

Clearly if continuity of a mapping f on a family of sets implies that f is continuous then the analog holds for equicontinuity of a family $\mathcal{F}$ (see the comment to condition (1)). Every uniform space is regular ([2], p. 180). In view of these considerations the following is a consequence of Th. 4.

THEOREM 5. *Let X be a k_3-space and $(Y, \mathcal{V})$ a uniform space. A family $\mathcal{F} \subset \mathcal{C}$ is precompact (compact) in $(Y^X, \mathcal{V}|\mathcal{K})$ iff*

(i) *$\mathcal{F}$ is equicontinuous,*

(ii) *$\mathcal{F}$ is precompact in $(Y^X, \mathcal{V}|\mathcal{S})$ (compact in $\mathcal{P}$).*

Now we are going to obtain more practical versions of the previous theorems.

LEMMA 6. *Let X be a topological space, $\mathcal{A} \subset 2^X$ and let $(Y, \mathcal{V})$ be a uniform space. If $\{x\} \in \mathcal{A}$ and $\mathcal{F} \subset Y^X$ is precompact (compact) in $(Y^X, \mathcal{V}|\mathcal{A})$ then $\mathcal{F}(x)$ is precompact (compact).*

Proof. Assume $\mathcal{F}$ is precompact. Then for each $V \in \mathcal{V}$ there exists a finite set $\mathcal{G} \subset \mathcal{F}$ such that $\mathcal{F} \subset V(\mathcal{G})$ on $\{x\}$, i.e. $\mathcal{F}(x) \subset V(\mathcal{G}(x))$. The set $\mathcal{G}(x)$ is finite and consequently $\mathcal{F}(x)$ is precompact. Now assume $\mathcal{F}$ is compact. If $\mathcal{U}$ is an open cover of $\mathcal{F}(x)$ then $\mathcal{U}$ consists of open neighbourhoods $U_{f(x)}$, $f \in \mathcal{F}$. Each $U_{f(x)}$ can be indentified with an open neighbourhood U_f in the uniform topology $\mathcal{J}_A$ for $A = \{x\}$. From the open cover $\{U_f : f \in \mathcal{F}\}$ of $\mathcal{F}$ we can choose a finite cover $\{U_g : g \in \mathcal{G}\}$ and hence $\{U_{g(x)} : g \in \mathcal{G}\}$ is a finite open cover of $\mathcal{F}(x)$.

From Lemmas 2, 6 we obtain

COROLLARY 7. *Let X be a topological space, $\mathcal{A} \subset 2^X$ and let $(Y, \mathcal{V})$ be a uniform space. If $\mathcal{F} \in \mathcal{C}_{\mathcal{A}}$ is precompact (compact) in $(Y^X, \mathcal{V}|\mathcal{A})$ then*

(i) *$\mathcal{F}$ is equicontinuous on members of $\mathcal{A}$.*

If in addition $\mathcal{A}$ contains all singletons then

(ii) *$\mathcal{F}(x)$ is precompact (compact), $x \in X$.*

Now we are ready to prove

THEOREM 8. *Let X be a topological space and $(Y, \mathcal{V})$ a uniform space. A family (closed in $\mathcal{C}_{\mathcal{K}}$ family) $\mathcal{F} \subset \mathcal{C}_{\mathcal{K}}$ is precompact (compact) in $(Y^X, \mathcal{V}|\mathcal{K})$ iff*

(i) *$\mathcal{F}$ is equicontinuous on compacta,*

(ii) *$\mathcal{F}(x)$ is precompact (compact) for each $x \in X$.*

Proof. In view of Corollary 7 conditions (i), (ii) are necessary. On the other hand if (i) holds then $\mathcal{J}_{\mathcal{K}} = \mathcal{P}$ on $\mathcal{F}$ (Theorem 3). Therefore (i) and (ii) are sufficient for the precompact case ([2], F p. 205). If $\mathcal{F}(x)$ is compact then it is precompact and complete ([2], Th. 32 p. 198). Consequently $\mathcal{F}$ is precompact. On the other hand family $\mathcal{F}$ is closed in $\mathcal{C}_{\mathcal{K}}$ which in turn is complete in $(Y^X, \mathcal{V}|\mathcal{K})$ ([2], p. 230) and therefore $\mathcal{F}$ is complete in Y^X. Now in view of ([2], Th. 32 p. 198) $\mathcal{F}$ is compact.

The following is a consequence of Th. 8 (compare remarks preceding Th. 5).

THEOREM 9. (cf. [2], Th. 18 p. 234) *Let X be a k_3-space and $(Y, \mathcal{V})$ a uniform space. A family (closed in $\mathcal{C}$ family) $\mathcal{F} \subset \mathcal{C}$ is precompact (compact) in $(Y^X, \mathcal{V}|\mathcal{K})$ iff*

(i) *$\mathcal{F}$ is equicontinuous,*

(ii) *$\mathcal{F}(x)$ is precompact (compact) for each $x \in X$.*

Now we are going to present versions of Theorems 4, 5, 8, 9 which suit the case of Y being a topological space.

If X, $(Y, \mathcal{T})$ are topological spaces then $\mathcal{T}_{\mathcal{K}}$ denotes the compact open topology on Y^X.

Due to ([2], p. 236) a family $\mathcal{F} \subset Y^X$, where X, Y are topological spaces is *evenly continuous on a set $B \subset X$* if the following is satisfied

(2) for each $x \in B$, $y \in Y$ and each neighbourhood V of y there is a neighbourhood U of x and W of y such that for each $f \in \mathcal{F}$ if $f(x) \in W$ then $f(B \cap U) \subset V$.

If $B = X$ then $\mathcal{F}$ is evenly continuous.

Condition (2) means that for each $x \in B$, $y \in Y$ and each neighbourhood V of y there exists a neighbourhood W of y such that x is an inner point in B of $B \cap \bigcap\{f^{-1}(V) : f \in \mathcal{F} \text{ and } f(x) \in W\}$.

LEMMA 10. *Let $X, (Y, \mathcal{T})$ be topological spaces and Y regular. If $\mathcal{F} \subset \mathcal{C}_{\mathcal{K}}$ is compact in $(Y^X, \mathcal{T}_{\mathcal{K}})$ then $\mathcal{F}$ is evenly continuous on compacta.*

Proof. Let $B \subset X$ be a compact set and let V be an open neighbourhood of a $y \in Y$. There exist closed neighbourhoods W, Z of y such that $W \subset \operatorname{Int} Z \subset Z \subset V$, Y being regular. Let $x \in B$ be arbitrary. If $h(x) \in Y \setminus W$ then $\{f \in \mathcal{F} : f(x) \in Y \setminus W\}$ is an open neighbourhood of h in $\mathcal{T}_{\mathcal{K}}$. Therefore $\mathcal{G} = \{g \in \mathcal{F} : g(x) \in W\}$ is closed in $\mathcal{F}$ and consequently $\mathcal{G}$ is compact. For simplicity of notations let us consider the restrictions of $\mathcal{F}$ and $\mathcal{G}$ to the subspace B of X. Let us adopt $\mathcal{D} = \{g^{-1}(Z) : g \in \mathcal{G}\}$. The open sets

$\{g \in \mathcal{G} : g(D) \subset V\}$, $D \in \mathcal{D}$ cover $\mathcal{G}$. In view of the compactness of $\mathcal{G}$ there exist sets $D_1, \dots, D_n \in \mathcal{D}$ such that for $\mathcal{G}_i = \{g \in \mathcal{G} : g(D_i) \subset V\}$, $\{\mathcal{G}_i : i = 1, \dots, n\}$ covers $\mathcal{G}$. For each $g \in \mathcal{G}$ we have $x \in g^{-1}(W) \subset g^{-1}(\operatorname{Int} Z) \subset \operatorname{Int} g^{-1}(Z)$ (g is continuous). By considering g_i applied to define D_i we obtain $x \in U = \operatorname{Int} D_1 \cap \dots \cap \operatorname{Int} D_n$. For any $f \in \mathcal{G}$ there exists an i such that $f \in \mathcal{G}_i$ and consequently $f(U) \subset f(D_i) \subset V$. Now it is clear that if $f \in \mathcal{F}$ and $f(x) \in W$ (i.e. $f \in \mathcal{G}$) then $f(B \cap U) \subset V$, i.e. (2) holds.

If X is a k_3-space and $(Y, \mathcal{T})$ is regular then $\mathcal{F} \subset \mathcal{C}_{\mathcal{K}}$ means $\mathcal{F} \subset \mathcal{C}$. On the other hand the neighbourhood W in the proof of Lemma 10 does not depend on B. Consequently (see the comment to condition (2)) under these additional assumptions compact family $\mathcal{F} \subset \mathcal{C}$ is evenly continuous. Hence we obtain

Lemma 11. ([1], Th. 2.2) *Let X be a k_3-space and $(Y, \mathcal{T})$ regular. If $\mathcal{F} \subset \mathcal{C}$ is compact in $(Y^X, \mathcal{T}_{\mathcal{K}})$ then $\mathcal{F}$ is evenly continuous.*

Theorem 12. (cp. [2], Th. 19 p. 235) *Let $X, (Y, \mathcal{T})$ be topological spaces. If $\mathcal{F} \subset Y^X$ is evenly continuous on compacta then $\mathcal{P} = \mathcal{T}_{\mathcal{K}}$ on $\mathcal{F}$.*

Proof. Clearly we have $\mathcal{P} \subset \mathcal{T}_{\mathcal{K}}$. Let us prove that $\mathcal{T}_{\mathcal{K}} \subset \mathcal{P}$ on $\mathcal{F}$. Let $\mathcal{G} = \{f \in \mathcal{F} : f(K) \subset V\}$ be an element of subbase of $\mathcal{T}_{\mathcal{K}}$ and such that $g \in \mathcal{G}$. We will show that $\mathcal{G}$ contains a neighbourhood of g in $\mathcal{P}$. For $x \in K$, $y = g(x) \in V$ and each $f \in \mathcal{F}$ if $f(x) \in W_y$ then $f(K \cap U_x) \subset V$ holds (see (2)). From the family $\{K \cap U_x : x \in K\}$ we can choose a finite cover $\{K \cap U_{x_i} : i = 1, \dots, n\}$ of K. Now if $f \in \mathcal{F}$ satisfies $f(x_i) \in W_{g(x_i)}, i = 1, \dots, n$ then we have $f(K) \subset V$, i.e. $\mathcal{G}$ contains a neighbourhood of g in $\mathcal{P}$.

From Lemma 10 and Theorem 12 we obtain

Theorem 13. *Let $X, (Y, \mathcal{T})$ be topological spaces and Y regular. A family $\mathcal{F} \subset \mathcal{C}_{\mathcal{K}}$ is compact in $(Y^X, \mathcal{T}_{\mathcal{K}})$ iff*

(i) *$\mathcal{F}$ is evenly continuous on compacta*

(ii) *$\mathcal{F}$ is compact in $\mathcal{P}$.*

In view of Lemma 11 the previous theorem implies

Theorem 14. *Let X be a k_3-space and $(Y, \mathcal{T})$ regular. A family $\mathcal{F} \subset \mathcal{C}$ is compact in $(Y^X, \mathcal{T}_{\mathcal{K}})$ iff*

(i) *$\mathcal{F}$ is evenly continuous*

(ii) *$\mathcal{F}$ is compact in $\mathcal{P}$.*

Lemma 15. *Let $X, (Y, \mathcal{T})$ be topological spaces. If $\mathcal{F}$ is compact in $(Y^X, \mathcal{T}_{\mathcal{K}})$ then $\mathcal{F}(x)$ is compact for each $x \in X$.*

Proof. If $\mathcal{V} \subset \mathcal{T}$ covers $\mathcal{F}(x)$ then from $\{f \in \mathcal{F} : f(x) \in V\}$, $V \in \mathcal{V}$ we can choose a finite cover of $\mathcal{F}$ in $\mathcal{T}_{\mathcal{K}}$ which consequently defines a finite cover $\mathcal{W} \subset \mathcal{V}$ of $\mathcal{F}(x)$.

Theorem 16. *Let $X, (Y, \mathcal{T})$ be topological spaces and Y regular. A family $\mathcal{F} \subset \mathcal{C}_{\mathcal{K}}$ closed in $\mathcal{C}_{\mathcal{K}}$ is compact in $(Y^X, \mathcal{T}_{\mathcal{K}})$ iff*

(i) *$\mathcal{F}$ is evenly continuous on compacta,*

(ii) *$\mathcal{F}(x)$ is compact for each $x \in X$.*

Proof. In view of Theorem 13 and Lemma 15 if $\mathcal{F} \subset \mathcal{C}_{\mathcal{K}}$ is compact in $(Y^X, \mathcal{T}_{\mathcal{K}})$ then (i) and (ii) hold. If (i) is satisfied for $\mathcal{F}$ then the same holds for the closure of $\mathcal{F}$ in $(Y, \mathcal{P})$ ([2], Th. 19 p. 235) which is contained in the closure of $\mathcal{F}$ in $(Y^X, \mathcal{T}_{\mathcal{K}})$ as we have $\mathcal{P} \subset \mathcal{T}_{\mathcal{K}}$. In view of (ii) and the Tychonoff Theorem ([2], Th. 13 p. 143) the set $\mathcal{H} = \{h \in Y^X : h(x) \in \mathcal{F}(x), x \in X\}$ is compact in $(Y^X, \mathcal{P})$ and so is $\overline{\mathcal{H}}$ ([2], H p. 133). Therefore $\overline{\mathcal{F}} \subset \mathcal{C}_{\mathcal{K}}$ is compact in $(Y, \mathcal{P})$ and in view of Th. 12 $\mathcal{F} = \overline{\mathcal{F}}$ is compact in $(Y^X, \mathcal{T}_{\mathcal{K}})$.

In consequence we obtain (cp. Th. 14)

Theorem 17. *Let X be a k_3-space and $(Y, \mathcal{T})$ regular. A family $\mathcal{F} \subset \mathcal{C}$ closed in $\mathcal{C}$ is compact in $(Y^X, \mathcal{T}_{\mathcal{K}})$ iff*

(i) *$\mathcal{F}$ is evenly continuous,*

(ii) *$\mathcal{F}(x)$ is compact for each $x \in X$.*

Remark 18. If a family of mappings is equicontinuous then it is evenly continuous ([2], Th. 22 p. 237). Therefore the "compact" versions of Theorems 4, 5, 8, 9 are respectively particular cases of Theorems 13, 14, 16, 17. On the other hand the closure of a compact set in a regular space is compact ([2], B (b) p. 161) and thus for Y being a uniform space we obtain the respective equivalences (see [2], Th. 23 p. 237).

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