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g-CHORDAL CURVES

Abstract. In this paper we introduce a notion of g-chordal curves which are a natural generalization of equichordal, equireciprocal and equipower curves. A Crofton-type integral formula and estimations of the area and the length of g-chordal curve are given. Moreover, a 1-parameter family of ovals with exactly four vertices in the class generated by the function $g(x) = x^m$ is constructed. A remark on the equichordal problem ends the paper.

1. Introduction

Let C be a plane closed simple regular curve and let $g : (0, +\infty) \to \mathbb{R}$ be a strictly monotonic function of the class C^1. Let $\|AB\|$ denote the distance between points A, B in the euclidean space $\mathbb{R}^2$.

Definition 1.1. A point P in $\mathbb{R}^2$ is called a g-chordal point of C if it has the following property: if a chord of C passes through P and joins points P_1, P_2 of C then

$$g(\|PP_1\|) + g(\|PP_2\|) = c \tag{1.1}$$

and the sum does not depend on the choice of a chord. The curve C will be called a g-chordal curve.

In this paper we will consider all g-chordal curves with g-chordal point at the origin O.

We denote by K the class of all plane simple closed curves given in a polar form

$$t \to r(t)e^{it} \quad \text{for} \quad t \in [0, 2\pi], \tag{1.2}$$

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where a function $r : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

$$\begin{cases} r \in C^1 \\ r(t) > 0 \\ r(t+2\pi) = r(t) \end{cases} \tag{1.3}$$

for $t \in \mathbb{R}$.

We associate with each function $r : \mathbb{R} \to \mathbb{R}$ a function $r_\pi : \mathbb{R} \to \mathbb{R}$ given by the formula

$$r_\pi(t) = r(t+\pi) \quad \text{for} \quad t \in \mathbb{R}. \tag{1.4}$$

A strictly monotonic function $g : (0, +\infty) \to \mathbb{R}$ of the class C^1 determines a subclass K_g of the class K by the following way: a curve of the form (1.2) belongs to K_g if and only if the function r satisfies the following condition

$$g \circ r + g \circ r_\pi = const \tag{1.5}$$

where $\circ$ denotes the composition of functions. A curve belonging to the class K_g is a g-chordal curve.

The class K_g generated by the function $g(u) = u$ contains all equichordal curves, see [6]. The class K_g generated by the function $g(u) = \frac{1}{u}$ contains all equireciprocal curves, see [4]. The class K_g generated by the function $g(u) = \ln u$ contains all equipower curves, see [15], [16], [7], [8], [9].

In this paper we will assume that a function g is a strictly increasing function.

The considerations in the fourth section are connected with ovals. We recall that a plane simple closed curve with a positive curvature will be called an oval, see [11], [14].

2. Crofton-type integral formula

We consider two curves C_n, $t \to r_n(t) e^{it}$ for $t \in [0, 2\pi]$, $(n = 1, 2)$ belonging to a class K_g and satisfying the condition

$$g \circ r_n + g \circ r_{n,\pi} = c_n. \tag{2.1}$$

We assume that C_1 is curve lying in the region bounded by C_2. Then $r_1(t) < r_2(t)$ for $t \in [0, 2\pi]$. The function g is strictly increasing so $g(r_1(t)) < g(r_2(t))$ for $t \in [0, 2\pi]$. This inequality implies immediately that $c_1 < c_2$. We denote by C_1C_2 a region bounded by C_1 and C_2. We consider a mapping $G : [0,1] \times [0, 2\pi] \to C_1C_2$ given by the formula

$$G(s,t) = g^{-1}(sg(r_2(t)) + (1-s)g(r_1(t)))e^{it}. \tag{2.2}$$

For each fixed $s \in [0,1]$ a curve $t \to G(s,t)$ is a g-chordal one. Indeed, let

$$r(s,t) = g^{-1}(sg(r_2(t)) + (1-s)g(r_1(t))). \tag{2.3}$$

Making use of (2.1) we get

$$g(r(s,t)) + g(r(s,t+\pi)) = sc_2 + (1-s)c_1$$

for a fixed $s \in [0,1]$ and for all $t \in [0,2\pi]$.

We note that $s = 0$ determines C_1 and $s = 1$ determines C_2.

Let $x \in \mathbb{R}^2$. We denote by $\|x\|$ the distance between x and the origin 0.

THEOREM 2.1. *Let $C_1, C_2 \in K_g$ where g is a positive-valued function. If C_1 lies in a region bounded by C_2 and the condition* (2.1) *holds then we have*

$$\iint\limits_{C_1C_2} \frac{g'(\|x\|)}{\|x\|}dx = \pi(c_2 - c_1). \tag{2.4}$$

Proof. Let

$$E = \{(s,t) : \ 0 < s < 1, 0 < t < 2\pi\}. \tag{2.5}$$

We denote by G_E a restriction of the function G to the set E. The function G_E maps bijectively E onto interior of the region C_1C_2 with deleted a line segment on x-axis.

We determine the jacobian $JG_E(s,t)$ of the function G_E at the point (s,t). By (2.2) we have

$$\frac{\partial G_E}{\partial s}(s,t) = \frac{g(r_2(t)) - g(r_1(t))}{g'(r(s,t))}e^{it},$$

$$\frac{\partial G_E}{\partial t}(s,t) = \frac{sg'(r_2(t))r_2'(t) + (1-s)g'(r_1(t))r_1'(t)}{g'(r(s,t))}e^{it} + r(s,t)ie^{it}$$

and

$$JG_E = (g \circ r_2 - g \circ r_1)\frac{r}{g' \circ r}. \tag{2.6}$$

We note that $JG_E(s,t) > 0$ for $(s,t) \in E$ and G_E is a diffeomorphism. For $x \in G(E)$ we have $x = r(s,t)e^{it}$ for some $(s,t) \in E$. Using the classical theorem on diffeomorphism we get

$$\iint\limits_{C_1C_2} \frac{g'(\|x\|)}{\|x\|}dx = \iint\limits_{G(E)} \frac{g'(\|x\|)}{\|x\|}dx$$

$$= \int_0^1\int_0^{2\pi} (g(r_2(t)) - g(r_1(t)))\,dtds$$

$$= \int_0^{2\pi} \left(g\left(r_2\left(t\right)\right) - g\left(r_1\left(t\right)\right)\right) dt$$

$$= \int_0^{\pi} \left(g\left(r_2\left(t\right)\right) - g\left(r_1\left(t\right)\right)\right) dt + \int_0^{\pi} \left(g\left(r_2\left(t+\pi\right)\right) - g\left(r_1\left(t+\pi\right)\right)\right) dt$$

$$= \int_0^{\pi} \left[\left(g\left(r_2\left(t\right)\right) + g\left(r_{2,\pi}\left(t\right)\right)\right) - \left(g\left(r_1\left(t\right)\right) + g\left(r_{1,\pi}\left(t\right)\right)\right)\right] dt = \pi\left(c_2 - c_1\right). \blacksquare$$

The formula (2.4) belongs to Crofton-type formulas. The original Crofton formulas can be found in [9].

Remark 2.1. For the function $g(u) = u^2$ we have

$$\pi\left(c_2 - c_1\right) = \iint_{C_1C_2} 2dx = 2\,\text{area}\,C_1C_2$$

but this result can be derived immediately by definition.

3. Estimations of the area and the length of a g-chordal curve

We assume that a function $g : (0, +\infty) \to \mathbb{R}$ satisfies the following conditions:

$$g'(u) > 0, \quad \text{for all} \quad u \in (0, +\infty), \tag{3.1}$$

$$g''(u) < 0, \quad \text{for all} \quad u \in (0, +\infty). \tag{3.2}$$

Let C, $t \to r(t)\, e^{it}$ for $t \in [0, 2\pi]$ be a g-chordal curve and

$$g\left(r\left(t\right)\right) + g\left(r\left(t+\pi\right)\right) = c. \tag{3.3}$$

In the sequel we will assume that the function r is 2π-periodic and $r \in C^2$. We introduce the following functions

$$\alpha(u) = g^{-1}\left(c - g(u)\right), \tag{3.4}$$

$$\varphi(u) = u^2 + \left[\alpha(u)\right]^2, \tag{3.5}$$

$$h(u) = \frac{u}{g'(u)} \tag{3.6}$$

for $u \in (0, +\infty)$.

Theorem 3.1. *Under assumptions* (3.1), (3.2) *and* (3.3) *the area of the region bounded by a g-chordal curve C satisfies the inequality*

$$\text{area}\, C \geq \pi \left[g^{-1}\left(\frac{c}{2}\right)\right]^2. \tag{3.7}$$

Proof. It follows from (3.1) and (3.2) that h is a strictly increasing function. We have $\alpha'(u) = \frac{-g'(u)}{g'(\alpha(u))}$ and $\varphi'(u) = 2g'(u)\left(h(u) - h(\alpha(u))\right)$.

Let $\varphi'(\widetilde{u}) = 0$. Then we have $h(\widetilde{u}) - h(\alpha(\widetilde{u})) = 0$ and $\widetilde{u} = \alpha(\widetilde{u})$. Hence we get immediately $\widetilde{u} = g^{-1}\left(\frac{c}{2}\right)$. Since we have $\alpha(\widetilde{u}) = \widetilde{u}$, $\alpha'(\widetilde{u}) = -1$ and $\varphi''(\widetilde{u}) = 4g'(\widetilde{u})\, h'(\widetilde{u}) > 0$ so φ attains the unique minimum at the point $\widetilde{u}$.

We have $\varphi(\widetilde{u}) = 2\widetilde{u}^2 = 2\left[g^{-1}\left(\frac{c}{2}\right)\right]^2$ and

$$\varphi(u) \geq 2\left[g^{-1}\left(\frac{c}{2}\right)\right]^2 \quad \text{for arbitrary} \quad u \in (0, +\infty).$$

Thus we have

$$2\,\mathrm{area}\, C = \int_0^{2\pi} [r(t)]^2\, dt = \int_0^{\pi} \varphi(r(t))\, dt \geq 2\pi \left[g^{-1}\left(\frac{c}{2}\right)\right]^2. \ \blacksquare$$

Remark 3.1. Condition (3.2) in Theorem 3.1 can be replaced by a condition $g''(u) \leq 0$ for $u \in (0, +\infty)$.

Theorem 3.2. *Let $C \in K_g$, $t \to z(t) = r(t)\, e^{it}$ for $t \in [0, 2\pi]$ and the radius vector $z(t)$ and the tangent vector $z'(t)$ to C at $z(t)$ are not collinear. Under assumptions* (3.1), (3.2) *and* (3.3) *the length L of a g-chordal curve C satisfies the inequality*

$$L \geq 2\pi g^{-1}\left(\frac{c}{2}\right). \tag{3.8}$$

Proof. We denote by $\beta(t)$ an oriented angle between the radius vector $z(t)$ of C and the tangent vector $z'(t)$ to C at $z(t)$. We note that

$$\cot \beta = \frac{r'}{r}. \tag{3.9}$$

We have

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{[r(t)]^2 + [r'(t)]^2}\, dt \\ &= \int_0^{2\pi} \frac{r(t)}{\sin \beta(t)}\, dt \geq \int_0^{2\pi} r(t)\, dt = \int_0^{\pi} [r(t) + r(t+\pi)]\, dt \\ &= \int_0^{\pi} \left[r(t) + g^{-1}(c - g(r(t)))\right] dt = \int_0^{\pi} [r(t) + \alpha(r(t))]\, dt. \end{aligned}$$

Let $\psi(u) = u + \alpha(u)$. If $\psi'(u_o) = 0$ then with respect to the condition (3.2) we get $u_o = g^{-1}\left(\frac{c}{2}\right)$. We have $\psi''(u_o) = \frac{-2g''(u_o)}{g'(u_o)} > 0$, so ψ attains the unique minimum at u_o and $\psi(u_o) = 2u_o = 2g^{-1}\left(\frac{c}{2}\right)$. Hence we obtain

$$L \geq \int_0^{\pi} \psi(r(t))\, dt \geq 2\pi g^{-1}\left(\frac{c}{2}\right). \ \blacksquare$$

4. One-parameter family of ovals with exactly four vertices in the class $K(m)$

Let $m \geq 1$. We denote by $K(m)$ the class K_g where $g(u) = u^m$ for $u \in (0, +\infty)$. We prove some general theorems on ovals.

Theorem 4.1. *Let a curve C, $t \to \rho(t) e^{it}$ for $t \in [0, 2\pi]$ be an oval. If $m \geq 1$ and*

$$m \geq \max_{[0,2\pi]} \left(\frac{\rho'}{\rho} \right)^2 \tag{4.1}$$

then a curve C_m, $t \to \rho(t)^{\frac{1}{m}} e^{it}$ for $t \in [0, 2\pi]$ is an oval.

Proof. C is an oval so its curvature is a positive-valued function and then

$$\varkappa = 2(\rho')^2 + \rho^2 - \rho\rho' > 0. \tag{4.2}$$

Let us fix a number $m > 1$ and let

$$r = \rho^{\frac{1}{m}}. \tag{4.3}$$

Differentiating r we obtain

$$r' = \frac{r}{m} \frac{\rho'}{\rho} \tag{4.4}$$

and

$$r'' = \frac{r}{m\rho^2} \left[\left(\frac{1}{m} - 1 \right) (\rho')^2 + \rho\rho'' \right]. \tag{4.5}$$

Let

$$k = 2(r')^2 + r^2 - rr''. \tag{4.6}$$

Making use of (4.2)-(4.5) we obtain

$$\begin{aligned} \frac{m^2\rho^2}{r^2} k &= \frac{m^2\rho^2}{r^2} \left\{ 2\frac{r^2}{m^2} \left(\frac{\rho'}{\rho} \right)^2 + r^2 - \frac{r^2}{m\rho^2} \left[\left(\frac{1}{m} - 1 \right) \left(\rho' \right)^2 + \rho\rho'' \right] \right\} \\ &= m\varkappa + \frac{m-1}{\rho^2} \left(m - \left(\frac{\rho'}{\rho} \right)^2 \right). \end{aligned}$$

If m satisfies the condition (4.1) then $k > 0$.

Using Theorem 4.1 we prove the following theorem

Theorem 4.2. *In each class $K(m)$ there exists a 1-parameter family of ovals with exactly four vertices.*

Proof. Case $m = 1$.

A curve C_a, $t \to (2 + a\cos t) e^{it}$ for $t \in [0, 2\pi]$ and a fixed $a \in (0, 1)$ belongs to $K(1)$. Let k denotes the curvature of C_a and

$$q(t) = \sqrt{(2 + a\cos t)^2 + a^2 \sin^2 t}.$$

For $a \in (0,1)$ we have

$$q(t)^3 k(t) = 2\left(a^2 + 3a\cos t + 2\right) \geq 2\left(a^2 - 3a + 2\right) = 2(1-a)(2-a) > 0$$

so C_a is an oval.

The equality

$$q(t)^5 k'(t) = 6a^2 (2\cos t + a)\sin t$$

implies immediately that C_a has exactly four vertices.

Case $m > 1$.

Let us fix $m > 1$ and $a \in (0,1)$. Let

$$r(t) = (2 + a\cos t)^{\frac{1}{m}}. \tag{4.7}$$

We note that a curve C_m, $t \to r(t)\, e^{it}$ for $t \in [0, 2\pi]$ belongs to $K(m)$. Using Theorem 4.1 we can show that C_m is an oval.

For the oval C_a, $t \to \rho(t)\, e^{it}$ for $t \in [0, 2\pi]$ where $\rho(t) = 2 + a\cos t$ we have

$$\left(\frac{\rho'(t)}{\rho(t)}\right)^2 = a^2 \frac{\sin^2 t}{(2 + a\cos t)^2}.$$

It is easy to verify that

$$a^2 \frac{\sin^2 t}{(2 + a\cos t)^2} \leq \frac{a^2}{4 - a^2} < \frac{1}{3},$$

so the inequality (4.1) is satisfied for $m \geq 1$. It means that C_m is an oval.

Now we prove that C_m has exactly four vertices. The curvature k of C_m is given by the formula

$$k(t) = \frac{2\,(r'(t))^2 + (r(t))^2 - r(t)\, r''(t)}{\left(\sqrt{[r(t)]^2 + [r'(t)]^2}\right)^3}. \tag{4.8}$$

Let

$$\omega(t) = \frac{-a}{m} \frac{\sin t}{2 + a\cos t} \tag{4.9}$$

and

$$\xi(t) = 2\frac{a^2 + a\cos t - 2}{(2 + a\cos t)^2}. \tag{4.10}$$

We have

$$\omega'(t) = \frac{-a}{m} \frac{2\cos t + a}{(2 + a\cos t)^2}, \tag{4.11}$$

$$\omega'' = \omega\xi \tag{4.12}$$

and

$$r' = \omega r. \tag{4.13}$$

We note that the formula (4.8) can be rewritten in the form

$$kr = \frac{\omega^2 + 1 - \omega'}{\left(\sqrt{1+\omega^2}\right)^3}. \tag{4.14}$$

Differentiating (4.14) we get

$$(k'r + kr')\left(\sqrt{1+\omega^2}\right)^5 = (2\omega\omega' - \omega'')(1+\omega^2) - 3(\omega^2 + 1 - \omega')\omega\omega'. \tag{4.15}$$

Using (4.12) and (4.13) we rewrite (4.15) in the form

$$(k' + k\omega)r\left(\sqrt{1+\omega^2}\right)^5 = \omega\{(2\omega' - \xi)(1+\omega^2) - 3\omega'(\omega^2 + 1 - \omega')\}. \tag{4.16}$$

With respect to (4.14) we have

$$k'r\left(\sqrt{1+\omega^2}\right)^5 = \omega\{3(\omega')^2 - (1+\omega^2)(\xi + \omega^2 + 1)\}.$$

The equality $\omega(t) = 0$ is satisfied in the interval $[0, 2\pi]$ for $t = 0$ and $t = \pi$ only. Thus it is necessary to show that the function

$$P = 3\left(\omega'\right)^2 - \left(1+\omega^2\right)\left(\xi + \omega^2 + 1\right) \tag{4.17}$$

has exactly two zeros in the interval $[0, 2\pi]$.

Substituting (4.9), (4.10) and (4.11) into (4.17) we obtain

$$\begin{aligned} P(t) = {} & \frac{3a^2}{m^2}\frac{(a + 2\cos t)^2}{(2 + a\cos t)^4} \\ & - \frac{m^2(2 + a\cos t)^2 + a^2\sin^2 t}{m^2(2 + a\cos t)^2}\left[\frac{2\left(a^2 + a\cos t - 2\right)}{(2 + a\cos t)^2}\right. \\ & \left. - \frac{m^2(2 + a\cos t)^2 + a^2\sin^2 t}{m^2(2 + a\cos t)^2}\right]. \end{aligned}$$

Hence we have

$$\begin{aligned} m^2(2 + a\cos t)^4 P(t) = {} & a^4\left(m^2 - 1\right)^2\cos^4 t + 6a^3m^2\left(m^2 - 1\right)\cos^3 t \\ & + 2a^2\left[(m^2 - 1)(6m^2 + a^2 - m^2a^2) + 4m^4 + 6\right]\cos^2 t \\ & + 4a\left(12m^4 + 2m^2a^2 - 2m^4a^2 + 3a^2\right)\cos t \\ & + \left(4m^2 + a^2\right)\left(8m^2 + a^2 - 2m^2a^2\right) + 3a^4. \end{aligned}$$

We consider the polynomial

$$\begin{aligned} B(x) = {} & a^4(M - 1)^2x^4 + 6a^3M(M - 1)x^3 \\ & + 2a^2\left[(M - 1)\left(6M + a^2 - Ma^2\right) + 4M^2 + 6\right]x^2 \\ & + 4a\left(12M^2 + 2Ma^2 - 2M^2a^2 + 3a^2\right)x \\ & + \left(4M + a^2\right)\left(8M + a^2 - 2Ma^2\right) + 3a^4, \end{aligned} \tag{4.18}$$

where

$$M = m^2. \tag{4.19}$$

We show that the polynomial B has exactly one root in the interval $[-1, 1]$. Since $a \in (0, 1)$ and $M = m^2 \geq 4$, so all coefficients of B are positive.

We have

$$\begin{aligned} B'(x) = {} & 4a^4 (M-1)^2 x^3 + 18a^3 M (M-1) x^2 \\ & +4a^2 \left[(M-1)\left(6M + a^2 - Ma^2\right) + 4M^2 + 6\right] x \\ & +4a \left(12M^2 + 2Ma^2 - 2M^2a^2 + 3a^2\right) \end{aligned}$$

and

$$\begin{aligned} B''(x) = {} & 12a^4 (M-1)^2 x^2 + 36a^3 M (M-1) x \\ & + 4a^2 \left[(M-1)\left(6M + a^2 - Ma^2\right) + 4M^2 + 6\right] \end{aligned}$$

B'' attains its minimum at the point

$$x_o = \frac{-3M}{2a(M-1)} < -1.$$

We note that

$$\frac{B''(-1)}{4a^2} = M^2(2a^2 - 9a + 10) - M(4a^2 - 9a + 6) + 2a^2 + 6 > 3M^2 - 6M + 6 > 0.$$

The conditions $x_o < -1$ and $B''(-1) > 0$ imply that $B''(x) > 0$ for all $x \in (-1, 1)$. Moreover, we note that

$$B'(-1) = \left(5a^2 - 20a + 24\right) M^2 + a(12 - 5a) M - 6a(2 - a) > 4M^2 - 12 > 0$$

for $M \geq 4$, B' is a strictly increasing function in $[-1, 1]$ and $B'(-1) > 0$, so $B'(x) > 0$ for $x \in [-1, 1]$. Thus the polynomial B is a strictly increasing function in the interval $(-1, 1)$. It follows from the four vertex theorem that the function P has at least two zeros in $(-1, 1)$, so B must have at least one root. Thus B has exactly one root in $(-1, 1)$ since it is a strictly increasing function.

With respect to the results of [2] Theorem 4.2 is not true in an arbitrary class K_g. For this reason we can formulate the following question:

Determine the family G_4 of all strictly monotonic functions $g : (0, +\infty) \to \mathbb{R}$ of the class C^1 such that if $g \in G_4$ then the class K_g contains a 1-parameter family of ovals with exactly four vertices.

5. Remark on the equichordal problem

The well-known equichordal problem was formulated by Fujiwara [5] in 1916 and independently by Blaschke, Rothe and Weitzenböck [1] in 1917. A literature connected to the equichordal problem is given in e.g. [3]. Gardner reminded a natural extension of the equichordal problem, see [6], Conjecture 3. We can extend the equichordal problem in the following form:

Determine the family G of all strictly monotonic functions $g : (0,+\infty) \to \mathbb{R}$ of the class C^1 for which there exists an oval with two g-chordal points.

The function $g(u) = \frac{1}{u}$ belongs to G, see [4]. In the case of the equichordal problem the function $g(u) = u$ does not belong to G, see [8].

Let us fix $g \in G$. We will assume that a chordal point of a g-chordal curve C coincides with the origin 0 and that C is given in the form $t \to r(t)\, e^{it}$ for $t \in [0, 2\pi]$. We consider the existence of a g-chordal curve with two g-chordal points. We recall that for $g(x) = \frac{1}{x}$ the ellipses are equireciprocal with two equireciprocal points [4].

THEOREM 5.1. *Let C be a g-chordal curve of the class C^2 with two g-chordal points F_1, F_2. We assume that $g(x) = x^\alpha$ for $\alpha \neq 0$ or $g \in C^2$ and $g^*(x) = x\frac{g''(x)}{g'(x)} > 0$. The chord of C passing through F_1, F_2 with ends at points A and B is orthogonal to the tangent lines to C at A and at B.*

Proof. We assume that the straight line passing through F_1 and F_2 coincides with the x-axe and F_1 coincides with the origin. Moreover, we assume that

1. the oriented angle between the x-axe and the tangent line to C at A is equal α_0,
2. the oriented angle between the x-axe and the tangent line to C at B is equal α_1.

Let $|AB| = m$, $|OA| = r(0) = r_0$, $|OB| = r(\pi) = m - r_0$.

We have $g \circ r + g \circ r_\pi = c$. Differentiating we obtain

$$r' g' \circ r + r'_\pi g' \circ r_\pi = 0$$

or

$$r g' \circ r \cot\alpha + r_\pi g' \circ r_\pi \cot\alpha_\pi = 0.$$

For $t = 0$ we have

$$r_0 g'(r_0) \cot\alpha_0 + (m - r_0)\, g'(m - r_0) \cot\alpha_1 = 0. \tag{5.1}$$

Let $|F_2 A| = r_1$, $|BF_2| = m - r_1$.

Of course, we have

$$r_1 g'(r_1) \cot\alpha_0 + (m - r_1)\, g'(m - r_1) \cot\alpha_1 = 0. \tag{5.2}$$

We note that if $\alpha_0 = \frac{\pi}{2}$ then $\alpha_1 = \frac{\pi}{2}$. We assume that $\alpha_0 \neq \frac{\pi}{2}$.

The relations (5.1) and (5.2) imply

$$\frac{r_0 g'(r_0)}{r_1 g'(r_1)} = \frac{(m - r_0)\, g'(m - r_0)}{(m - r_1)\, g'(m - r_1)}$$

or

$$\frac{r_0 g'(r_0)}{(m-r_0)\, g'(m-r_0)} = \frac{r_1 g'(r_1)}{(m-r_1)\, g'(m-r_1)}. \tag{5.3}$$

Let us consider a function $a : (0, m) \to \mathbb{R}$ given by the formula

$$a(x) = \frac{x g'(x)}{(m-x)\, g'(m-x)}.$$

We note that the relation (5.3) can be rewritten in the form

$$a(r_0) = a(r_1) \tag{5.4}$$

and

$$a\left(\frac{m}{2}\right) = 1.$$

We have

the nominator of $a'(x)$

$$\begin{aligned}
&= \left[g'(x) + x g''(x)\right](m-x)\, g'(m-x) \\
&\quad - x g'(x)\left[-g'(m-x) - (m-x)\, g''(m-x)\right] \\
&= (m-x)\, g'(x)\, g'(m-x) + x(m-x)\, g'(m-x)\, g''(x) \\
&\quad + x g'(x)\, g'(m-x) + x(m-x)\, g'(x)\, g''(m-x) \\
&= g'(x)\, g'(m-x)\left[m + (m-x)\, g^*(x) + x g^*(m-x)\right].
\end{aligned}$$

It is easy to see that

1. if $g^* > 0$ then the function a is increasing,
2. if $g(x) = x^\alpha$, $\quad(\alpha \neq 0)$ then $g^*(x) = \alpha - 1$,

and

$$\begin{aligned}
\text{the nominator of } a'(x) &= g'(x)\, g'(m-x)\left[m + (m-x)(\alpha-1) + x(\alpha-1)\right] \\
&= g'(x)\, g'(m-x)\, m\alpha \neq 0.
\end{aligned}$$

In both cases the function a is bijective. The equation (5.4) implies that $r_0 = r_1$.We have $r_0 = |0A| < |F_2 A| = r_1$. The contradiction ends the proof. ∎

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