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A CHARACTERIZATION OF u-UNIFORMLY COMPLETENESS OF RIESZ SPACES IN TERMS OF STATISTICAL u-UNIFORMLY PRE-COMPLETENESS

Abstract. In this paper we introduce statistically u-uniformly convergent sequences in Riesz spaces (vector lattices) and then we give a characterization of u-uniformly completeness of Riesz spaces.

The notion of statistical convergence of sequences was introduced by Steinhauss [6] at a conference held at Wroclaw University, Poland, in 1949 (see also [1]). A sequence (x_n) of real numbers is said to be *statistically convergent* to a real number x if

$$\lim_{n\to\infty}\frac{1}{n}|\{k : k \leq n,\ |x_k - x| \geq \epsilon\}| = 0$$

for each $0 < \epsilon$, where the vertical bars denote the cardinality of the set which they enclose. Maddox [4] has generalized the notion statistical convergent sequence for locally convex spaces as follows: A sequence (x_n) in a locally convex space X which determined by the seminorms $(q_i)_{i\in I}$, is said to be statistical convergent to $x \in X$ if:

$$\lim_{n\to\infty}\frac{1}{n}|\{k : k \leq n, q_i(x_n - x) \geq \epsilon\}| = 0$$

for each $0 < \epsilon$ and $i \in I$.

A vector space X with a partial order $\leq$ is called an *ordered vector space* if $\alpha x + z \leq \alpha y + z$ for each $z \in X$ whenever $x \leq y$, $0 \leq \alpha \in \mathbf{R}$. An ordered vector space X is called a *Riesz space* (or *vector lattice*) if supremum of $x, y \in X$, $x \vee y := sup\{x, y\}$ exists for each $x, y \in X$. $C(K)$-spaces, l_p-spaces and c_0 are natural examples of Riesz spaces. In a Riesz space X we write,

$$x^+ = x \vee 0, \quad x^- = (-x) \vee 0 \quad \text{and} \quad |x| = x \vee (-x)$$

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for each $x \in X$. Let X be a Riesz space, $u, x \in X$, $0 < u$ and (x_n) be a sequence in X. (x_n) is said to be *u-uniformly convergent* to x (we write $x_n \to x(u)$) if for each $0 < \epsilon$, there exists n_0 such that $|x_{n_0+k} - x| \leq \epsilon u$ for each k. (x_n) is called *u-uniformly Cauchy sequence* if for each $0 < \epsilon$, there exists k such that $|x_{n+k} - x_{m+k}| \leq \epsilon u$ for each n, m. X is called *relatively uniformly complete* if each u-uniformly Cauchy sequence is u-uniformly convergent to some x. It is obvious that each u-uniformly convergent sequence is an u-uniformly Cauchy sequence. For more detail about u-uniformly convergent sequences, we refer to [5].

We can modify the definition of statistically convergent sequence as follows.

Definition 1. Let $p > 0$ be a real number and let X be a Riesz space, $0 < u \in X$ and (x_n) be a sequence in X. We say that (x_n) is p-*statistical u-uniformly convergent to x* if:

$$\lim_{n\to\infty} \frac{1}{n^p} |\{k : k \leq n, (|x_k - x| - \epsilon u)^+ > 0\}| = 0$$

for each $0 < \epsilon$ and we write $x_n \overset{p-st}{\to} x(u)$. If $p = 1$ then we say that (x_n) is statistical u-uniformly Cauchy sequence and we write $x_n \overset{st}{\to} x(u)$.

The proof of the following theorem immediately follows from the basic inequalities in Riesz spaces, so we omit the proof.

Theorem 2. *Let X be a Riesz space and suppose that $x_n \overset{st}{\to} x(u)$ and $y_n \overset{st}{\to} y(v)$. Then we have*

$$\alpha x_n + y_n \overset{st}{\to} \alpha x + y(u+v), \quad x_n \vee y_n \overset{st}{\to} x \vee y(u+v).$$

In particular,

$$x_n^+ \overset{st}{\to} x^+(u), \quad x_n^- \overset{st}{\to} x^-(u) \quad \textit{and} \quad |x_n| \overset{st}{\to} |x|(u).$$

A Riesz space X is called *Archimedean* if $0 \leq x \leq n^{-1}y$ (for each n) implies that $x = 0$. By using the previous theorem we can prove that in an Archimedean Riesz space statistical limit is unique.

Corollary 3. *A Riesz space is Archimedean if and only if the limit of each statistical u-uniformly convergent sequence is unique for each $0 \leq u$.*

Proof. Let X be an Archimedean Riesz space, $x_n \overset{st}{\to} x(u)$ and $x_n \overset{st}{\to} y(u)$. Then $|x_n - x| + |x_n - y| \overset{st}{\to} 0(u)$. Suppose that $0 < (|x - y| - \epsilon u)^+$ for some $0 < \epsilon$. This implies that

$$0 < (|x_n - x| + |x_n - y| - \epsilon u)^+$$

for each n, which is a contradiction. Hence $|x - y| \leq \epsilon u$ for each $0 < \epsilon$. It implies that $x = y$. Conversely, suppose that $0 \leq x \leq n^{-1}y$ for each n. Then $x_n = x \xrightarrow{st} x(y)$ and $x_n = x \xrightarrow{st} 0(y)$. Since the limit is unique, we get $x = 0$.

A relation between u-uniformly convergent sequences and statistical u-uniformly convergent sequences can be given as follows.

THEOREM 4. *A sequence (x_n) in a Riesz space X is u-uniformly convergent if and only if it is u-uniformly Cauchy sequence and statistical u-uniformly convergent.*

Proof. By the definition, if (x_n) is u-uniformly convergent then it is statistically u-uniformly convergent and u-uniformly Cauchy sequence. Conversely, suppose that $x_n \xrightarrow{st} x(u)$ and (x_n) is u-uniformly Cauchy sequence. Then there exists a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \to x(u)$. Indeed, there exists r_1 such that

$$|\{k : k \leq r_1, (|x_k - x| - 2^{-1}u)^+ > 0\}| \leq 2^{-1}r_1$$

so there exists $n_1 \leq r_1$ satisfying $|x_{n_1} - x| \leq 2^{-1}u$. Now we can construct r_2 such $n_1 < 2r_2$ such that

$$|\{k : k \leq 3r_2, (|x_k - x| - 3^{-1}u)^+ > 0\}| \leq r_2$$

so

$$n_1 < 2r_2 \leq |\{k : k \leq 3r_2, |x_i - x| \leq 3^{-1}u\}|$$

hence there exists $n_1 < n_2$ such that $|x_{n_2} - x| \leq 3^{-1}u$. Now by induction, we can claim that there exists a subsequence (x_{n_k}) such that

$$|x_{n_k} - x| \leq (k+1)^{-1}u.$$

Since each u-uniformly Cauchy sequence which has a u-uniformly convergent subsequence is u-uniformly convergent, the proof is completed.

COROLLARY 5. *In an Archimedan Riesz space X, if a sequence (x_n) is increasing, and $x_n \xrightarrow{st} x(u)$ then $x = \sup x_n$ (In symbols, if $x_n \uparrow$ and $x_n \xrightarrow{st} x(u)$ then $x_n \uparrow x$).*

Proof. From the proof of the above theorem we know that (x_n) has a subsequence (x_{k_n}) with $x_{k_n} \to x(u)$, so $x = \sup x_{k_n}$. Since (x_n) is increasing we have $x = \sup x_n$

A real sequence (x_n) is called *statistically Cauchy sequence* if for each $0 < \epsilon$ there exists a positive integer $n(\epsilon)$ such that

$$\lim_{n\to\infty} \frac{1}{n^2}|\{k : k \leq n, \epsilon \leq |x_k - x_{n(\epsilon)}|\}| = 0$$

A sequence (x_n) is statistically Cauchy sequence if and only if it is statistically convergent (see [3, Theorem 1.]). There is another type statistical

Cauchy sequence, that is weaker than being statistical Cauchy sequence, namely *statistical pre-Cauchy sequence* (introduced in [2]), that is, a sequence (x_n) is called *statistical pre-Cauchy sequence* if

$$\lim_{n\to\infty} \frac{1}{n^2}|\{(i,j) : i,j \leq n, \epsilon < |x_i - x_j|\}| = 0$$

for each $0 < \epsilon$. Each statistically pre-Cauchy sequence is statistically convergent. (see [2], Theorem 2). In Riesz space statistically pre-Cauchy sequence can be modified as follows.

DEFINITION 6. Let $p > 0$ be a real number. A sequence (x_n) in a Riesz space X is said to be *statistical* u-uniformly pre-Cauchy sequence ($0 < u \in X$) if for each $0 < \epsilon$,

$$\lim_{n\to\infty} \frac{1}{n^2}|\{(i,j) : i,j \leq n, (|x_i - x_j| - \epsilon u)^+ > 0\}| = 0$$

and we say that X is p-*statistical* u-uniformly pre-complete if each statistical u-uniformly pre-Cauchy sequence is p-statistical u-uniformly convergent.

LEMMA 7. *In a Riesz space X, each statistical u-uniformly pre-Cauchy sequence (x_n) has a u-uniformly Cauchy subsequence (y_k).*

Proof. By induction we can construct a strictly increasing sequence (n_k) of natural numbers such that $\alpha_k = {n_k}^2 - 2^{-k}n_k^2 - n_k \leq |A_k|$, where

$$A_k = \{(i,j) : i,j \leq n_k, i \neq j \quad \text{and} \quad |x_i - x_j| \leq 2^{-k}u\}.$$

As $\lim \alpha_k = \infty$, it is clear that $\cup_k A_k$ is infinite. Let us denote

$$B_1 = A_1 \quad \text{and} \quad B_k = A_k - \cup A_{i=1}^{k-1} \quad (1 < k).$$

Then each B_k is finite and $\cup_n B_k$ is infinite. Now there exists a strictly increasing sequence (r_k) of natural numbers such that $(i_{r_n}, j_{r_k}) \in B_{r_k}$ with $i_{r_k} < j_{r_k}$. By taking $n_{2k-1} = i_{r_k}$ and $n_{2k} = j_{r_k}$ we get

$$|x_{n_i} - x_{n_{i+1}}| \leq 2^{-i}u$$

for each i. Let $y_k = x_{n_k}$. Then (y_k) is a subsequence of (x_n) and satisfies the inequality

$$|y_k - y_{k+1}| \leq 2^{-k}u.$$

This implies that $|y_k - y_{k+i}| \leq 2^{-k+1}u$, so (y_k) is an u-uniformly Cauchy subsequence of (x_n).

THEOREM 8. *A Riesz space X is u-uniformly complete if and only if it is 2-statistical u-uniformly pre-complete.*

Proof. Suppose that X is u-uniformly complete and (x_n) be a statistical u-uniformly pre-Cauchy sequence. From the lemma (x_n) has an u-uniformly

Cauchy sequence (x_{k_n}). From the hypothesis we have $x_{k_n} \to x(u)$ for some $x \in X$. Let

$$A_k = \{i : i \leq k, 0 < (|x_i - x| - \epsilon u)^+\}$$

and

$$B_k = \{(i,j) : i,j \leq k, 0 < |x_i - x_j| - 2^{-1}\epsilon u)^+\}$$

for a given $0 < \epsilon$. Choose n_{k_0}, p_0 such that

$$k_{n_0} \leq p_0 + i, \quad (p_0 + i)^{-2}|B_{p_0+i}| \leq \epsilon \quad \text{and} \quad |x_{k_{n_0}} - x| < 2^{-1}\epsilon u$$

for each i. It is obvious that for each k, if $p \in A_{p_0+k}$, then $p \leq p_0 + k$ and from the inequality

$$0 < (|x_p - x| - \epsilon u)^+ \leq (|x_p - x_{k_{n_0}}| - 2^{-1}\epsilon u)^+ + (|x_{k_{n_0}} - x| - 2^{-1}\epsilon u)^+$$

we have $(p, k_{n_0}) \in B_{p_0+k}$, so $|A_{p_0+i}| \leq |B_{p_0+i}|$ for each i. Hence

$$(p_0 + i)^{-2}|A_{p_0+i}| \leq (p_0 + i)^{-2}|B_{p_0+i}| \leq \epsilon$$

for each i and it completes the proof in one direction. Conversely, suppose that X is 2-statistical u-uniformly pre-complete and (x_n) be an u-uniformly Cauchy sequence. Then it is u-statistical uniformly convergent sequence, so $x_n \xrightarrow{st} x(u)$ and there exists a subsequence (x_{k_n}) of (x_n) with $x_{k_n} \to x(u)$ for some $x \in X$. Since (x_n) is u-uniformly Cauchy sequence, this completes the proof.

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