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SUBHARMONIC AND MULTIPLE PERIODIC SOLUTIONS FOR HAMILTONIAN SYSTEMS WITH LOCAL PARTIAL SUBLINEAR NONLINEARITY

Abstract. The existence of subharmonic and multiple periodic solutions as well as the minimality of periods are obtained for the nonautonomous Hamiltonian systems $\dot{x} = JH'(t,x)$ with locally and partially sublinear Hamiltonian H; that is, there exist a decomposition $\mathbb{R}^{2N} = A \oplus B$ of $\mathbb{R}^{2N}$, an $\alpha \in]0,1[$ and two periodic functions $a \in L^{\frac{2}{1-\alpha}}([0,T],\mathbb{R}^+)$ and $b \in L^2([0,T],\mathbb{R}^+)$ such that $|H'(t,u+v)| \leq a(t)|v|^\alpha + b(t)$ for all $(t,u,v) \in [0,T] \times A \times B$ and $\frac{H(t,u+v)}{|v|^{2\alpha}} \to +\infty$ or $-\infty$ as $|v| \to \infty$ in B, uniformly in $u \in A$ for a.e. t in some non empty open subset C of $[0,T]$. For the resolution we use an analogy of Egorov's theorem and a generalized saddle point theorem.

1. Introduction

Let $H : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$, $(t,x) \mapsto H(t,x)$ be a continuous function T-periodic $(T > 0)$ in the first variable, differentiable in the second variable, and $H'(t,x) = \frac{\partial H}{\partial x}(t,x)$ is continuous. Consider the Hamiltonian system of ordinary differential equations:

$$(\mathcal{H}) \qquad\qquad \dot{x} = JH'(t,x)$$

where $x \in C^1(\mathbb{R}, \mathbb{R}^{2N})$, J is the standard symplectic matrix:

$$J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix},$$

I_N being the identity matrix of order N.

It has been proved that the system $(\mathcal{H})$ has subharmonic solutions by using many different techniques, for example Morse theory, minimax theory. Many solvability conditions are given, such as the coercivity condition (see [3, 6, 7, 10]), the convexity condition (see [2, 9, 11]), the boundedness con-

dition (see [8]), the sublinear condition (see [1, 8]). Specially, under the conditions

(1) There exist $\alpha \in \,]0,1[$ and two constants $a, b > 0$ such that
$$\left| H'(t,x) \right| \le a \left| x \right|^\alpha + b, \ \forall x \in \mathbb{R}^{2N}, \ \text{a.e. } t \in \mathbb{R}.$$

(2) $$\lim_{|x| \to \infty} \frac{H(t,x)}{|x|^{2\alpha}} \to +\infty \ or \ -\infty, \ \text{uniformly in } t \in \mathbb{R},$$

Silva [8] proved the existence of subharmonics for problem $(\mathcal{H})$ (see theorem 1.2 in [8]). Recently, Daouas and Timoumi [1] generalized the result mentioned above. In this paper, we suppose that the nonlinearity $H'(t,x)$ is partially sublinear; that is, there exists a decomposition $\mathbb{R}^{2N} = A \oplus B$ of $\mathbb{R}^{2N}$ such that H is periodic in A and there exist $0 < \alpha < 1$ and two periodic functions $a \in L^{\frac{2}{1-\alpha}}([0,T], \mathbb{R}^+)$ and $b \in L^2([0,T], \mathbb{R}^+)$ satisfying the following condition

(1′) $$\left| H'(t, u+v) \right| \le a(t) \left| v \right|^\alpha + b(t), \ \forall u \in A, \ \forall v \in B, \ \text{a.e. } t \in \mathbb{R},$$

and there exists a non empty open subset C of $[0,T]$ such that H satisfies the local partial sublinearity:

(2′) $$\lim_{|v| \to \infty} \frac{H(t, u+v)}{|v|^{2\alpha}} \to +\infty \ or \ -\infty, \ \text{uniformly in } u \in A, \ \text{a.e. } t \in C.$$

Under these conditions, we obtain some existence of subharmonics and multiple periodic solutions for the system $(\mathcal{H})$. Furthermore, we study the minimality of periods. For the resolution, we use an analogy of Egorov's theorem, the Least action principle and a Generalized saddle point theorem [4].

2. Preliminaries

We will recall a minimax theorem: "Generalized saddle point theorem [4]", which will be useful in the proof of our results.

Given a Banach space E and a complete connected Finsler manifold V of class C^2, we consider the space $X = E \times V$. Let $E = W \oplus Z$ (topological direct sum) and $E_n \oplus Z_n$ be a sequence of closed subspaces with $Z_n \subset Z$, $W_n \subset W$, $1 \le \dim W_n < \infty$. Define

$$X_n = E_n \times V.$$

For $f \in C^1(X, \mathbb{R})$, we denote by $f_n = f_{|X_n}$. Then we have $f_n \in C^1(X_n, \mathbb{R})$, for all $n \ge 1$.

DEFINITION 2.1. Let $f \in C^1(X, \mathbb{R})$. The function f satisfies the Palais-Smale condition with respect to (X_n) at a level $c \in \mathbb{R}$ if every sequence (x_n) satisfying

$$x_n \in X_n, \ f_n(x_n) \to c, \ f'_n(x_n) \to 0$$

has a subsequence which converges in X to a critical point of f. The above property will be referred as the $(PS)_c^*$ condition with respect to (X_n).

DEFINITION 2.2. Let Y be a closed subset of a space X. The cuplength of X relative to Y is the largest integer n such that there exist $\alpha_0 \in H^*(X, Y)$, $* \geq 0$ and $\alpha_1, \ldots, \alpha_n \in H^*(X)$, $* \geq 1$ with

$$\alpha_0 \cup \alpha_1 \cup \cdots \cup \alpha_n \neq 0.$$

We write then cuplenth(X,Y)=n. We set cuplength$(X, Y) = -\infty$ if no such integer exists. Here H^* denotes the singular cohomology over the real field $\mathbb{R}$.

THEOREM 2.1 (Generalized saddle point theorem). *Assume that there exist $r > 0$ and $\alpha < \beta \leq \gamma$ such that*

  a) *f satisfies the $(PS)_c^*$ condition with respect to (X_n) for every $c \in [\beta, \gamma]$,*
  b) *$f(w, v) \leq \alpha$ for every $(w, v) \in W \times V$ such that $||w|| = r$,*
  c) *$f(z, v) \geq \beta$ for every $(z, v) \in Z \times V$,*
  d) *$f(w, v) \leq \gamma$ for every $(w, v) \in W \times V$ such that $||w|| \leq r$.*

Then $f^{-1}([\beta, \gamma])$ contains at least cuplength $(V) + 1$ critical points of f.

Now, for giving a variational formulation of $(\mathcal{H})$, some preliminary materials on functional spaces and norms are needed.

Consider the Hilbert space $E = W^{\frac{1}{2}, 2}(S^1, \mathbb{R}^{2N})$, where $S^1 = \mathbb{R}/T\mathbb{Z}$, and the quadratic form Q defined in E by

$$Q(x) = \frac{1}{2} \int_0^T \langle J\dot{x}, x \rangle dt$$

where $\langle ., . \rangle$ inside the sign integral is the inner product in $\mathbb{R}^{2N}$. If $x \in E$, then x has a Fourier expansion

$$x(t) \approx \sum_{m \in \mathbb{Z}} \exp\left(\frac{2\pi}{T} m t J\right) \hat{x}_m,$$

where $\hat{x}_m \in \mathbb{R}^{2N}$ and $\sum_{m \in \mathbb{Z}} (1 + |m|)|\hat{x}_m|^2 < \infty$. By an easy calculation, we obtain

(2.1) $$Q(x) = -\pi \sum_{m \in \mathbb{Z}} m |\hat{x}_m|^2.$$

Therefore Q is a continuous quadratic form on E.

Consider the subspaces of E:

$$E^0 = \mathbb{R}^{2N},$$

$$E^- = \left\{ x \in E \ / \ x(t) = \sum_{m \geq 1} \exp\left(\frac{2\pi}{T} m t J\right) \hat{x}_m \ a.e. \right\},$$

$$E^+ = \left\{ x \in E \ / \ x(t) = \sum_{m \leq -1} \exp\left(\frac{2\pi}{T} mtJ\right) \hat{x}_m \ a.e. \right\}.$$

Then $E = E^0 \oplus E^+ \oplus E^-$. It is not difficult to verify that E^0, E^-, E^+ are respectively the subspaces of E on which Q is null, negative definite and positive definite, and these subspaces are orthogonal with respect to the bilinear form:

$$B(x, y) = \frac{1}{2} \int\limits_0^T < J\dot{x}, y > dt, \quad x, y \in E$$

associated with Q. If $x \in E^+$ and $y \in E^-$, then $B(x,y) = 0$ and $Q(x+y) = Q(x) + Q(y)$. It is also easy to check that E^0, E^- and E^+ are mutually orthogonal in $L^2(S^1, \mathbb{R}^{2N})$. It follows that if $x = x^+ + x^- + x^0 \in E$, then the expression

$$||x||^2 = Q(x^+) - Q(x^-) + |x^0|^2$$

is an equivalent norm in E.

PROPOSITION 2.2. *E is compactly embedded in* $L^2(S^1, \mathbb{R}^{2N})$. *In particular there is a constant $\alpha > 0$ such that*

$$||x||_{L^2} \leq \alpha ||x||$$

for all $x \in E$.

3. Main results

Consider a decomposition $\mathbb{R}^{2N} = A \oplus B$ of $\mathbb{R}^{2N}$ with

$$A = space \{e_{i_1}, \ldots, e_{i_p}\}, \quad B = space \{e_{i_{p+1}}, \ldots, e_{i_{2N}}\}$$

where $0 \leq p \leq 2N - 1$ and $(e_i)_{1 \leq i \leq 2N}$ is the standard basis of $\mathbb{R}^{2N}$. Here, P_A (resp. P_B) denotes the projection of $\mathbb{R}^{2N}$ into A (*resp.B*).

Let $H : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$, $(t, x) \mapsto H(t, x)$ be a continuous function T-periodic ($T > 0$) in the first variable, differentiable with respect to the second variable, and $H'(t, x) = \frac{\partial H}{\partial x}(t, x)$ is continuous. Consider the following assumptions:

(H_0) H is periodic in the variables $x_{i_1}, \ldots, x_{i_p}$.

(H_1) There exist $\alpha \in]0, 1[$ and two T-periodic functions $a \in L^{\frac{2}{1-\alpha}}([0,T], \mathbb{R}^+)$ and $b \in L^2([0,T], \mathbb{R}^+)$ such that

$$\|H'(t,x)\| \leq a(t) |P_B(x)|^\alpha + b(t), \forall x \in \mathbb{R}^{2N}, \text{ a.e. } t \in [0,T].$$

(H_2) There exists a non empty open subset C of $[0,T]$ and a T-periodic function f integrable in $[0,T]$ such that either

(i) $\lim\limits_{|P_B(x)| \to +\infty} \dfrac{H(t,x)}{|P_B(x)|^{2\alpha}} = +\infty$, uniformly in $P_A(x) \in A$, a.e. $t \in C$,

and

$$H(t,x) \geq f(t), \ \forall x \in \mathbb{R}^{2N}, \ \text{a.e.} \ t \in [0, T],$$

or

(ii) $\quad \lim\limits_{|P_B(x)| \to +\infty} \dfrac{H(t,x)}{|P_B(x)|^{2\alpha}} = -\infty, \ \text{uniformly in} \ P_A(x) \in A, \ \text{a.e.} \ t \in C,$

and

$$H(t,x) \leq f(t), \ \forall x \in \mathbb{R}^{2N}, \ \text{a.e.} \ t \in [0, T].$$

Our first main result conserns the multiplicity of periodic solutions and is:

THEOREM 3.1. *Assume* $(H_0) - (H_2)$ *hold. Then the Hamiltonian system*

$(\mathcal{H})$ $\qquad\qquad\qquad\qquad \dot{x} = JH'(t,x)$

possesses at least $(p+1)$ *T-periodic solutions geometrically distinct.*

If $0 \leq p \leq 2N - 2$, we consider the assumption:

(A) there exists $i_0 \in \{1, \ldots, N\}$ such that $e_{i_0}, e_{i_0+N} \in B$.

Our second main result conserns the subharmonics and is:

THEOREM 3.2. *Suppose that* H *verifies* (A) *and* (H_0)–(H_2)*, then for all integer* $k \geq 1$*, the system* $(\mathcal{H})$ *possesses at least* $(p+1)$ *kT-periodic solutions* $x_k^1, \ldots, x_k^{p+1}$ *geometrically distinct such that for all* $i = 1, \ldots, p+1$,

$$\lim\limits_{k \to \infty} \left\| x_k^i \right\|_\infty = +\infty.$$

REMARK 3.1. The Theorem 3.2 generalizes the Theorem 1.2 in [8].

Now, consider the assumptions:

(H_1') There exist $\alpha \in]0, 1[$, $\beta > 2\alpha$, $a \in L^{\frac{2\beta}{\beta-2\alpha}}(0, T; \mathbb{R}^+)$ and $b \in L^2(0, T; \mathbb{R}^+)$ such that

$$\left\| H'(t,x) \right\| \leq a(t) |P_B(x)|^\alpha + b(t), \ \forall x \in \mathbb{R}^{2N}, \ \text{a.e.} \ t \in [0, T].$$

(H_2') There exist a non empty open subset C of $[0, T]$, two positive constants c, d and a T-periodic function f integrable in $[0, T]$ such that either

(i) $\quad \langle H'(t,x), P_B(x) \rangle \geq c |P_B(x)|^\beta - d, \ \forall x \in \mathbb{R}^{2N}, \ \text{a.e.} \ t \in C,$

and

$$\langle H'(t,x), P_B(x) \rangle \geq f(t), \ \forall x \in \mathbb{R}^{2N}, \ \text{a.e.} \ t \in [0, T],$$

or

(ii) $\quad \langle H'(t,x), P_B(x) \rangle \leq -c |P_B(x)|^\beta + d, \ \forall x \in \mathbb{R}^{2N}, \ \text{a.e.} \ t \in C,$

and

$$\langle H'(t,x), P_B(x) \rangle \leq f(t), \ \text{for all} \ x \in \mathbb{R}^{2N}, \ \text{a.e.} \ t \in [0, T].$$

Our third main result conserns the minimality of periods and is:

THEOREM 3.3. *Assume* (A), (H_0), (H_1') *and* (H_2') *hold. Then for all integer* $k \geq 1$, *the system* $(\mathcal{H})$ *possesses at least* $(p+1)$ kT-*periodic solutions* $x_k^1, \ldots, x_k^{p+1}$ *geometrically distinct such that for all* $i = 1, \ldots, p+1$, $\lim_{k \to \infty} \left\| x_k^i \right\|_\infty = +\infty$. *Moreover, for all* $i = 1, \ldots, p+1$ *and for any sufficiently large prime number* k, kT *is the minimal period of* x_k^i.

EXAMPLE 3.1. Let $A : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$, $(t, r) \to A(t, r)$ be a continuous periodic function in (t, r), differentiable in r and $\frac{\partial A}{\partial r}(t, r)$ is continuous. Let $0 < \alpha < 1$, $0 < \epsilon < 1 - \alpha$ and consider the Hamiltonian:

$$H(t, r, p) = a(t)[1 + |p - A(t, r)|^2]^{\alpha + \frac{\epsilon}{2}}$$

where $a : \mathbb{R} \to \mathbb{R}$ is a continuous periodic function not identically null and having a constant sign. Then the Hamiltonian H satisfies all the above assumptions.

Proof of Theorem 3.1. To begin let us observe the following

REMARK 3.2. We remark that if x is a periodic solution of $\dot{x} = JH'(t, x)$ then $y(t) = x(-t)$ is a periodic solution of $\dot{y} = -JH'(-t, y)$. Moreover, $-H(-t, x)$ satisfies $(H_2)(i)$ whenever $H(t, x)$ satisfies $(H_2)(ii)$. Hence it suffices to assume that H satisfies $(H_2)(i)$.

The following two lemmas, which are analogeous to Egorov's lemma, will be needed in the proof of our results. The first lemma treats the sequence case and the second does the continuous function case. They deal with tending to $+\infty$.

LEMMA 3.1. *Suppose that* C *is a non empty open subset of* $\mathbb{R}$ *with* $\mathrm{meas}(C) < \infty$, F *is a given set and* $f_n(t, u)$ *is a sequence of functions defined in* $C \times F$, *continuous in* t *such that* $f_n(t, u) \to +\infty$ *as* $n \to +\infty$, *uniformly in* $u \in F$, *for a.e.* $t \in C$. *Then, for any* $\rho > 0$, *there exists a subset* C_ρ *of* C *with* $\mathrm{meas}(C - C_\rho) < \rho$ *such that* $f_n(t, u) \to +\infty$ *as* $n \to +\infty$, *uniformly for all* $(t, u) \in C_\rho \times F$.

Proof. Without loss of generality, we may assume that $f_n(t, u) \to +\infty$ as $n \to +\infty$, uniformly in $u \in F$, for all $t \in C$. For every $r > 0$ and every positive integer n, define

$$C(n, r) = \bigcap_{k=n+1}^{\infty} \{t \in C / f_k(t, u) \geq r, \forall u \in F\}.$$

Then $C(n, r)$ is measurable and

$$C(m, r) \subset C(n, r) \text{ if } m < n.$$

Hence we have
$$C = \bigcup_{n=1}^{\infty} C(n, r)$$
because that $f_n(t, u) \to +\infty$ as $n \to +\infty$, uniformly in $u \in F$, for all $t \in C$. By the properties of Lebesgue's measure one has
$$\text{meas}(C) = \lim_{n \to \infty} \text{meas}(C(n, r))$$
which implies that
$$\lim_{n \to \infty} \text{meas}(C - C(n, r)) = 0.$$
Hence for any $\rho > 0$ and for every integer i there exists $n_i \in \mathbb{N}$ such that
$$\text{meas}(C - C(n_i, i)) < \frac{\rho}{2^i}.$$
Set
$$C_\rho = \bigcap_{i=1}^{\infty} C(n_i, i).$$
Then one has
$$\text{meas}(C - C_\rho) = \text{meas}(C - \bigcap_{i=1}^{\infty} C(n_i, i)) = \text{meas}(\bigcup_{i=1}^{\infty}(C - C(n_i, i))$$
$$\leq \sum_{i=1}^{\infty} meas(C - C(n_i, i)) < \sum_{i=1}^{\infty} \frac{\rho}{2^i} = \rho.$$
Futhermore, $f_n(t, u) \to +\infty$ as $n \to +\infty$, uniformly for all $(t, u) \in C_\rho \times F$. Indeed, for every $r > 0$, choose $i_0 \geq r$. Then we have $C_\rho \subset C(n_{i_0}, i_0)$, which implies that
$$f_n(t, u) \geq i_0 \geq r$$
for all $n \geq n_{i_0}$ and all $(t, u) \in C_\rho \times F$.

LEMMA 3.2. *Suppose that H satisfies assumption $(H_2)(i)$. Then for every $\rho > 0$ there exists a subset C_ρ of C with $meas(C - C_\rho) < \rho$ such that*

$$(3.1) \quad \lim_{|P_B(x)| \to +\infty} \frac{H(t, x)}{|P_B(x)|^{2\alpha}} = +\infty, \quad \text{uniformly for all } (t, P_A(x)) \in C_\rho \times A.$$

Proof. Suppose that H satisfies $(H_2)(i)$ and set for $t \in [0, T]$, $(u, v) \in A \times B$
$$f(t, u, v) = \frac{H(t, u + v)}{|v|^{2\alpha}}$$
and
$$f_n(t, u) = \inf_{|v| \geq n} f(t, u, v)$$

for all integer $n \geq 1$, $t \in [0, T]$ and $u \in A$. By the continuity of $f(t, u, v)$ in v for almost every $t \in C$ and all $u \in A$ one has

$$f_n(t, u) = \inf\{f(t, u, v)/ |v| \geq n,$$
$$v = \xi_{p+1}e_{i_{p+1}} + \cdots + \xi_{2N}e_{i_{2N}}/\xi_{p+1}, \ldots, \xi_{2N} \in \mathbb{Q}\}$$

for all $n \geq 1$, almost every $t \in C$ and all $u \in A$, which implies that $f_n(., u)$ is mesurable for all positive integer n and $u \in A$.

Now the fact

$$f_n(t, u) \to +\infty \ as \ n \to \infty$$

uniformly in $u \in A$, for almost every $t \in C$ follows from the same property of $f(t, u, v)$. By Lemma 3.1 there exists, for every $\rho > 0$, a subset C_ρ with $meas(C - C_\rho) < \rho$ such that $f_n(t, u) \to +\infty$ as $n \to +\infty$ uniformly for all $(t, u) \in C_\rho \times A$, which implies the desired property of $f(t, u, v)$.

Assume that $0 \leq p \leq 2N - 1$. We are interested in the existence and multiplicity of periodic solutions to the system $(\mathcal{H})$. By making the change of variables $t \to \frac{t}{k}$, the system $(\mathcal{H})$ transforms to

$(\mathcal{H}_k)$ $\qquad\qquad\qquad \dot{u} = kJH'(kt, u).$

Hence, to find kT-periodic solutions of $(\mathcal{H})$, it suffises to find T-periodic solutions of $(\mathcal{H}_k)$.

Associate to the systems $(\mathcal{H}_k)$ the family of functionals (ϕ_k) defined on the space $E = H^{\frac{1}{2}}(S^1, \mathbb{R}^{2N})$ by:

$$\phi_k(u) = \frac{1}{2} \int_0^T \langle J\dot{u}, u) \rangle dt + k \int_0^T H(kt, u) dt.$$

It is well known that every functional ϕ_k is continuously differentiable in E and critical points of ϕ_k on E correspond to the T-periodic solutions of the system $(\mathcal{H}_k)$, moreover one has

$$\phi_k'(u)h = \int_0^T \langle J\dot{u}, h \rangle dt + k \int_0^T \langle H'(kt, u), h \rangle dt$$

for all $u, h \in E$. Consider the space $X = (W \oplus Z) \times V$ with $W = E^-$, $Z = E^+ \oplus B$ and V the quotient space

$$V = A/ \{x \sim x + e_i, \ i = i_1, \ldots, i_p\}$$

which is nothing but the torus T^p. Here, B $(\subset Z)$ consists of constant B-valued functions. We regard the functional ϕ_k as defined on the space $X = (W \oplus Z) \times V$ as follows

(3.1) $\qquad \phi_k(u + v) = \frac{1}{2} \int_0^T \langle J\dot{u}, u \rangle dt + k \int_0^T H(kt, u + v) dt.$

To find critical points of ϕ_k we will apply Theorem 2.1 to this functional with respect to the sequence of subspaces $X_n = E_n \times V$, where

$$E_n = \left\{ x \in E /\ x(t) = \sum_{|m| \leq n} \exp\left(\frac{2\pi}{T} mtJ\right) \hat{u}_m \text{ a.e.} \right\}, \quad n \geq 0.$$

Let us check the Palais-Smale condition.

LEMMA 3.3. *For all level $c \in \mathbb{R}$, the functional ϕ_k satisfies the $(PS)^*_c$ condition with respect to the sequence $(X_n)_{n \in \mathbb{N}}$.*

Proof. Let $(u_n, v_n)_{n \in \mathbb{N}}$ be a sequence such that for all $n \in \mathbb{N}$, $(u_n, v_n) \in X_n$ and

$$(3.2) \qquad \phi_k(u_n + v_n) \to c \quad \text{and} \quad \phi'_{k,n}(u_n + v_n) \to 0 \text{ as } n \to \infty,$$

where $\phi_{k,n}$ is the functional ϕ_k restricted to X_n. Set $u_n = u_n^+ + u_n^- + u_n^0 + v_n$, with $u_n^+ \in E^+$, $u_n^- \in E^-$, $u_n^0 \in B$ and $v_n \in V$. We have the relation

$$(3.3) \qquad \phi'_{k,n}(u_n + v_n).u_n^+ = \left\| u_n^+ \right\|^2 + k \int_0^T \langle H'(kt, u_n + v_n), u_n^+ \rangle dt.$$

Since $\phi'_{k,n}(u_n + v_n) \to 0$ as $n \to \infty$, there exists a constant $c_1 > 0$ such that

$$(3.4) \qquad \forall n \in \mathbb{N}, \quad \left| \phi'_{k,n}(u_n + v_n).u_n^+ \right| \leq c_1 \left\| u_n^+ \right\|.$$

By assumption (H_1) and Hölder's inequality, with $p = \frac{1}{\alpha}$, $q = \frac{1}{1-\alpha}$, we have

$$(3.5) \qquad \left| \int_0^T \langle H'(kt, u_n + v_n), u_n^+ \rangle dt \right|$$

$$\leq \int_0^T [a(kt) |P_B(u_n(t))|^\alpha + b(kt)] \left| u_n^+ \right| dt$$

$$\leq \left\| u_n^+ \right\|_{L^2} [(\int_0^T [a^2(kt) |P_B(u_n(t))|^{2\alpha} dt)^{\frac{1}{2}} + (\int_0^T b^2(kt) dt)^{\frac{1}{2}}]$$

$$\leq \left\| u_n^+ \right\|_{L^2} [\|a\|_{L^{\frac{2}{1-\alpha}}} \|P_B(u_n)\|_{L^2}^\alpha + \|b\|_{L^2}].$$

Then by (3.3), (3.4), (3.5) and Proposition 2.2, there exist two constants $c_2, c_3 > 0$ such that

$$(3.6) \qquad \left\| u_n^+ \right\| \leq c_2 \left\| P_B(u_n) \right\|^\alpha + c_3.$$

Observing that a similar result holds for (u_n^-):

$$(3.7) \qquad \left\| u_n^- \right\| \leq c_2 \left\| P_B(u_n) \right\|^\alpha + c_3.$$

We conclude from (3.6) and (3.7) that the sequence (u_n) is bounded if and only if the sequence $(P_B(u_n))$ is bounded. Assume that $(P_B(u_n))$ is

not bounded, we can assume, by going to a subsequence if necessary, that $\|P_B(u_n)\| \to \infty$ as $n \to \infty$. Since $0 < \alpha < 1$, we conclude by (3.6) and (3.7) that

$$(3.8) \qquad \frac{u_n^+}{\|P_B(u_n)\|} \to 0, \quad \frac{u_n^-}{\|P_B(u_n)\|} \to 0 \; as \; n \to \infty.$$

Therefore, we have

$$(3.9) \qquad y_n = \frac{u_n}{\|P_B(u_n)\|} \to y \in B, \; |y| = 1 \; as \; n \to \infty.$$

It follows that

$$(3.10) \qquad \frac{|u_n^0|}{\|P_B(u_n)\|} \to 1 \; as \; n \to \infty.$$

Consequently, by (3.6), (3.7) and (3.10), we can find a positive constant c_4 such that

$$(3.11) \qquad \|u_n^i\| \le c_4 \, |u_n^0|^\alpha \, , \; i = +, -.$$

Now, we apply the fact that $(\phi_k(u_n))$ is bounded to get

$$(3.12) \qquad \frac{\|u_n^+\|^2 - \|u_n^-\|^2}{|u_n^0|^{2\alpha}} + k \int_0^T \frac{H(kt, u_n)}{|u_n^0|^{2\alpha}} dt \le \frac{c_5}{|u_n^0|^{2\alpha}}.$$

where c_5 is a positive constant. Using (3.11) and (3.12), we can find a constant c_6 satisfying

$$(3.13) \qquad \int_0^T \frac{H(kt, u_n^0)}{|u_n^0|^{2\alpha}} dt = \int_0^T \frac{H(kt, u_n)}{|u_n^0|^{2\alpha}} dt + \int_0^T \frac{H(kt, u_n^0) - H(kt, u_n)}{|u_n^0|^{2\alpha}} dt$$

$$\le c_6 + \int_0^T \frac{H(kt, u_n^0) - H(kt, u_n)}{|u_n^0|^{2\alpha}} dt.$$

On the other hand, by mean value theorem, assumption (H_1) and Hölder's inequality, we have

$$(3.14) \qquad \int_0^T [H(kt, u_n^0) - H(kt, u_n)] dt$$

$$= -\int_0^T < H'(kt, u_n^0 + \theta(u_n^+ + u_n^- + v_n)), u_n^+ + u_n^- + v_n > dt$$

$$\le \int_0^T [a(kt) \left| P_B(u_n^0 + \theta(u_n^+ + u_n^- + v_n)) \right|^\alpha + b(kt)] \left| u_n^+ + u_n^- + v_n \right| dt$$

$$\le [\|a\|_{L^{\frac{2}{1-\alpha}}} \left\| P_B(u_n^0 + \theta(u_n^+ + u_n^-)) \right\|_{L^2}^\alpha + \|b\|_{L^2}] \left\| u_n^+ + u_n^- + v_n \right\|_{L^2}.$$

By considering (3.14) and Sobolev's embedding $E \hookrightarrow L^2(0,T;\mathbb{R}^{2N})$, we can find a constant $c_7 > 0$ such that

$$(3.15) \quad \int_0^T [H(kt, u_n^0) - H(kt, u_n)]dt$$

$$\leq c_7[|u_n^0|^\alpha + \|u_n^+\|^\alpha + \|u_n^-\|]^\alpha + 1][\|u_n^+\| + \|u_n^-\| + 1].$$

After combining (3.11), (3.13) and (3.15), we get

$$(3.16) \quad \int_0^T \frac{H(kt, u_n^0)}{|u_n^0|^{2\alpha}}dt \leq c_8$$

for some positive constant c_8. However, the condition (3.16) contradicts $(H_2)(i)$ because $|u_n^0| \to \infty$ as $n \to \infty$. Consequently, (u_n) is bounded in X. Going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$, $u_n^0 \to u^0$ and $v_n \to v$. Notice that

$$(3.17) \quad Q(u_n^+ - u^+) = (\phi_{k,n}'(u_n + v_n) - \phi_{k,n}'(u + v)).(u_n^+ - u^+)$$

$$-k\int_0^T \langle H'(kt, u_n + v_n) - H'(kt, u + v), u_n^+ - u^+\rangle dt$$

which implies that $u_n^+ \to u^+$ in E. Similarly, $u_n^- \to u^-$ in E. It follows that $(u_n, v_n) \to (u, v)$ in X and $\phi_k'(u + v) = 0$. So ϕ_k satisfies the $(PS)_c^*$ condition for all $c \in \mathbb{R}$. The Lemma 3.3 is proved.

Now, let us prove that for all $k \geq 1$, the functional ϕ_k satisfies the conditions a), b) and c) of Theorem 2.1.

a) Let $(u, v) \in W \times V$, with $u = u^- \in E^-$, we have by using mean value Theorem, assumption (H_1) and Proposition 2.2

$$(3.18) \quad \phi_k(u + v) = -\|u^-\|^2 + k\int_0^T H(kt, u^- + v)dt$$

$$= -\|u^-\|^2 + k\int_0^T H(kt, v)dt + k\int_0^T H'(kt, v + \theta u^-)u^- dt$$

$$\leq -\|u^-\|^2 + k\int_0^T H(kt, v)dt + k\int_0^T [a(kt)|P_B(u^-)|^\alpha + b(kt)]|u^-| dt$$

$$\leq -\|u^-\|^2 + k\int_0^T H(kt, v)dt + k\|u^-\|_{L^2}[(\int_0^T a^2(kt)|u^-|^{2\alpha})^{\frac{1}{2}} + \|b\|_{L^2}]$$

$$\leq -\|u^-\|^2 + \|u^-\|[c_9\|u^-\|^\alpha + c_{10}] + c_{11}$$

where c_9, c_{10}, c_{11} are three positive constants. So

(3.19) $\phi_k(u + v) \to -\infty$ as $u \in W$, $\|u\| \to \infty$, uniformly in $v \in V$.

b) Let $(u, v) \in Z \times V$, with $u = u^+ + u^0$, we have by using mean value theorem

$$\text{(3.20)} \quad \phi_k(u + v) = \|u^+\|^2 + k \int_0^T H(kt, u^+ + u^0 + v)dt$$

$$= \|u^+\|^2 + k \int_0^T H(kt, u^0 + v)dt + k \int_0^T \langle H'(kt, u^0 + v + \theta u^+), u^+ \rangle dt.$$

By assumption (H_1) and Proposition 2.2, we can find a constant $c_{12} > 0$ such that

$$\text{(3.21)} \quad \left| k \int_0^T \langle H'(kt, u^0 + v + \theta u^+), u^+ \rangle dt \right|$$

$$\leq k \int_0^T [a(kt) |P_B(u^0 + \theta u^+)|^\alpha + b(kt)] |u^+| \, dt$$

$$\leq k \|u^+\|_{L^2} \left(\left[\int_0^T a^2(kt)(|u^0| + |u^+|)^{2\alpha} dt \right]^{\frac{1}{2}} + \|b\|_{L^2} \right)$$

$$\leq c_{12} \|u^+\| \, [|u^0|^\alpha + \|u^+\|^\alpha + 1].$$

Therefore, by using (3.20) and (3.21) we obtain

$$\text{(3.22)} \quad \phi_k(u) \leq \|u^+\|^2 + k \int_0^T H(kt, u^0 + v)dt - c_{12} \|u^+\| \, [|u^0|^\alpha + \|u^+\|^\alpha + 1].$$

Now let $d > \frac{c_{12}^2}{2}$, then by assumption $(H_2)(i)$, there exists a constant $e > 0$ such that

$$\text{(3.23)} \quad k \int_0^T H(kt, u^0 + v)dt \geq d |u^0|^{2\alpha} - e.$$

So by (3.22) and (3.23), we have

$$\text{(3.24)} \quad \phi_k(u) \geq \|u^+\|^2 + d |u^0|^{2\alpha} - e - c_{12} \|u^+\| \, [|u^0|^\alpha + \|u^+\|^\alpha + 1]$$

$$\geq \frac{1}{2} \|u^+\|^2 - c_{12}[\|u^+\|^{\alpha+1} - \|u^+\|]$$

$$+ \frac{1}{2} \left[\|u^+\| - c_{12} |u^0|^\alpha \right]^2 + \left[d - \frac{c_{12}^2}{2} \right] |u^0|^{2\alpha} - e$$

$$\geq \left[\frac{1}{2} \|u^+\|^2 - c_{12} \|u^+\|^{\alpha+1} - c_{12} \|u^+\| \right] + \left[d - \frac{c_{12}^2}{2} \right] |u^0|^{2\alpha} - e.$$

Therefore

(3.25) $\phi_k(u + v) \to \infty$ *as* $u \in Z$, $\|u\| \to \infty$, uniformly in $v \in V$.

Hence by Lemma 3.3 and properties (3.19), (3.25), we deduce that, for all $k \geq 1$, the functional ϕ_k satisfies all the assumptions of Theorem 2.1. Therefore, for all integer $k \geq 1$, the Hamiltonian system $(\mathcal{H}_k)$ possesses at least $(p + 1)$ T-periodic solutions $u_k^1, \ldots, u_k^{p+1}$ geometrically distincts. The proof of Theorem 3.1 and part 1 of Theorem 3.2 are proved.

Proof of Theorem 3.2. In the following, we will suppose that $0 \leq p \leq 2N - 2$.

By Theorem 2.1 and Remark 2.1, the sequences (u_k^i) obtained in the proof of Theorem 3.1 satisfy for all $i = 1, \ldots, p + 1$

(3.26) $$\phi_k(u_k^i) = b_k^i \geq \inf_{u \in Z \times V} \phi_k(\sqrt{k}\varphi + u)$$

where $\varphi(t) = exp(\frac{2\pi}{T}tJ)e_{i_0} \in W$, with $x_k^i(t) = u_k^i(\frac{t}{k})$ a kT-periodic solution of $(\mathcal{H})$.

We will prove, that for all $i = 1, \ldots, p + 1$, the sequence (u_k^i) has the following property

(3.27) $$\lim_{k \to \infty} \frac{1}{k}\phi_k(u_k^i) = +\infty.$$

This will be obtained by using the estimates (3.26) on the critical levels of ϕ_k, and implies that for all $i = 1, \ldots, p + 1$, the sequence $(\|u_k^i\|_\infty)_{k \in \mathbb{N}}$ goes to infinity as k goes to infinity. For this, we will need the following two lemmas.

LEMMA 3.4. *Let* $i_0 \in \{1, \ldots, N\}$ *be such that* e_{i_0}, $e_{i_0+N} \in B$ *and given*

$$u(t) = \exp\left(\frac{2\pi}{T}tJ\right)e_{i_0} + u^+(t) + u^0 + v$$

with $u^+ \in X^+$, $u^0 \in B$, $v \in V$. *Then we have*

(3.28) $P_B(u(t)) \neq 0$, *for a.e.* $t \in [0, T]$.

Proof. Arguing by contradiction and assume that $P_B(u(t)) = 0$ for a.e. $t \in [0, T]$. We have

$$u^+(t) \approx \sum_{m \leq -1} \exp\left(\frac{2\pi}{T}mtJ\right)\hat{u}_m \text{ a.e. } t \in [0, T]$$

$$\approx \sum_{m \leq -1}\left[\cos\left(\frac{2\pi}{T}mt\right)\hat{u}_m + \sin\left(\frac{2\pi}{T}mt\right)J\hat{u}_m\right] \text{ a.e. } t \in [0, T],$$

where $\hat{u}_m \in \mathbb{R}^{2N}$. Denote $\hat{u}_m = \sum_{j=1}^{N} \alpha_{m,j} e_j + \sum_{j=1}^{N} \beta_{m,j} e_{j+N}$, with $\alpha_{m,j}, \beta_{m,j} \in \mathbb{R}$, then

$$u^+(t) \approx \sum_{j=1}^{N} \sum_{m \leq -1} \left[\alpha_{m,j} \cos\left(\frac{2\pi}{T}mt\right) - \beta_{m,j} \sin\left(\frac{2\pi}{T}mt\right) \right] e_j$$

$$+ \sum_{j=1}^{N} \sum_{m \leq -1} \left[\beta_{m,j} \cos\left(\frac{2\pi}{T}mt\right) + \alpha_{m,j} \sin\left(\frac{2\pi}{T}mt\right) \right] e_{j+N} \text{ a.e.}$$

But $P_B(u(t)) = 0$ implies $P_{e_{i_0}}(u(t)) = 0$ and $P_{e_{i_0+N}}(u(t)) = 0$, which gives us

$$\cos\left(\frac{2\pi}{T}t\right) + \sum_{m \leq -1} \left[\alpha_{m,i_0} \cos\left(\frac{2\pi}{T}mt\right) - \beta_{m,i_0} \sin\left(\frac{2\pi}{T}mt\right) \right] + P_{e_{i_0}}(u^0) = 0$$

and

$$\sin\left(\frac{2\pi}{T}t\right) + \sum_{m \leq -1} \left[\beta_{m,i_0} \cos\left(\frac{2\pi}{T}mt\right) + \alpha_{m,i_0} \sin\left(\frac{2\pi}{T}mt\right) \right] + P_{e_{i_0+N}}(u^0) = 0$$

then we obtain $1 + \alpha_{-1,i_0} = 0$, $1 - \alpha_{-1,i_0} = 0$ which is impossible. The proof of lemma 3.4 is complete.

LEMMA 3.5. *Suppose that $H \in C^1(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ satisfies $(H_2)(i)$, then*

$$(3.29) \qquad \inf_{u \in Z \times V} \frac{\phi_k(\sqrt{k}\varphi + u)}{k} \to +\infty, \text{ as } k \to +\infty,$$

where $\varphi(t) = \frac{1}{\sqrt{\pi}} \exp(\frac{2\pi}{T}tJ)e_{i_0}$, with $i_0 \in \{1, \ldots, N\}$ is such that $e_{i_0}, e_{i_0+N} \in B$.

Proof. Arguing by contradiction and assume that there exist sequences $k_j \to \infty$, $(u_j) \subset Z \times V$, and a constant $c_1 \in \mathbb{R}$ such that

$$(3.30) \qquad \phi_{k_j}(\sqrt{k_j}\varphi + u_j) \leq k_j c_1, \forall j \in \mathbb{N}.$$

Taking $u_j = \sqrt{k_j}(u_j^+ + u_j^0 + v_j)$ with $u_j^+ \in E^+$, $u_j^0 \in B$, $v_j \in V$, we obtain, by an easy calculation

$$(3.31) \quad \phi_{k_j}(\sqrt{k_j}\varphi + u_j) = k_j[\|u_j^+\|^2 - 1 + \int_0^T H(k_j t, \sqrt{k_j}(\varphi + u_j^+ + u_j^0 + v_j))dt].$$

By assumption $(H_2)(i)$, the Hamiltonian H is bounded from below, so we deduce from (3.31) that there exists a constant $c_2 > 0$ such that

$$(3.32) \qquad \phi_{k_j}(\sqrt{k_j}\varphi + u_j) \geq k_j[\|u_j^+\|^2 - c_2].$$

Combining this with (3.30), we conclude that (u_j^+) is bounded in X. Taking a subsequence if necessary we can find $u^+ \in E^+$ such that

$$(3.33) \qquad u_j^+(t) \to u^+(t) \text{ as } j \to \infty \text{ for a.e. } t \in [0, T].$$

We claim that (u_j^0) is also bounded in X. Indeed, if we suppose otherwise, (3.33) implies that

(3.34) $\quad \sqrt{k_j}|P_B(\varphi(t)+u_j^+(t)+u_j^0+v_j)| \to +\infty$ as $j \to \infty$ for a.e. $t \in [0, T]$.

Let $\rho > 0$ and C_ρ be a mesurable subset of C defined as in lemma 3.2, we have

(3.35) $\quad \displaystyle\int_{C_\rho} H(k_j t, \sqrt{k_j}(\varphi(t) + u_j^+(t) + u_j^0 + v_j))dt \to \infty$ as $j \to +\infty$.

In the other hand, we have

(3.36) $\quad \displaystyle\int_{[0,T]-C_\rho} H(k_j t, \sqrt{k_j}(\varphi(t) + u_j^+(t) + u_j^0 + v_j))dt \geq \int_{[0,T]-C_\rho} f(k_j T)$

$$\geq -\int_0^T |f(t)|\, dt.$$

Therefore, we have by (3.35) and (3.36)

(3.37) $\quad \displaystyle\int_0^T H(k_j t, \sqrt{k_j}(\varphi(t) + u_j^+(t) + u_j^0 + v_j))dt \to \infty$ as $j \to +\infty$,

and we deduce from equality (3.31) that

(3.38) $\qquad \dfrac{1}{k_j}\phi_{k_j}(\sqrt{k_j}\varphi + u_j) \to \infty$ as $j \to \infty$

which contradicts (3.30) and proves our claim. Taking a subsequence if necessary, we can assume that there exists $u^0 \in B$ and $v \in V$ such that for almost every $t \in [0, T]$

(3.39). $\qquad u_j^+(t) + u_j^0 + v_j \to u(t) = u^+(t) + u^0 + v$, as $j \to \infty$.

By Lemma 3.4, we know that $P_B(\varphi(t) + u(t)) \neq 0$ for almost every $t \in [0, T]$. Therefore

(3.40) $\quad \sqrt{k_j}|P_B(\varphi(t) + u_j^+(t) + u_j^0 + v_j)| \to +\infty$ as $j \to \infty$, for a.e. $t \in [0, T]$.

As above, by using (3.40), $(H_2)(i)$ and Lemma 3.2, we obtain (3.37), which contradicts (3.30). That concludes the proof of Lemma 3.5.

We claim that $\left\|x_k^i\right\|_\infty = \left\|u_k^i\right\|_\infty \to \infty$ as $k \to \infty$. Indeed, if we suppose otherwise, (u_k^i) possesses a bounded subsequence $(u_{k_n}^i)$. Since

(3.41) $\qquad \dfrac{\phi_{k_n}(u_{k_n}^i)}{k_n} = -\dfrac{1}{2}\int_0^T \langle H'(k_n t, u_{k_n}^i), u_{k_n}^i\rangle dt + \int_0^T H(k_n t, u_{k_n}^i)dt$

the sequence $(\dfrac{b_{k_n}^i}{k_n})$ is bounded, contrary to (3.26) with (3.28). Consequently,

we have

(3.42) $$\lim_{k \to \infty} \left\| x_k^i \right\|_\infty = \lim_{k \to \infty} \left\| u_k^i \right\|_\infty = +\infty,$$

that concludes the proof of Theorem 3.2.

Proof of Theorem 3.3. As in Remark 3.2, we can assume without loss of generality that the Hamiltonian H satisfies $(H_2')(i)$. The following estimate will be needed later.

LEMMA 3.6. *If assumptions* (H_0), (H_1') *and* (H_2') *hold, then for all* $x \in \mathbb{R}^{2N}$ *such that* $|P_B(x)| \geq 1$ *and almost every* $t \in C$, *we have*

(3.43) $$H(t, x) \geq H(t, P_A(x)) + \frac{c}{\beta}(|P_B(x)|^\beta - 1)$$

$$+ d \operatorname{Log} |P_B(x)| - \frac{a(t)}{\alpha + 1} - b(t).$$

Proof. For $x \in \mathbb{R}^{2N}$ such that $|P_B(x)| \geq 1$ and for almost every $t \in C$, we have

(3.44) $$H(t, x) = H(t, P_A(x)) + \int_0^{\frac{1}{|P_B(x)|}} \langle H'(t, P_A(x) + sP_B(x)), P_B(x) \rangle ds$$

$$+ \int_{\frac{1}{|P_B(x)|}}^1 \langle H'(t, P_A(x) + sP_B(x)), P_B(x) > ds.$$

By (H_1'), we have

(3.45) $$\left| \int_0^{\frac{1}{|P_B(x)|}} \langle H'(t, P_A(x) + sP_B(x)), P_B(x) \rangle ds \right|$$

$$\leq \int_0^{\frac{1}{|P_B(x)|}} [a(t) |sP_B(x)|^\alpha + b(t)] |P_B(x)| \, ds = \frac{a(t)}{\alpha + 1} + b(t).$$

On the other hand by $(H_2')(i)$, we have

(3.46) $$\int_{\frac{1}{|P_B(x)|}}^1 \langle H'(t, P_A(x) + sP_B(x)), P_B(x) \rangle ds$$

$$\geq \int_{\frac{1}{|P_B(x)|}}^1 [c |sP_B(x)|^\beta + d] \frac{ds}{s} = \frac{c}{\beta}(|P_B(x)|^\beta - 1) + d Log(|P_B(x)|).$$

Then property (3.43) follows from properties (3.44)–(3.46), which completes the proof of Lemma 3.6.

Now, since (H_1') implies (H_1) and (3.43) with (H_0) and $(H_2')(i)$ imply $(H_2)(i)$, we deduce from Theorem 3.2 that for all integer $k \geq 1$, the system $(\mathcal{H})$ possesses $(p+1)$ kT-periodic solutions $x_k^1, \ldots, x_k^{p+1}$ geometrically distinct such that for all $i = 1, \ldots, p+1$, $\lim_{k \to \infty} \left\| x_k^i \right\|_\infty = +\infty$. It remains to study the minimal period of x_k^i for all $i = 1, \ldots, p+1$.

The following lemma will be needed (see Proposition 3.2 in [5] for a proof).

LEMMA 3.7. *If x is a T-periodic solution of $(\mathcal{H})$, then we have*

$$(3.47) \qquad \int_0^T \langle H'(t,x), x \rangle \, dt \leq \frac{T}{2\pi} \int_0^T \left| H'(t,x) \right|^2 dt.$$

Consider the family of functionals

$$\psi_k(x) = \frac{1}{2} \int_0^{kT} \langle J\dot{x}, x \rangle \, dt + \int_0^{kT} H(t,x) \, dt$$

defined respectively on the spaces X_k defined as the space X introduced above, with $E = H^{\frac{1}{2}}(S^1, \mathbb{R}^{2N})$ and $S^1 = \mathbb{R}/kT\mathbb{Z}$ in this case.

It is easy to see that for all $k \geq 1$ and for all $i = 1, \ldots, p+1$, x_k^i is a critical point of ψ_k and by (3.27), we have

$$(3.48) \qquad \lim_{k \to \infty} \frac{1}{k} \psi_k(x_k^i) = +\infty.$$

In a first step, we will show that the set S_T of T-periodic solutions of $(\mathcal{H})$ is bounded in X. Assume by contradiction that there exists a sequence (x_k) in S_T such that $\|x_k\| \to \infty$ in X as $k \to \infty$. Let us write $x_k = x_k^+ + x_k^- + x_k^0 + v_k$ where $x_k^j \in E^j$, $j = 0, -, +$ and $v_k \in V$. Multiplying both sides of the identity

$$J\dot{x}_k + H'(t, x_k) = 0$$

by x_k^+ and integrating, we obtain

$$(3.49) \qquad \left\| x_k^+ \right\|^2 + \int_0^T < H'(t, x_k), x_k^+ > dt = 0.$$

By Hölder's inequality, assumption (H_1') and Proposition 2.2, there exist two constants $c_1 > 0$ and c_2 such that

$$(3.50) \qquad \left\| x_k^+ \right\| \leq c_1 \left\| P_B(x_k) \right\|^\alpha + c_2.$$

Similarly

$$(3.51) \qquad \left\| x_k^- \right\| \leq c_1 \left\| P_B(x_k) \right\|^\alpha + c_2.$$

We conclude from (3.50) and (3.51) that the sequence (x_k) is bounded if and only if the sequence $(P_B(x_k))$ is bounded. Assume that $(P_B(x_k))$ is

not bounded, we can assume, by going to a subsequence if necessary, that $\|P_B(x_k)\| \to \infty$ as $k \to \infty$. We deduce as in the proof of Lemma 3.3 that

$$(3.52) \qquad y_k = \frac{x_k}{\|P_B(x_k)\|} \to y \in B, \ \|y\| = 1, \text{ as } k \to \infty.$$

Since the embedding $X \hookrightarrow L^2$, $x \mapsto x$ is compact, we may assume without loss of generality that

$$(3.53) \qquad y_k(t) \to y \text{ as } k \to \infty \text{ for a.e. } t \in [0, T]$$

and consequently

$$(3.54) \qquad |P_B(x_k)(t)| \to +\infty \text{ as } k \to \infty \text{ for a.e. } t \in [0, T].$$

So by Fatou's lemma, we obtain

$$(3.55) \qquad \int_0^T |P_B(x_k)(t)|^\beta \, dt \to +\infty \text{ as } k \to \infty.$$

On the other hand, by $(H_2')(i)$ we have

$$(3.56) \quad c \int_0^T |P_B(x_k)(t)|^\beta \le \int_0^T \langle H'(t, x_k), P_B(x_k)\rangle dt - dT$$

$$= \int_0^T \langle H'(t, x_k), x_k\rangle dt - \int_0^T \langle H'(t, x_k), P_A(x_k)\rangle dt - dT.$$

By Lemma 3.7, assumption (H_1') and Hölder's inequality, we can find two positive constants c_3, c_4 such that

$$(3.57) \quad \int_0^T \langle H'(t, x_k), x_k\rangle dt \le \frac{T}{2\pi} \int_0^T |H'(t, x_k)|^2 \, dt$$

$$\le \frac{T}{2\pi} \int_0^T [a(t) |P_B(x_k)|^\alpha + b(t)]^2 dt$$

$$\le \frac{T}{4\pi} \int_0^T [a^2(t) |P_B(x_k)|^{2\alpha} + b^2(t)] dt$$

$$\le \frac{T}{4\pi} \Big(\int_0^T a^{\frac{2\beta}{\beta-2\alpha}}(t)\Big)^{\frac{\beta-2\alpha}{2\beta}} \Big(\int_0^T |P_B(x_k)(t)|^\beta \, dt\Big)^{\frac{2\alpha}{\beta}} + 2\|b\|_{L^2}^2$$

$$= c_3 \|P_B(x_k)\|_{L^\beta}^{2\alpha} + c_4.$$

On the other hand, as in (3.11), there exists a constant $c_5 > 0$ such that

$$(3.58) \qquad \|x_k^i\| \le c_5 |x_k^0|^\alpha, \ i = -, +.$$

Therefore, by Proposition 2.2, there exist two positive constants c_6, c_7 such that

$$(3.59) \qquad \|x_k\|_{L^2} \le c_6 \|P_B(x_k)\|_{L^\beta}^\alpha + c_7.$$

By Hölder's inequality, (3.57) and (3.59), we have

$$(3.60) \quad \left| \int_0^T \langle H'(t, x_k), P_A(x_k) \rangle dt \right| \le \|P_A(x_k)\|_{L^2} \left(\int_0^T |H'(t, x_k)|^2 \, dt \right)^{\frac{1}{2}}$$

$$\le [c_6 \|P_B(x_k)\|_{L^\beta}^\alpha + c_7][c_3 \|P_B(x_k)\|_{L^\beta}^{2\alpha} + c_4]^{\frac{1}{2}}$$

$$\le c_8 \|P_B(x_k)\|_{L^\beta}^{2\alpha} + c_9$$

where c_8, c_9 are two positive constants.

Combining (3.56), (3.57) and (3.60), we can find two constants $c_{10}, c_{11} > 0$ such that

$$(3.61) \qquad \|P_B(x_k)\|_{L^\beta}^\beta \le c_{10} \|P_B(x_k)\|_{L^\beta}^{2\alpha} + c_{11}.$$

However (3.61) contradicts (3.55) because $\beta > 2\alpha$. Hence S_T is bounded and as a consequence $\psi_1(S_T)$ is bounded. Since for any $x \in S_T$ one has $\psi_k(x) = k\psi_1(x)$, there exists a positive constant M such that

$$(3.62) \qquad \forall x \in S_T, \ \forall k \ge 1, \ \frac{1}{k} |\psi_k(x_k)| \le M.$$

Consequently, (3.48) and (3.62) show that for all $i = 1, \ldots, p+1$ and for all k sufficiently large, we have $x_k^i \notin S_T$. So if k is chosen to be prime number, the minimal period of x_k^i has to be kT and the proof of Theorem 3.3 is complete.

References

[1] A. Daouas, M. Timoumi, *Subharmonics for not uniformly coercive Hamiltonian systems*, Nonlinear Anal. 66 (2007), 571–581.

[2] I. Ekeland, H. Hofer, *Subharmonics for convex nonautonomous hamiltonian systems*, Comm. Pure Appl. Math. 40 (1987), 1–36.

[3] A. Fonda, A. C. Lazer, *Subharmonic solutions of conservative systems with non convex potential*, Amer. Math. Soc. 115 (1990), 183–190.

[4] G. Fournier, D. Lupo, M. Ramos, M.Willem, *Limit relative category and critical point theory*, Dynam. Report. 3 (1994), 1–24.

[5] J. Mawhin, M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer, 1989.

[6] Z. Q. Ou, C. L. Tang, *Periodic and subharmonic solutions for a class of superquadratic Hamiltonian systems*, Nonlinear Anal. (2004), 245–258.

[7] P. H. Rabinowitz, *On subharmonic solutions of Hamiltonian systems*, Com. Pure Appl. Math. 33 (1980), 609–633.

[8] E. Silva, *Subharmonic solutions for subquadratic Hamiltonian systems*, J. Differential Equations 115 (1995), 120–145.

[9] M. Timoumi, *Subharmonics of a Hamiltonian systems class*, Demonstratio Math. 37 (2004), 977–990.

[10] M. Timoumi, *Subharmonics of nonconvex Hamiltonian systems*, Arch. Math. (1999), 422–429.

[11] M. Willem, *Subharmonic oscillations of convex hamiltonian systems*, Nonlinear Anal. 9 (1985), 1303–1311.

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