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Bogdan Szal

APPLICATION OF THE $MRBVS$ CLASSES TO EMBEDDING RELATIONS OF THE BESOV CLASSES

Abstract. L. Leindler obtained a necessary and sufficient condition in order to a function $f \in L^p$ having Fourier coefficients of rest bounded variation belong to the Besov class. In the present paper the analogue of this result is proved with function having Fourier coefficients of mean rest bounded variation. We also discuss embedding relations between the Besov classes.

1. Introduction

The properties of the Besov classes have been studied by many authors (see [7], [13], [14]). First we recall some results concerning classes of sequences.

In [10] Leindler defined a new such class.

Definition 1. Let $\gamma := (\gamma_n)$ be a positive sequence. A null sequence $c := (c_n)$ of real numbers satisfying the inequality

$$\sum_{n=m}^{\infty} |c_n - c_{n+1}| \leq K(c)\,\gamma_m, \quad m = 1, 2, ...$$

with a positive constant $K(c)$ is said to be a sequence of γ Rest Bounded Variation, in symbol: $c \in \gamma RBVS$.

If $\gamma \equiv c$ and $c_n > 0$, then we call the sequence c the Rest Bounded Variation Sequence; and briefly we write $c \in RBVS$. In [9] and [11] Leindler introduced the class of Mean Rest Bounded Variation Sequences ($MRBVS$), where γ is defined by a certain arithmetical mean of the coefficients, e.g.,

$$\gamma_m^* := \frac{1}{m} \sum_{n \geq m/2}^{m} c_n \tag{1.1}$$

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or

$$\overline{\gamma}_m := \frac{1}{m}\sum_{n=m}^{2m-1} c_n.$$

It is easy to see that the class $\gamma_m^* MRBVS$ includes the class $RBVS$, consequently the almost monotone and monotone sequences, too; but $\overline{\gamma}_m MRBS$ does not, in general. In [16] we proved that $RBVS \neq \gamma_m^* MRBVS$. Namely, we shown that the sequence

$$(1.2)\qquad d_n := \begin{cases} 1 & \text{if } n = 1, \\ \dfrac{1+m+(-1)^n m}{(2^{\mu_m})^2 m} & \text{if } \mu_m \leq n < \mu_{m+1}, \end{cases}$$

where $\mu_m = 2^m$ for $m = 1, 2, 3...$, belongs to the class $\gamma_m^* MRBVS$ but it does not belong to the class $RBVS$.

The aim of the present paper is to show $\overline{\gamma}_m MRBVS \subset \gamma_m^* MRBVS$ and $\overline{\gamma}_m MRBVS \neq \gamma_m^* MRBVS$. In [7] and [11] L. Leindler generalized the results of Potapov [13] and Potapov and Berisha [14] to the class $RBVS$ and $\overline{\gamma}_m MRBVS$, respectively. In present paper we shall also prove that these results are true for the class $\gamma_m^* MRBVS$.

Let L^p, $1 \leq p \leq \infty$, be the space of 2π periodic, measurable functions f with the norm

$$\|f\|_p := \begin{cases} \left(\int_0^{2\pi} |f(x)|^p \, dx\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \|f\|_C := \max_x |f(x)| & p = \infty. \end{cases}$$

Denote by $E_n(f)_p$ the best approximation of f in L^p-metric by trigonometric polynomials T_n of the degree at most n and by $\omega_k(f;t)_p$ the modulus of continuity of order $k \in \mathbb{N}$

$$\omega_k(f;t)_p = \sup_{|h|\leq t}\left\|\sum_{v=0}^{k}(-1)^{k-v} C_k^{\nu} f(x+\nu h)\right\|_p, \quad C_k^v = \binom{k+v}{k}.$$

A function α is called σ-type if it is measurable on $[0,1]$, integrable on $[\delta, 1]$ for every $\delta \in (0,1)$, and there exist positive constants C_1 and C_2 such that

(i) $\alpha(t) \geq C_1$ for all $t \in [0,1]$,

(ii) $\int_0^{\delta} \alpha(t)\, t^{\sigma} dt \leq C_2 \delta^{\sigma} \int_{\delta}^{2\delta} \alpha(t)\, dt$ for all $\delta \in (0, \delta_0)$, where $0 < \delta_0 \leq \frac{1}{2}$ is given.

A positive function α is said to satisfy λ-condition, $\lambda > 0$, if there exists a positive constant C_3 such that

$$\int_{2\delta}^{1} \alpha(t)\, t^{\lambda} dt \le C_3 \delta^{\lambda} \int_{\delta}^{2\delta} \alpha(t)\, dt, \quad \text{for all } \delta \in (0, \delta_0).$$

We say that $f \in B(p, \theta, \alpha)$ if

(i) $f \in L^p$,
(ii) $0 < \theta < \infty$,
(iii) $\alpha(t)$ is σ-type,
(iv) $\int_0^1 \alpha(t)\, \omega_k^{\theta}(f; t)_p\, dt < \infty$, $k \ge \frac{\sigma}{\theta}$.

We write $I_1 \ll I_2$ if there exists a positive constant K such that $I_1 \le K I_2$. If $I_1 \ll I_2$ and $I_2 \ll I_1$ hold simultaneously, then we write $I_1 \asymp I_2$.

Throughout the paper we shall also use the following notation:

$$S(a_\nu, q, k, n) := \begin{cases} |a_n| & \text{if } q = 1, \\ n^{-k}\left(\sum\limits_{\nu=1}^{n} |a_\nu|^q \nu^{(k+1)q-2}\right)^{\frac{1}{q}} + \left(\sum\limits_{\nu=n+1}^{\infty} |a_\nu|^q \nu^{q-2}\right)^{\frac{1}{q}} & \text{if } 1 < q < \infty, \\ n^{-k} \sum\limits_{\nu=1}^{n} |a_\nu|\, \nu^k + \sum\limits_{\nu=n+1}^{\infty} |a_\nu| & \text{if } q = \infty. \end{cases}$$

2. Main results

Now, we formulate our results.

Theorem 1. *The class $\gamma_m^* MRBVS$ includes the class $\overline{\gamma}_m MRBVS$ but $\gamma_m^* MRBVS \ne \overline{\gamma}_m MRBVS$.*

Theorem 2. *Let for $1 < p < q \le \infty$, the function α satisfies λ-condition with*

$$\lambda = \left(\frac{1}{p} - \frac{1}{q}\right)\theta,\ 0 < \theta < \infty,\ \ \alpha^*(t) = \alpha(t)\, t^{\lambda}.$$

If $(a_n) \in \gamma_m^* MRBVS$ and f has the Fourier expansion

$$f(x) \sim \sum_{n=1}^{\infty} a_n \cos nx, \tag{2.1}$$

then the Besov classes $B(p, \theta, \alpha)$ and $B(q, \theta, \alpha^*)$ coincide. Furthermore, for any

$$k_1 \ge \frac{\sigma}{\theta},\ \ k_2 \ge \frac{\sigma^*}{\theta},\ \ k_3 \ge \frac{\sigma^*}{\theta},\ \ \sigma^* = \sigma - \lambda,$$

we have

$$\int_0^1 \alpha^*(t)\,\omega_{k_2}^{\theta}(f;t)_q\,dt \ll \int_0^1 \alpha(t)\,\omega_{k_1}^{\theta}(f;t)_p\,dt \ll \int_0^1 \alpha^*(t)\,\omega_{k_3}^{\theta}(f;t)_q\,dt. \tag{2.2}$$

Theorem 3. *If $f \in L^p$, $1 < p < \infty$, $(a_n) \in \gamma_m^* MRBVS$ and f has the Fourier expansion* (2.1)*, then*

$$S(a_\nu, p, k, n) \ll \omega_k\left(f; \frac{1}{n}\right)_p \ll S(a_\nu, p, k, n). \tag{2.3}$$

Theorem 4. *Let $f \in L^p$, $1 < p < \infty$, $(a_n) \in \gamma_m^* MRBVS$ and f has the Fourier expansion* (2.1)*, $\alpha(t) = t^{-r\theta-1}$ and $k > r$. $f \in B(p,\theta,\alpha)$ with $\theta \geq 1$ if and only if*

$$J_1 := \sum_{n=1}^{\infty} a_n^{\theta} n^{r\theta+\theta-\frac{\theta}{p}-1} < \infty.$$

Corollary 1. *Let $f \in L^p$, $1 < p < \infty$, $(a_n) \in \gamma_m^* MRBVS$ and f has the Fourier expansion* (2.1)*, $\alpha(t) = t^{-r\theta-1}$ and $k > r$. If $\theta \leq 1$, then a sufficient condition for $f \in B(p,\theta,\alpha)$ is*

$$J_2 := \sum_{n=1}^{\infty} a_n^{\theta} n^{r\theta-\frac{\theta}{p}} < \infty$$

and a necessary condition is

$$J_1 = \sum_{n=1}^{\infty} a_n^{\theta} n^{r\theta+\theta-\frac{\theta}{p}-1} < \infty.$$

Remark 1. Theorem 1 implies that Theorems 2, 3, 4 improve the results of [7] and [11].

3. Lemmas

To prove our theorem, the following lemmas will be needed.

Lemma 1. [5, Corollary 1] *If $\lambda_n > 0$ and $a_n \geq 0$, then*

$$\sum_{n=1}^{\infty} \lambda_n \left(\sum_{k=1}^{n} a_k\right)^p \leq p^p \sum_{n=1}^{\infty} \lambda_n^{1-p} a_n^p \left(\sum_{k=n}^{\infty} \lambda_k\right)^p \tag{3.1}$$

and

$$\sum_{n=1}^{\infty} \lambda_n \left(\sum_{k=1}^{\infty} a_k\right)^p \leq p^p \sum_{n=1}^{\infty} \lambda_n^{1-p} a_n^p \left(\sum_{k=1}^{n} \lambda_k\right)^p \tag{3.2}$$

hold for any $p \geq 1$; while if $0 < p < 1$, then the inequality in (3.1) *and* (3.2) *hold with opposite direction.*

Lemma 2. [3, Theorem 19] *If $a_n \geq 0$ and $0 < \alpha < \beta < \infty$, then*

$$\left(\sum_{n=1}^{\infty} a_n^{\beta}\right)^{\frac{1}{\beta}} \leq \left(\sum_{n=1}^{\infty} a_n^{\alpha}\right)^{\frac{1}{\alpha}}.$$

Lemma 3. [1, p. 293] *If $f \in L^{\infty} \equiv C$ and*

$$f(x) = \sum_{n=1}^{\infty} a_n \cos nx, \quad x \in [-\pi, \pi],$$

where $a_n \geq 0$, then

$$\sum_{k=2n}^{\infty} a_k \leq 4E_n(f)_C.$$

Lemma 4. [17] *If $f \in L^p$, $1 < p \leq 2$, then*

$$\omega_k\left(f; \frac{1}{n}\right)_p \ll n^{-k}\left(\sum_{\nu=1}^{n} \nu^{kp-1} E_{\nu}^{p}(f)_p\right)^{\frac{1}{p}};$$

while if $p \geq 2$, then the reverse inequality holds.

Lemma 5. [12, p. 847] *If $f \in L^p$, $1 \leq p \leq \infty$, $0 < \theta < \infty$, α is a σ-type function and $k \geq \frac{\sigma}{\theta}$, then*

$$E_0^{\theta}(f)_p + E_1^{\theta}(f)_p + \sum_{\nu=1}^{\infty} \mu(\nu) E_{2^{\nu}}^{\theta}(f)_p \asymp \int_0^1 \alpha(t)\, \omega_k^{\theta}(f; t)_p\, dt,$$

where

$$\mu(n) := \int_{2^{-n}}^{2^{-n+1}} \alpha(t)\, dt, \ \ n \geq 1 \ \ \text{and} \ \ \mu(0) = 1.$$

Lemma 6. *If α is a σ-type function, then*

$$\mu(n+1) \ll \mu(n) \tag{3.3}$$

and

$$\mu(n) \ll \mu(1) \ll 1 \tag{3.4}$$

hold for all natural number n.

Proof. First we prove (3.3). If α is a σ-type function, then for $n \geq 1$ we have

$$\mu(n+1) = \int_{2^{-n-1}}^{2^{-n}} \alpha(t)\, dt \leq \left(2^{n+1}\right)^{\sigma} \int_{2^{-n-1}}^{2^{-n}} \alpha(t)\, t^{\sigma} dt \leq \left(2^{n+1}\right)^{\sigma} \int_{0}^{2^{-n}} \alpha(t)\, t^{\sigma} dt$$

$$\leq 2^{\sigma} C_2 \int_{2^{-n}}^{2^{-n+1}} \alpha(t)\, dt \ll \mu(n).$$

Since $\mu(1) = \int_{\frac{1}{2}}^{1} \alpha(t)\,dt \ll 1$, then by (3.3) we obtain that (3.4) also holds. This completes the proof. ∎

Lemma 7. [2, p. 37] *Let f has the Fourier expansion* (2.1), *where $a_n \geq 0$ and let $\frac{1}{p} - 1 < \beta < \frac{1}{p}$, then a sufficient condition for $x^{-\gamma} f(x) \in L^p$ is*

$$\sum_{n=1}^{\infty} n^{p+p\beta-2} \left(\sum_{k=n}^{\infty} |a_k - a_{k+1}| \right)^p < \infty \tag{3.5}$$

and a necessary condition is

$$\sum_{n=1}^{\infty} n^{p+p\beta-2} \left(\sum_{k=n}^{\infty} k^{-1} a_k \right)^p < \infty. \tag{3.6}$$

Lemma 8. *Let f has the Fourier expansion* (2.1) *with $(a_n) \in \gamma_m^* MRBVS$. If $1 < p < \infty$ and $\frac{1}{p} - 1 < \beta < \frac{1}{p}$, then $x^{-\beta} f(x) \in L^p$ if and only if*

$$\sum_{n=1}^{\infty} n^{p+p\beta-2} \left(\gamma_n^*\right)^p < \infty. \tag{3.7}$$

Proof. The sufficiency of condition (3.7) follows easily from Lemma 7, since (3.7) and $(a_n) \in \gamma_m^* MRBVS$ imply (3.5).

Now, we prove the necessity of (3.7). It is clear that

$$\sum_{k=n}^{\infty} \frac{a_k}{k} \geq \sum_{k=n}^{2n} \frac{a_k}{k} \geq \frac{1}{2n} \sum_{k=n}^{2n} a_k \geq \frac{1}{2} \gamma_{2n}^*. \tag{3.8}$$

Similarly, we get that

$$\sum_{k=n}^{\infty} \frac{a_k}{k} \geq \sum_{k=n}^{2n+1} \frac{a_k}{k} \geq \frac{1}{2n+1} \sum_{k=\frac{2n+1}{2}}^{2n+1} a_k \geq \frac{1}{2} \gamma_{2n+1}^*. \tag{3.9}$$

Using inequalities (3.8) and (3.9) into (3.6), we see that (3.6) implies (3.7).

Thus, if $x^{-\gamma} f(x) \in L^p$, then by Lemma 7 (3.6) holds whence (3.7) is also true. The proof is complete. ∎

In special case when $\beta = 0$ the above lemma leads us to the following remark.

Remark 2. If $1 < p < \infty$ and $(a_n) \in \gamma_m^* MRBVS$, then $f \in L^p$ if and only if

$$\sum_{n=1}^{\infty} n^{p-2} \left(\gamma_n^*\right)^p < \infty$$

or, more precisely

$$\|f\|_p^p \asymp \sum_{n=1}^{\infty} n^{p-2} (\gamma_n^*)^p .$$

Lemma 9. *Assume that f has the Fourier expansion (2.1), where $(a_n) \in \gamma_m^* MRBVS$. If $1 < p < \infty$ and*

$$\sum_{n=1}^{\infty} n^{p-2} (\gamma_n^*)^p < \infty,$$

then

$$E_n(f)_p \ll \gamma_{n+1}^* (n+1)^{1-\frac{1}{p}} + \left(\sum_{k=n+1}^{\infty} k^{p-2} (\gamma_k^*)^p \right)^{\frac{1}{p}}.$$

The above lemma is a special case of Theorem 1 in [8] for cosine series with $\gamma \equiv \gamma_m^*$.

Lemma 10. [4, Theorem 3] *If $f \in L^p$, $1 < p < \infty$, $a_n \geq 0$ and f has the Fourier expansion (2.1), then*

$$\sum_{\nu=2n}^{\infty} k^{-2} a_\nu \leq n^{-2+\frac{1}{p}} E_n(f)_p.$$

Lemma 11. [13, Theorem 2] *If $f \in B(p,\theta,\alpha)$, $1 < p < q \leq \infty$ and α satisfies λ-condition with $\lambda = \left(\frac{1}{p} - \frac{1}{q}\right)\theta$, then $f \in B(q,\theta,\alpha^*)$, where*

$$\alpha^*(t) := \alpha(t)\, t^{\lambda}, \quad \text{that is,} \quad B(p,\theta,\alpha) \subset B(q,\theta,\alpha^*);$$

furthermore,

$$\int_0^1 \alpha^*(t)\, \omega_{k_2}^{\theta}(f;t)_q\, dt \ll \int_0^1 \alpha(t)\, \omega_{k_1}^{\theta}(f;t)_p\, dt$$

for any

$$k_1 \geq \frac{\sigma}{\theta}, \quad k_2 \geq \frac{\sigma^*}{\theta} \quad \text{and} \quad \sigma^* := \sigma - \left(\frac{1}{p} - \frac{1}{q}\right) + \varepsilon, \quad \varepsilon > 0.$$

Lemma 12. *If $f \in L^p$, $1 < p < \infty$, $(a_n) \in \gamma_m^* MRBVS$ and f has the Fourier expansion (2.1), then*

$$E_n(f)_p \gg \left(\sum_{\nu=4n}^{\infty} \nu^{p-2} (\gamma_\nu^*)^p \right)^{\frac{1}{p}}.$$

Proof. We apply Remark 2 to the function

$$f(x) - \sum_{k=1}^{4n-1} a_k \cos kx + a_{4n} \sum_{k=1}^{4n-1} \cos kx.$$

Thus

$$\sum_{k=4n}^{\infty} k^{p-2}\left(\gamma_k^*\right)^p \ll a_{4n}^p \left\| \sum_{k=1}^{4n-1} \cos(k\cdot) \right\|_p^p + \|f - S_{4n-1}(f)\|_p^p .$$

Since

$$\left\| \sum_{k=1}^{4n-1} \cos(k\cdot) \right\|_p^p = 2\int_0^{\pi} \left| \sum_{k=1}^{4n-1} \cos kx \right|^p dx = 2\left(\int_0^{\frac{\pi}{n}} + \int_{\frac{\pi}{n}}^{\pi} \right) \left| \frac{\cos 2nx \sin \frac{4n-1}{2}x}{\sin \frac{x}{2}} \right|^p dx$$
$$\ll n^p \int_0^{\frac{\pi}{n}} dx + \int_{\frac{\pi}{n}}^{\pi} \frac{1}{x^p} dx \ll n^{p-1},$$

by the theorem of M. Riesz [15], we get

$$\sum_{k=4n}^{\infty} k^{p-2}\left(\gamma_k^*\right)^p \ll E_{4n-1}^p(f)_p + a_{4n}^p n^{p-1} \le E_n^p(f)_p + a_{4n}^p n^{p-1}. \tag{3.10}$$

If $(a_n) \in \gamma_m^* MRBVS$ then applying Lemma 10 we obtain

$$\begin{aligned} a_{4n}^p n^{p-1} &\le n^{p-1}\left(\sum_{k=4n}^{\infty} |a_k - a_{k+1}| \right)^p \ll n^{p-1}\left(\frac{1}{2n} \sum_{k=2n}^{4n} a_k \right)^p \\ &\ll n^{2p-1}\left(\sum_{k=2n}^{4n} k^{-2} a_k \right)^p \ll n^{2p-1}\left(\sum_{k=2n}^{\infty} k^{-2} a_k \right)^p \ll E_n^p(f)_p. \end{aligned} \tag{3.11}$$

The inequalities (3.10) and (3.11) imply the assertion of Lemma 12. ■

Lemma 13. *Assume that $f \in L^p$, $1 < p < q \le \infty$, has the Fourier expansion* (2.1) *with $(a_n) \in \gamma_m^* MRBVS$. If $q < \infty$, then*

$$S_1 := \sum_{k=4n}^{\infty} k^{\frac{q}{p}-2} E_k^q(f)_p \ll E_n^q(f)_q$$

and if $q = \infty$, then

$$S_2 := \sum_{k=4n}^{\infty} k^{\frac{1}{p}-1} E_k(f)_p \ll E_n(f)_q.$$

Proof. By Lemma 9, we have

$$S_1 \ll \sum_{k=4n}^{\infty} k^{\frac{q}{p}-2}\left(\gamma_{k+1}^*\right)^q (k+1)^{q\left(1-\frac{1}{p}\right)} + \sum_{k=4n}^{\infty} k^{\frac{q}{p}-2}\left(\sum_{l=k+1}^{\infty} \left(\gamma_l^*\right) l^{p-2} \right)^{\frac{q}{p}}.$$

Using the inequality (3.2) of Lemma 1 and Lemma 12, we get

$$S_1 \ll \sum_{k=4n}^{\infty} \left(\gamma_{k+1}^*\right)^q (k+1)^{q-2} + \sum_{k=4n}^{\infty} \left(\gamma_{k+1}^*\right)^q (k+1)^{q-2+\frac{q}{p}-\left(\frac{q}{p}\right)^2} \left(\sum_{l=1}^{k+1} l^{\frac{q}{p}-2}\right)^{\frac{q}{p}}$$
$$\ll \sum_{k=4n}^{\infty} (\gamma_k^*)^q k^{q-2} \ll E_n^q(f)_q.$$

To estimate S_2 we apply Lemma 9. Thus

$$S_2 \ll \sum_{k=4n}^{\infty} k^{\frac{1}{p}-1} \gamma_{k+1}^* (k+1)^{1-\frac{1}{p}} + \sum_{k=4n}^{\infty} k^{\frac{1}{p}-1} \left(\sum_{l=k+1}^{\infty} (\gamma_l^*)^p l^{p-2}\right)^{\frac{1}{p}}$$

$$= \sum_{k=4n}^{\infty} \gamma_{k+1}^* + \sum_{k=4n}^{\infty} k^{\frac{1}{p}-1} \left(\sum_{l=k+1}^{\infty} (\gamma_l^*)^p l^{p-2}\right)^{\frac{1}{p}} = S_{21} + S_{22}.$$

Then

$$S_{21} \le \sum_{k=4n}^{\infty} \gamma_k^* = \sum_{i=2}^{\infty} \sum_{k=2^i n}^{2^{i+1}} \gamma_k^* \ll \sum_{i=2}^{\infty} \sum_{k=2^i n}^{2^{i+1}} \frac{1}{k} \sum_{v=\frac{k}{2}}^{k} a_v$$

$$\ll \sum_{i=2}^{\infty} \sum_{v=2^{i-1}n}^{2^{i+1}n} a_v \sum_{k=2^i n}^{2^{i+1}} \frac{1}{k} \ll \sum_{i=2}^{\infty} \sum_{v=2^{i-1}n}^{2^{i+1}n} a_v$$

and

$$S_{22} \le \sum_{m=2}^{\infty} \sum_{k=2^m n}^{2^{m+1}n} k^{\frac{1}{p}-1} \left(\sum_{l=2^m n}^{\infty} (\gamma_l^*)^p l^{p-2}\right)^{\frac{1}{p}}$$
$$= \sum_{m=2}^{\infty} \sum_{k=2^m n}^{2^{m+1}n} k^{\frac{1}{p}-1} \left(\sum_{i=m}^{\infty} \sum_{l=2^i n}^{2^{i+1}n} (\gamma_l^*)^p l^{p-2}\right)^{\frac{1}{p}}$$
$$\ll \sum_{m=2}^{\infty} \sum_{k=2^m n}^{2^{m+1}n} k^{\frac{1}{p}-1} \left(\sum_{i=m}^{\infty} \sum_{l=2^i n}^{2^{i+1}n} \left(\sum_{v=\frac{l}{2}}^{l} a_v\right)^p l^{-2}\right)^{\frac{1}{p}}$$
$$\ll \sum_{m=2}^{\infty} (2^m n)^{\frac{1}{p}} \sum_{i=m}^{\infty} \left(\sum_{l=2^i n}^{2^{i+1}n} \left(\sum_{v=\frac{l}{2}}^{l} a_v\right)^p l^{-2}\right)^{\frac{1}{p}}$$
$$\ll \sum_{m=2}^{\infty} (2^m n)^{\frac{1}{p}} \sum_{i=m}^{\infty} \sum_{v=2^{i-1}n}^{2^{i+1}n} a_v \left(\sum_{l=2^i n}^{2^{i+1}n} l^{-2}\right)^{\frac{1}{p}}$$

$$\ll \sum_{m=2}^{\infty} (2^m n)^{\frac{1}{p}} \sum_{i=m}^{\infty} \left(2^i n\right)^{-\frac{1}{p}} \sum_{v=2^{i-1}n}^{2^{i+1}n} a_v$$

$$\ll \sum_{i=2}^{\infty} \left(2^i n\right)^{-\frac{1}{p}} \sum_{v=2^{i-1}n}^{2^{i+1}n} a_v \sum_{m=2}^{i} (2^m n)^{\frac{1}{p}} \ll \sum_{i=2}^{\infty} \sum_{v=2^{i-1}n}^{2^{i+1}n} a_v.$$

Hence

$$S_2 \ll \sum_{i=2}^{\infty} \sum_{v=2^{i-1}n}^{2^{i+1}n} a_v = \sum_{i=2}^{\infty} \left(\sum_{v=2^{i-1}n}^{2^i n-1} a_v + a_{2^i n} + \sum_{v=2^i n+1}^{2^{i+1}n} a_v \right)$$

$$\leq \sum_{v=2n}^{\infty} a_v + \sum_{v=4n}^{\infty} a_v + \sum_{v=4n+1}^{\infty} a_v \ll \sum_{v=2n}^{\infty} a_v$$

and by Lemma 3

$$S_2 \ll E_n(f)_q.$$

Thus our proof is complete. ∎

Lemma 14. [14, Theorem 1] *If $f \in L^p$, $1 \leq p \leq \infty$, f has the Fourier expansion* (2.1) *and $\theta_1 := \min(2,p)$, $\theta_2 := \max(2,p)$, then*

$$S(a_\nu, \theta_1, k, n) \ll \omega_k \left(f; \frac{1}{n}\right)_p \ll S(a_\nu, \theta_2, k, n).$$

4. Proofs of Theorems

4.1. Proof of Theorem 1

First we show that $c \in \overline{\gamma}_m MRBVS$ implies that $\overline{\gamma}_k \ll \overline{\gamma}_l$ if $k \geq l$. Indeed, if $\mu \geq l$, then

$$l^{-1} \sum_{\nu=l}^{2l-1} c_v \gg \sum_{\nu=l}^{\infty} |c_\nu - c_{\nu+1}| \geq \sum_{\nu=\mu}^{\infty} |c_\nu - c_{\nu+1}| \geq c_\mu.$$

Thus

$$\sum_{\mu=k}^{2k-1} l^{-1} \sum_{\nu=l}^{2l-1} c_\nu \gg \sum_{\mu=k}^{2k-1} c_\mu,$$

whence

$$\overline{\gamma}_l = l^{-1} \sum_{\nu=l}^{2l-1} c_\nu \gg k^{-1} \sum_{\mu=k}^{2k-1} c_\mu = \overline{\gamma}_k.$$

Using this we have

$$\sum_{n=m}^{\infty} |c_n - c_{n+1}| \ll \overline{\gamma}_m \ll \overline{\gamma}_{\frac{m}{2}} = \frac{2}{m} \sum_{n\geq m/2}^{m-1} c_n \ll \gamma_m^*,$$

therefore $c \in \gamma_m^* MRBVS$.

In [16] we proved that the sequence d, defined in (1.2), belongs to the class $\gamma_m^* MRBVS$. Now, we show that $d \notin \overline{\gamma}_m MRBVS$. Namely, for $m \geq 1$ we have

$$\sum_{i=\mu_m}^{\infty} |d_i - d_{i+1}| \geq \sum_{i=\mu_m}^{2\mu_m - 2} |d_i - d_{i+1}| \geq \sum_{i=\mu_m}^{2\mu_m - 2} \frac{2}{\left(2^{\mu_m}\right)^2} = \frac{2\left(\mu_m - 1\right)}{\left(2^{\mu_m}\right)^2}.$$

Since

$$\overline{\gamma}_m = \frac{1}{\mu_m} \sum_{i=\mu_m}^{2\mu_m - 1} d_i = \frac{1}{\mu_m}\left(\frac{\mu_m}{2}\frac{1+2m}{\left(2^{\mu_m}\right)^2 m} + \frac{\mu_m}{2}\frac{1}{\left(2^{\mu_m}\right)^2 m}\right) = \frac{1+m}{m\left(2^{\mu_m}\right)^2},$$

the inequality

$$\sum_{i=m}^{\infty} |d_i - d_{i+1}| \leq K\left(d\right)\overline{\gamma}_m$$

does not hold, that is, (d_n) does not belong to $\overline{\gamma}_m MRBVS$. Hence $\gamma_m^* MRBVS \neq \overline{\gamma}_m MRBVS$ and our proof is complete. ■

4.2. Proof of Theorem 2

By Lemma 11, the first inequality in (2.2) holds, whence

$$B\left(p, \theta, \alpha\right) \subset B\left(q, \theta . \alpha^*\right).$$

To prove the second inequality in (2.2), we use Lemma 5. If $f \in B\left(q, \theta, \alpha^*\right)$, then

$$I_q := E_0^\theta(f)_q + E_1^\theta(f)_q + \sum_{n=1}^{\infty} \mu^*\left(n\right) E_{2^n}^\theta(f)_q \ll \int_0^1 \alpha^*\left(t\right) \omega_{k3}^\theta\left(f;t\right)_q dt < \infty,$$

where $k_3 \geq \frac{\sigma^*}{\theta}$ and

$$\mu^*\left(n\right) := \int_{2^{-n}}^{2^{-n+1}} \alpha^*\left(t\right) dt \quad \text{for} \quad n \geq 1 \quad \text{and} \quad \mu^*\left(0\right) = 1.$$

Since $q > p > 1$, then for $\lambda = \left(\frac{1}{p} - \frac{1}{q}\right)\theta$ we have

$$\mu\left(n\right) \ll \mu^*\left(n\right) 2^{n\left(\frac{1}{p} - \frac{1}{q}\right)\theta}.$$

From this and Lemma 6, we get

$$\begin{aligned} I_p &:= E_0^\theta(f)_p + E_1^\theta(f)_p + \sum_{n=1}^{\infty} \mu\left(n\right) E_{2^n}^\theta(f)_p \\ &= E_0^\theta(f)_p + E_1^\theta(f)_p + \sum_{n=1}^{3} \mu\left(n\right) E_{2^n}^\theta\left(f\right)_p + \sum_{n=1}^{\infty} \mu\left(n+3\right) E_{2^{n+3}}^\theta(f)_p \end{aligned}$$

$$\ll E_0^\theta(f)_p + E_1^\theta(f)_p + \sum_{n=1}^{\infty} \mu(n)\, E_{2^{n+3}}^\theta(f)_p$$

$$\ll E_0^\theta(f)_p + E_1^\theta(f)_p + \sum_{n=1}^{\infty} \mu^*(n) \left(2^{n\left(\frac{1}{p}-\frac{1}{q}\right)} E_{2^{n+3}}(f)_p\right)^\theta$$

$$\ll E_0^\theta(f)_p + E_1^\theta(f)_p + \sum_{n=1}^{\infty} \mu^*(n) \left(\sum_{k=2^{n+2}}^{2^{n+3}} k^{\left(\frac{1}{p}-\frac{1}{q}\right)-1} E_k(f)_p\right)^\theta .$$

Hence, if $q = \infty$, by Lemma 13

$$I_p \ll E_0^\theta(f)_q + E_1^\theta(f)_q + \sum_{n=1}^{\infty} \mu^*(n)\, E_{2^n}^\theta(f)_q$$

and immediately $I_p \ll I_q$. If $q < \infty$, then applying Hölder's inequality with q and $\frac{q}{q-1}$ and Lemma 13, we obtain

$$I_p \ll E_0^\theta(f)_p + E_1^\theta(f)_p + \sum_{n=1}^{\infty} \mu^*(n) \left(\sum_{k=2^{n+2}}^{2^{n+3}} k^{\frac{q}{p}-2} E_k^q(f)_p\right)^{\frac{\theta}{q}}$$

$$\ll E_0^\theta(f)_q + E_1^\theta(f)_q + \sum_{n=1}^{\infty} \mu^*(n)\, E_{2^n}^\theta(f)_q,$$

and thus $I_p \ll I_q$.

Finally, we use again Lemma 5, whence

$$\int_0^1 \alpha(t)\, \omega_{k_1}^\theta (f;t)_p\, dt \ll I_p \ll I_q \ll \int_0^1 \alpha^*(t)\, \omega_{k_3}^\theta (f;t)_q\, dt < \infty$$

follows with $k_1 \geq \frac{\sigma}{\theta}$. Thus we get the second inequality in (2.2), and therefore

$$B(q,\theta,\alpha^*) \subset B(p,\theta,\alpha)$$

also holds. This completes the proof. ∎

4.3. Proof of Theorem 3

As the begin we verify the first inequality in (2.3). If $1 < p \leq 2$, then (2.3) follows from Lemma 14.

If $2 \leq p < \infty$, then using (1.1), Lemma 12, Jackson's theorem and the property of $\omega_k(f;\delta)_p$, we get

$$\left(\sum_{v=n+1}^{\infty} a_v v^{p-2}\right)^{\frac{1}{p}} \leq \left(\sum_{v=n+1}^{\infty} \left(\sum_{i=v}^{\infty} |a_i - a_{i+1}|\right)^p v^{p-2}\right)^{\frac{1}{p}} \tag{4.1}$$

$$\ll \left(\sum_{v=n+1}^{\infty} (\gamma_v^*)^p v^{p-2}\right)^{\frac{1}{p}} \ll E_{\left[\frac{n}{4}\right]}(f)_p \ll \omega_k\left(f;\frac{1}{n}\right)_p .$$

Since $(a_n) \in \gamma_m^* MRBVS$, using Hölder's inequality with p and $\frac{p}{p-1}$, and applying again Lemma 12, we obtain

$$\begin{aligned}
v^{p-1}a_v^p &\ll v^{p-1}\left(\frac{1}{v}\sum_{l=\frac{v}{2}}^{v} a_l\right)^p \ll \left(\sum_{l=\frac{v}{2}}^{v} l^{-\frac{1}{p}} a_l\right)^p \\
&\ll \sum_{l=\frac{v}{2}}^{v} a_l^p \left(\sum_{l=\frac{v}{2}}^{v} l^{-\frac{1}{p-1}}\right)^{p-1} \ll v^{p-2}\sum_{l=\frac{v}{2}}^{v} a_l^p \ll \sum_{l=\frac{v}{2}}^{v} l^{p-2} a_l^p \\
&\ll \sum_{l=\frac{v}{2}}^{v} (\gamma_l^*)^p l^{p-2} \ll \sum_{l=\frac{v}{2}}^{\infty} (\gamma_l^*)^p\, l^{p-2} \ll E_{\left[\frac{v}{8}\right]}^p (f)_p .
\end{aligned}$$

Putting it into the following sum and applying Lemma 4, we get the following estimate

$$\begin{aligned}
n^{-k}\left(\sum_{v=1}^{n} a_v^p v^{(k+1)p-2}\right)^{\frac{1}{p}} &\ll n^{-k}\left(\sum_{v=1}^{n} E_{\left[\frac{v}{8}\right]}^p (f)_p v^{kp-1}\right)^{\frac{1}{p}} \\
&\ll n^{-k}\left(\sum_{\mu=0}^{n} E_\mu^p(f)_p(\mu+1)^{kp-1}\right)^{\frac{1}{p}} \ll \omega_k\left(f;\frac{1}{n}\right)_p .
\end{aligned} \tag{4.2}$$

From (4.1) and (4.2) the first inequality in (2.3) also holds for $2 \le p < \infty$, thus it is true for any $1 < p < \infty$.

Now, we prove that

$$\omega_k\left(f;\frac{1}{n}\right)_p \ll S\left(a_v, p, k, n\right) \tag{4.3}$$

also holds. For $2 \le p < \infty$ it follows from Lemma 14. If $1 < p \le 2$, then using Lemma 4 and Lemma 9 we get

$$\begin{aligned}
\omega_k\left(f;\frac{1}{n}\right)_p &\ll n^{-k}\left(\sum_{v=1}^{n} E_v^p(f)_p v^{kp-1}\right)^{\frac{1}{p}} \\
&\ll n^{-k}\left(\sum_{v=1}^{n} v^{kp-1}(\gamma_{\nu+1}^*)^p(v+1)^{p-1} + \sum_{v=1}^{n} v^{kp-1}\sum_{l=v+1}^{\infty}(\gamma_l^*)^p l^{p-2}\right)^{\frac{1}{p}} \\
&\ll n^{-k}\left(\sum_{v=1}^{n} v^{(k+1)p-2}(\gamma_{\nu+1}^*)^p \right.\\
&\quad \left. + \sum_{v=1}^{n} v^{kp-1}\sum_{l=v+1}^{n}(\gamma_l^*)^p\, l^{p-2} + \sum_{v=1}^{n} v^{kp-1}\sum_{l=n+1}^{\infty}(\gamma_l^*)^p l^{p-2}\right)^{\frac{1}{p}}
\end{aligned}$$

$$\begin{aligned}
&\ll n^{-k}\Bigg(\sum_{v=1}^{n} v^{(k+1)p-2}\left(\gamma_{\nu+1}^{*}\right)^{p} \\
&\qquad+\sum_{l=2}^{n}\left(\gamma_{l}^{*}\right)^{p} l^{p-2} \sum_{v=1}^{l} v^{kp-1}+n^{kp} \sum_{l=n+1}^{\infty}\left(\gamma_{l}^{*}\right)^{p} l^{p-2}\Bigg)^{\frac{1}{p}} \\
&\ll n^{-k}\left(\sum_{v=1}^{n} v^{(k+1)p-2}\left(\gamma_{\nu+1}^{*}\right)^{p}+n^{kp} \sum_{l=n+1}^{\infty}\left(\gamma_{l}^{*}\right)^{p} l^{p-2}\right)^{\frac{1}{p}} \\
&:=n^{-k}\left(\Omega_{1}+n^{kp} \Omega_{2}\right)^{\frac{1}{p}} .
\end{aligned}$$

Let $N=N(n)$ be such that $2^{N} \leq n-1<2^{N+1}$, then

$$\begin{aligned}
\Omega_{1} &=\sum_{v=1}^{3} v^{(k+1)p-2}\left(\gamma_{\nu+1}^{*}\right)^{p}+\sum_{v=4}^{n} v^{(k+1)p-2}\left(\gamma_{\nu+1}^{*}\right)^{p} \\
&\ll \sum_{v=1}^{3}\left(\gamma_{\nu+1}^{*}\right)^{p}+\sum_{l=2}^{N} \sum_{i=2^{l}}^{2^{l+1}} i^{(k+1)p-2}\left(\gamma_{i+1}^{*}\right)^{p} \\
&\ll \sum_{v=2}^{4}\left(\gamma_{\nu+1}^{*}\right)^{p}+\sum_{l=2}^{N} \sum_{i=2^{l}}^{2^{l+1}}\left(i^{(k+1)-\frac{2}{p}-1} \sum_{\mu=\frac{i+1}{2}}^{i+1} a_{\mu}\right)^{p} \\
&\ll \sum_{v=2}^{4}\left(\gamma_{\nu+1}^{*}\right)^{p}+\sum_{l=2}^{N} \sum_{i=2^{l}}^{2^{l+1}}\left(\sum_{\mu=\frac{i+1}{2}}^{i+1} \mu^{(k+1)-\frac{2}{p}-1} a_{\mu}\right)^{p}
\end{aligned}$$

and by Hölder's inequality with p and $\frac{p}{p-1}$ we have

$$\begin{aligned}
\text{(4.4)}\quad \Omega_{1} &\ll \sum_{v=2}^{4}\left(\gamma_{\nu+1}^{*}\right)^{p}+\sum_{l=2}^{N} \sum_{i=2^{l}}^{2^{l+1}} i^{(k+1)p-3} \sum_{\mu=\frac{i+1}{2}}^{i+1} a_{\mu}^{p} \\
&\ll \sum_{v=2}^{4}\left(\gamma_{\nu+1}^{*}\right)^{p}+\sum_{l=2}^{N} \sum_{i=2^{l}}^{2^{l+1}} i^{-1} \sum_{\mu=\frac{i+1}{2}}^{i+1} \mu^{(k+1)p-2} a_{\mu}^{p} \\
&\ll \sum_{v=2}^{4}\left(\gamma_{\nu+1}^{*}\right)^{p}+\sum_{l=2}^{N} \sum_{\mu=2^{l-1}}^{2^{l+1}+1} \mu^{(k+1)p-2} a_{\mu}^{p} \\
&\ll \sum_{\mu=1}^{4} a_{\mu}^{p}+\sum_{l=2}^{N}\left(\sum_{\mu=2^{l-1}}^{2^{l}-1} \mu^{(k+1)p-2} a_{\mu}^{p}+a_{2^{l}}^{p} 2^{l((k+1)p-2)}\right.
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\mu=2^l+1}^{2^{l+1}} \mu^{(k+1)p-2} a_\mu^p + a_{2^{l+1}+1}^p \left(2^{l+1}+1\right)^{(k+1)p-2}\Bigg) \\
&\ll \sum_{\mu=1}^{4} a_\mu^p + \sum_{\mu=1}^{2^N-1} \mu^{(k+1)p-2} a_\mu^p + \sum_{\mu=4}^{2^N} \mu^{(k+1)p-2} a_\mu^p + \sum_{\mu=5}^{2^{N+1}} \mu^{(k+1)p-2} a_\mu^p \\
&\quad + \sum_{\mu=9}^{2^{N+1}+1} \mu^{(k+1)p-2} a_\mu^p \ll \sum_{\mu=1}^{2^{n+1}+1} \mu^{(k+1)p-2} a_\mu^p \ll \sum_{\mu=1}^{2n} \mu^{(k+1)p-2} a_\mu^p.
\end{aligned}
$$

Now, we estimate Ω_2. Using Hölder's inequality with p and $\frac{p}{p-1}$, we get

$$
\begin{aligned}
\Omega_2 &= \sum_{l=n+1}^{\infty} \left(\gamma_{l+1}^*\right)^p (l+1)^{p-2} \ll \sum_{m=0}^{\infty} \sum_{i=2^m n}^{2^{m+1}n} i^{p-2} \left(\gamma_{i+1}^*\right)^p \\
&\ll \sum_{m=0}^{\infty} \sum_{i=2^m n}^{2^{m+1}n} \left(i^{-\frac{2}{p}} \sum_{\mu=\frac{i+1}{2}}^{i+1} a_\mu \right)^p \ll \sum_{m=0}^{\infty} \sum_{i=2^m n}^{2^{m+1}n} \left(\sum_{\mu=\frac{i+1}{2}}^{i+1} \mu^{-\frac{2}{p}} a_\mu \right)^p \\
&\ll \sum_{m=0}^{\infty} \sum_{i=2^m n}^{2^{m+1}n} \sum_{\mu=\frac{i+1}{2}}^{i+1} a_\mu^p \left(\sum_{\mu=\frac{i+1}{2}}^{i+1} \mu^{-\frac{2}{p-1}} \right)^{p-1} \\
&\ll \sum_{m=0}^{\infty} \sum_{i=2^m n}^{2^{m+1}n} i^{p-3} \sum_{\mu=\frac{i+1}{2}}^{i+1} a_\mu^p \ll \sum_{m=0}^{\infty} \sum_{i=2^m n}^{2^{m+1}n} i^{-1} \sum_{\mu=\frac{i+1}{2}}^{i+1} \mu^{p-2} a_\mu^p \\
&\ll \sum_{m=0}^{\infty} \sum_{\mu=\frac{2^m n+1}{2}}^{2^{m+1}n+1} \mu^{p-2} a_\mu^p \sum_{i=2^m n}^{2^{m+1}n} i^{-1} \ll \sum_{m=0}^{\infty} \sum_{\mu=2^{m-1}n}^{2^{m+1}n+1} \mu^{p-2} a_\mu^p \\
&= \sum_{m=0}^{\infty} \Bigg(\sum_{\mu=\frac{2^m n+1}{2}}^{2^m n-1} \mu^{p-2} a_\mu^p + a_{2^m n+1}^p 2^{m(p-2)} \\
&\quad + \sum_{\mu=2^m n+1}^{2^{m+1}n} \mu^{p-2} a_\mu^p + a_{2^{m+1}n+1}^p \left(2^{m+1}n+1\right)^{p-2} \Bigg) \\
&\le \sum_{\mu=\frac{n+1}{2}}^{\infty} \mu^{p-2} a_\mu^p + \sum_{\mu=n}^{\infty} \mu^{p-2} a_\mu^p + \sum_{\mu=n+1}^{\infty} \mu^{p-2} a_\mu^p + \sum_{\mu=2n+1}^{\infty} \mu^{p-2} a_\mu^p \\
&\ll \sum_{\mu=\frac{n+1}{2}}^{\infty} \mu^{p-2} a_\mu^p.
\end{aligned} \tag{4.5}
$$

From (4.4), (4.5), applying Lemma 2 we obtain

$$\omega_k\left(f;\frac{1}{n}\right)_p \ll n^{-k}\left(\sum_{\mu=1}^{2n}\mu^{(k+1)p-2}a_\mu^p + n^{kp}\sum_{\mu=\frac{n+1}{2}}^{\infty}\mu^{p-2}a_\mu^p\right)^{\frac{1}{p}}$$

$$= n^{-k}\left(\sum_{\mu=1}^{n}\mu^{(k+1)p-2}a_\mu^p + \sum_{\mu=n+1}^{2n}\mu^{(k+1)p-2}a_\mu^p + n^{kp}\sum_{\mu=\frac{n+1}{2}}^{n}\mu^{p-2}a_\mu^p\right.$$

$$\left. + n^{kp}\sum_{\mu=n+1}^{\infty}\mu^{p-2}a_\mu^p\right)^{\frac{1}{p}}$$

$$\ll n^{-k}\left(\sum_{\mu=1}^{n}\mu^{(k+1)p-2}a_\mu^p + n^{kp}\sum_{\mu=n+1}^{2n}\mu^{p-2}a_\mu^p\right.$$

$$\left. + \sum_{\mu=\frac{n+1}{2}}^{n}\mu^{(k+1)p-2}a_\mu^p + n^{kp}\sum_{\mu=n+1}^{\infty}\mu^{p-2}a_\mu^p\right)^{\frac{1}{p}}$$

$$\ll n^{-k}\left(\sum_{\mu=1}^{n}\mu^{(k+1)p-2}a_\mu^p\right)^{\frac{1}{p}} + \left(\sum_{\mu=n+1}^{\infty}\mu^{p-2}a_\mu^p\right)^{\frac{1}{p}} \ll S\left(a_\mu,p,k,n\right).$$

This proves (4.3) for $1 < p \leq 2$, and consequently (4.3) is true for any $1 < p < \infty$. The proof is complete. ∎

4.4. Proof of Theorem 4

In the proof we use the following inequality

$$J := \int_0^1 t^{-r\theta-1}\omega_k^\theta\left(f;t\right)_p dt \asymp \sum_{n=1}^{\infty} n^{r\theta-1}\omega_k^\theta\left(f;\frac{1}{n}\right)_p. \tag{4.6}$$

Hence, by Theorem 3, we obtain

$$J \ll \sum_{n=1}^{\infty} n^{r\theta-1}\left(n^{-k}\left(\sum_{\mu=1}^{n}\mu^{(k+1)p-2}a_\mu^p\right)^{\frac{1}{p}} + \left(\sum_{\mu=n+1}^{\infty}\mu^{p-2}a_\mu^p\right)^{\frac{1}{p}}\right)^\theta$$

$$\ll \sum_{n=1}^{\infty} n^{(r-k)\theta-1}\left(\sum_{\mu=1}^{n}\mu^{(k+1)p-2}a_\mu^p\right)^{\frac{\theta}{p}} + \sum_{n=1}^{\infty} n^{r\theta-1}\left(\sum_{\mu=n+1}^{\infty}\mu^{p-2}a_\mu^p\right)^{\frac{\theta}{p}}.$$

If $\frac{\theta}{p} \geq 1$, then by Lemma 1

$$J \ll \sum_{n=1}^{\infty} n^{r\theta+\theta-1-\frac{\theta}{p}-\frac{(r-k)\theta^2}{p}} a_n^\theta\left(\sum_{\mu=n}^{\infty}\mu^{(r-k)\theta-1}\right)^{\frac{\theta}{p}}$$

$$+ \sum_{n=1}^{\infty} n^{r\theta+\theta-1-\frac{\theta}{p}-\frac{r\theta^2}{p}} a_n^\theta\left(\sum_{\mu=1}^{n}\mu^{r\theta-1}\right)^{\frac{\theta}{p}} \ll \sum_{n=1}^{\infty} a_n^\theta n^{r\theta+\theta-\frac{\theta}{p}-1}.$$

From the above estimate we get that $J \ll J_1$.

An easy consideration, by (4.6) yields that

$$J \asymp \sum_{n=0}^{\infty} 2^{nr\theta} \omega_k^{\theta}\left(f; \frac{1}{2^n}\right)_p . \tag{4.7}$$

Hence, if $\frac{\theta}{p} \le 1$, then using again Theorem 3 and (4.7), we obtain that

$$\begin{aligned}
J &\ll \sum_{n=0}^{\infty} 2^{n(r-k)\theta}\left(\sum_{\mu=1}^{2^n} \mu^{(k+1)p-2} a_\mu^p\right)^{\frac{\theta}{p}} + \sum_{n=0}^{\infty} 2^{nr\theta}\left(\sum_{\mu=2^n}^{\infty} \mu^{p-2} a_\mu^p\right)^{\frac{\theta}{p}} \\
&\ll \sum_{n=0}^{\infty} 2^{n(r-k)\theta}\left(\sum_{l=0}^{n}\sum_{\mu=2^l}^{2^{l+1}} \mu^{(k+1)p-2} a_\mu^p\right)^{\frac{\theta}{p}} + \sum_{n=0}^{\infty} 2^{nr\theta}\left(\sum_{l=n}^{\infty}\sum_{\mu=2^l}^{2^{l+1}} \mu^{p-2} a_\mu^p\right)^{\frac{\theta}{p}} \\
&\ll \sum_{n=0}^{\infty} 2^{n(r-k)\theta}\left(\sum_{l=0}^{n}\sum_{\mu=2^l}^{2^{l+1}} \mu^{(k+1)p-2} \left(\frac{1}{\mu}\sum_{i=\frac{\mu}{2}}^{\mu} a_i\right)^p\right)^{\frac{\theta}{p}} \\
&\quad + \sum_{n=0}^{\infty} 2^{nr\theta}\left(\sum_{l=n}^{\infty}\sum_{\mu=2^l}^{2^{l+1}} \mu^{p-2} \left(\frac{1}{\mu}\sum_{i=\frac{\mu}{2}}^{\mu} a_i\right)^p\right)^{\frac{\theta}{p}} \\
&\ll \sum_{n=0}^{\infty} 2^{n(r-k)\theta}\left(\sum_{l=0}^{n}\left(\frac{1}{2^l}\sum_{i=2^{l-1}}^{2^{l+1}} a_i\right)^p 2^{l((k+1)p-1)}\right)^{\frac{\theta}{p}} \\
&\quad + \sum_{n=0}^{\infty} 2^{nr\theta}\left(\sum_{l=n}^{\infty}\left(\frac{1}{2^l}\sum_{i=2^{l-1}}^{2^{l+1}} a_i\right)^p 2^{l(p-1)}\right)^{\frac{\theta}{p}} \\
&\ll \sum_{n=0}^{\infty} 2^{n(r-k)\theta}\sum_{l=0}^{n}\left(\frac{1}{2^l}\sum_{i=2^{l-1}}^{2^{l+1}} a_i\right)^{\theta} 2^{l\left((k+1)\theta-\frac{\theta}{p}\right)} \\
&\quad + \sum_{n=0}^{\infty} 2^{nr\theta}\sum_{l=n}^{\infty}\left(\frac{1}{2^l}\sum_{i=2^{l-1}}^{2^{l+1}} a_i\right)^{\theta} 2^{l\left(\theta-\frac{\theta}{p}\right)}.
\end{aligned} \tag{4.8}$$

If $\theta \ge 1$, then

$$\begin{aligned}
J &\ll \sum_{n=0}^{\infty} 2^{n(r-k)\theta}\sum_{l=0}^{n} 2^{l\left((k+1)\theta-\frac{\theta}{p}-1\right)}\sum_{i=2^{l-1}}^{2^{l+1}} a_i^{\theta} + \sum_{n=0}^{\infty} 2^{nr\theta}\sum_{l=n}^{\infty} 2^{l\left(\theta-\frac{\theta}{p}-1\right)}\sum_{i=2^{l-1}}^{2^{l+1}} a_i^{\theta} \\
&\ll \sum_{l=0}^{\infty} 2^{l\left((k+1)\theta-\frac{\theta}{p}-1\right)}\sum_{i=2^{l-1}}^{2^{l+1}} a_i^{\theta}\sum_{n=l}^{\infty} 2^{n(r-k)\theta} + \sum_{l=0}^{\infty} 2^{l\left(\theta-\frac{\theta}{p}-1\right)}\sum_{i=2^{l-1}}^{2^{l+1}} a_i^{\theta}\sum_{n=0}^{l} 2^{nr\theta}
\end{aligned}$$

$$\ll \sum_{l=0}^{\infty} 2^{l\left(r\theta+\theta-\frac{\theta}{p}-1\right)} \sum_{i=2^{l-1}}^{2^{l+1}} a_i^{\theta} \ll \sum_{l=0}^{\infty} \sum_{i=2^{l-1}}^{2^{l+1}} i^{r\theta+\theta-\frac{\theta}{p}-1} a_i^{\theta}$$
$$\ll \sum_{i=1}^{\infty} i^{r\theta+\theta-\frac{\theta}{p}-1} a_i^{\theta} \ll J_1.$$

To prove that $J \gg J_1$ when $\frac{\theta}{p} \geq 1$, we start again with (4.7) and using Theorem 3 we get that

$$J \gg \sum_{n=0}^{\infty} 2^{n(r-k)\theta} \left(\sum_{\mu=1}^{2^n} \mu^{(k+1)p-2} a_\mu^p \right)^{\frac{\theta}{p}} + \sum_{n=0}^{\infty} 2^{nr\theta} \left(\sum_{\mu=2^n}^{\infty} \mu^{p-2} a_\mu^p \right)^{\frac{\theta}{p}}$$
$$= \sum_{n=0}^{\infty} 2^{n(r-k)\theta} \left(\sum_{l=0}^{n-1} \sum_{\mu=2^l}^{2^{l+1}} \mu^{(k+1)p-2} a_\mu^p \right)^{\frac{\theta}{p}} + \sum_{n=0}^{\infty} 2^{nr\theta} \left(\sum_{l=n}^{\infty} \sum_{\mu=2^l}^{2^{l+1}} \mu^{p-2} a_\mu^p \right)^{\frac{\theta}{p}}$$
$$\gg \sum_{n=0}^{\infty} 2^{n(r-k)\theta} \left(\sum_{l=0}^{n-1} 2^{l((k+1)p-2)} \sum_{\mu=2^l}^{2^{l+1}} a_\mu^p \right)^{\frac{\theta}{p}} + \sum_{n=0}^{\infty} 2^{nr\theta} \left(\sum_{l=n}^{\infty} 2^{l(p-2)} \sum_{\mu=2^l}^{2^{l+1}} a_\mu^p \right)^{\frac{\theta}{p}}$$
$$\gg \sum_{n=0}^{\infty} 2^{n(r-k)\theta} \sum_{l=0}^{n-1} 2^{l\left((k+1)\theta-\frac{\theta}{p}\right)} \left(\frac{1}{2^l} \sum_{\mu=2^l}^{2^{l+1}} a_\mu^p \right)^{\frac{\theta}{p}}$$
$$+ \sum_{n=0}^{\infty} 2^{nr\theta} \sum_{l=n}^{\infty} 2^{l\left(\theta-\frac{\theta}{p}\right)} \left(\frac{1}{2^l} \sum_{\mu=2^l}^{2^{l+1}} a_\mu^p \right)^{\frac{\theta}{p}}$$
$$\gg \sum_{n=0}^{\infty} 2^{n(r-k)\theta} \sum_{l=0}^{n-1} 2^{l\left((k+1)\theta-\frac{\theta}{p}\right)} \left(\frac{1}{2^l} \sum_{\mu=2^l}^{2^{l+1}} a_\mu \right)^{\theta}$$
$$+ \sum_{n=0}^{\infty} 2^{nr\theta} \sum_{l=n}^{\infty} 2^{l\left(\theta-\frac{\theta}{p}\right)} \left(\frac{1}{2^l} \sum_{\mu=2^l}^{2^{l+1}} a_\mu \right)^{\theta}.$$

Since $(a_n) \in \gamma_m^* MRBVS$, we have that

$$J \gg \sum_{n=0}^{\infty} 2^{n(r-k)\theta} \sum_{l=0}^{n-1} 2^{l\left((k+1)\theta-\frac{\theta}{p}\right)} \left(\sum_{\mu=2^{l+1}}^{\infty} |a_\mu - a_{\mu+1}| \right)^{\theta}$$
$$+ \sum_{n=0}^{\infty} 2^{nr\theta} \sum_{l=n}^{\infty} 2^{l\left(\theta-\frac{\theta}{p}\right)} \left(\sum_{\mu=2^{l+1}}^{\infty} |a_\mu - a_{\mu+1}| \right)^{\theta}$$
$$\gg \sum_{l=0}^{\infty} 2^{l\left((k+1)\theta-\frac{\theta}{p}\right)} \left(\sum_{\mu=2^{l+1}}^{\infty} |a_\mu - a_{\mu+1}| \right)^{\theta} \sum_{n=l}^{\infty} 2^{n(r-k)\theta}$$

$$+\sum_{l=n}^{\infty}2^{l\left(\theta-\frac{\theta}{p}\right)}\left(\sum_{\mu=2^{l+1}}^{\infty}|a_\mu-a_{\mu+1}|\right)^{\theta}\sum_{n=0}^{l}2^{nr\theta}$$

$$\gg\sum_{l=0}^{\infty}2^{l\left(r\theta+\theta-\frac{\theta}{p}\right)}\left(\sum_{\mu=2^{l+1}}^{\infty}|a_\mu-a_{\mu+1}|\right)^{\theta}$$

$$\gg\sum_{l=0}^{\infty}2^{l\left(r\theta+\theta-\frac{\theta}{p}-1\right)}\sum_{v=2^{l+1}}^{2^{l+2}-1}\left(\sum_{\mu=v}^{\infty}|a_\mu-a_{\mu+1}|\right)^{\theta}\gg\sum_{l=0}^{\infty}2^{l\left(r\theta+\theta-\frac{\theta}{p}-1\right)}\sum_{v=2^{l+1}}^{2^{l+2}-1}a_v^{\theta}$$

$$\gg\sum_{l=0}^{\infty}\sum_{v=2^{l+1}}^{2^{l+2}-1}v^{r\theta+\theta-\frac{\theta}{p}-1}a_v^{\theta}\gg\sum_{v=2}^{\infty}v^{r\theta+\theta-\frac{\theta}{p}-1}a_v^{\theta}\gg J_1.$$

In the case $\frac{\theta}{p}\le 1$, using (4.6), the Theorem 3 and Lemma 1, we get

(4.9)

$$J\gg\sum_{n=1}^{\infty}n^{(r-k)\theta-1}\left(\sum_{\mu=1}^{n}\mu^{(k+1)p-2}a_\mu^p\right)^{\frac{\theta}{p}}+\sum_{n=1}^{\infty}n^{r\theta-1}\left(\sum_{\mu=n+1}^{\infty}\mu^{p-2}a_\mu^p\right)^{\frac{\theta}{p}}$$

$$\gg\sum_{n=1}^{\infty}n^{r\theta+\theta-\frac{\theta}{p}-1-\frac{(r-k)\theta^2}{p}}a_n^{\theta}\left(\sum_{\mu=n}^{\infty}\mu^{(r-k)\theta-1}\right)^{\frac{\theta}{p}}$$

$$+\sum_{n=1}^{\infty}n^{r\theta+\theta-\frac{\theta}{p}-1-\frac{r\theta^2}{p}}a_n^{\theta}\left(\sum_{\mu=1}^{n}\mu^{r\theta-1}\right)^{\frac{\theta}{p}}\gg\sum_{n=1}^{\infty}a_n^{\theta}n^{r\theta+\theta-\frac{\theta}{p}-1}\gg J_1$$

and the proof is complete. ∎

4.5. Proof of Corollary 1

The proof of Corollary 1 goes analogously as the proof of Theorem 4. First we prove the necessity. If $\theta\le 1$ then $\frac{\theta}{p}\le 1$ and by (4.9) we obtain that $J\gg J_1$.

Consequently, we prove the sufficiency. If $\theta\le 1$ then by (4.8) we get

$$J\ll\sum_{n=0}^{\infty}2^{n(r-k)\theta}\sum_{l=0}^{n}2^{l\left(k\theta-\frac{\theta}{p}\right)}\left(\sum_{i=2^{l-1}}^{2^{l+1}}a_i\right)^{\theta}+\sum_{n=0}^{\infty}2^{nr\theta}\sum_{l=n}^{\infty}2^{l\left(-\frac{\theta}{p}\right)}\left(\sum_{i=2^{l-1}}^{2^{l+1}}a_i\right)^{\theta}$$

$$\ll\sum_{n=0}^{\infty}2^{n(r-k)\theta}\sum_{l=0}^{n}2^{l\left(k\theta-\frac{\theta}{p}\right)}\sum_{i=2^{l-1}}^{2^{l+1}}a_i^{\theta}+\sum_{n=0}^{\infty}2^{nr\theta}\sum_{l=n}^{\infty}2^{l\left(-\frac{\theta}{p}\right)}\sum_{i=2^{l-1}}^{2^{l+1}}a_i^{\theta}$$

$$\ll\sum_{l=0}^{\infty}2^{l\left(r\theta-\frac{\theta}{p}\right)}\sum_{i=2^{l-1}}^{2^{l+1}}a_i^{\theta}\ll\sum_{l=0}^{\infty}\sum_{i=2^{l-1}}^{2^{l+1}}i^{r\theta-\frac{\theta}{p}}a_i^{\theta}\ll J_2,$$

where J_2 is defined in Corollary 1. This ends our proof. ∎

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UNIVERSITY OF ZIELONA GÓRA
FACULTY OF MATHEMATICS, COMPUTER SCIENCE AND ECONOMETRICS
ul. Szafrana 4a
65-516 ZIELONA GÓRA, POLAND
E-mail: B.Szal@wmie.uz.zgora.pl

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