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SEVERAL EXTENDED ANALOGUES OF HILBERT'S INEQUALITIES

Abstract. By introducing the function $\frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \min\{x,y\}}$ with real numbers α, β, γ, we get several extended analogues of Hilbert's inequalities.

1. Introduction

If f, g are real functions such that

$$0 < \int_0^\infty f^2(x)dx < \infty \text{ and } 0 < \int_0^\infty g^2(x)dx < \infty, \tag{1.1}$$

then we have the following well known Hilbert's integral inequality [1],

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{1/2}$$

where the constant factor π is the best possible. Furthermore, we have also the following Hardy-Hilbert's type inequality [1, Th 341, Th 342],

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}} dxdy < 4 \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{1/2},$$

$$\int_0^\infty \int_0^\infty \frac{\log x - \log y}{x-y} f(x)g(y)dxdy < \pi^2 \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{1/2},$$

where the constant factors 4 and π^2 are both the best possible.

There are numerous literatures to study the Hilbert's and Hardy-Hilbert's type inequalities from different directions [4, 5, 6, 7]. Recently, Li-Wu-He [3]

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obtained the following inequality: if (1.1) is satisfied, then we have

$$(1.2)\qquad \int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}}dxdy < c\left\{\int_0^\infty f^2(x)dx\int_0^\infty g^2(x)dx\right\}^{1/2},$$

where the constant factor $c = 1.7408\ldots$ is the best possible.

He-Qian-Li [2] proved the following inequality: if (1.1) is satisfied, then we have

$$(1.3)\quad \int_0^\infty\int_0^\infty \frac{|\log x-\log y|}{x+y+\min\{x,y\}}f(x)g(y)dxdy < c\left\{\int_0^\infty f^2(x)dx\int_0^\infty g^2(x)dx\right\}^{1/2},$$

where the constant factor $c = 6.88947\ldots$ is the best possible.

In this short paper, we will give several extended analogues of Hilbert's inequalities which include the case (1.3).

2. Main results

Before giving our main results, we need mention the following

Lemma 1. *Let γ,α,β be three real numbers. Then we have the following equations*

$$\int_0^\infty \frac{|\log x-\log y|^\gamma}{\alpha x+\beta y+\min\{x,y\}}\left(\frac{x}{y}\right)^{1/2}dy = \int_0^\infty \frac{|\log x-\log y|^\gamma}{\alpha x+\beta y+\min\{x,y\}}\left(\frac{y}{x}\right)^{1/2}dx$$

$$= \int_0^1 \frac{2^{\gamma+1}|\log t|^\gamma}{t^2(\beta+1)+\alpha}dt + \int_0^1 \frac{2^{\gamma+1}|\log t|^\gamma}{t^2(\alpha+1)+\beta}dt =: A,$$

where $A\in[0,\infty]$.

Proof. For any given y, let $y = tx$, then it follows that

$$\int_0^\infty \frac{|\log x-\log y|^\gamma}{\alpha x+\beta y+\min\{x,y\}}\left(\frac{x}{y}\right)^{1/2}dy$$

$$= \int_0^\infty \frac{|\log t|^\gamma}{\alpha+t\beta+\min\{1,t\}}\left(\frac{1}{t}\right)^{1/2}dt$$

$$= \int_0^1 \frac{|\log t|^\gamma}{t(\beta+1)+\alpha}\left(\frac{1}{t}\right)^{1/2}dt + \int_1^\infty \frac{|\log t|^\gamma}{t\beta+(1+\alpha)}\left(\frac{1}{t}\right)^{1/2}dt$$

$$= \int_0^1 \frac{|\log t|^\gamma}{t(\beta+1)+\alpha}\left(\frac{1}{t}\right)^{1/2}dt + \int_0^1 \frac{|\log t|^\gamma}{t(\alpha+1)+\beta}\left(\frac{1}{t}\right)^{1/2}dt$$

$$= \int_0^1 \frac{2^{\gamma+1}|\log t|^\gamma}{t^2(\beta+1)+\alpha}dt + \int_0^1 \frac{2^{\gamma+1}|\log t|^\gamma}{t^2(\alpha+1)+\beta}dt$$

which implies the desired result. ∎

THEOREM 1. *If f, g are real functions such that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$. Furthermore, let $A \in (0, \infty)$, then we have*

$$\int_0^\infty\int_0^\infty \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \min\{x,y\}} f(x)g(y)dxdy < A\left(\int_0^\infty f^2(x)dx\right)^{1/2}\left(\int_0^\infty g^2(y)dy\right)^{1/2}, \tag{2.1}$$

where A is defined in Lemma 1 and is the best possible.

Proof. By Hölder's inequality and Lemma 1, we get

$$\begin{aligned}\int_0^\infty\int_0^\infty & \frac{|\log x - \log y|}{x + y + \min\{x,y\}} f(x)g(y)dxdy \\ &\le \left\{\int_0^\infty\left(\int_0^\infty \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \min\{x,y\}}\left(\frac{x}{y}\right)^{1/2} dy\right) f^2(x)dx\right\}^{1/2} \\ &\quad\times \left\{\int_0^\infty\left(\int_0^\infty \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \min\{x,y\}}\left(\frac{y}{x}\right)^{1/2} dx\right) g^2(y)dy\right\}^{1/2} \\ &\le A\left(\int_0^\infty f^2(x)dx\right)^{1/2}\left(\int_0^\infty g^2(y)dy\right)^{1/2}.\end{aligned} \tag{2.2}$$

If the equality in (2.2) holds, then there exist two constant c and d, such that they are not both zero (without loss of generality, suppose that $c \neq 0$) and

$$c\frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \min\{x,y\}}\left(\frac{x}{y}\right)^{1/2} f^2(x) = d\frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \min\{x,y\}}\left(\frac{y}{x}\right)^{1/2} g^2(y),$$

a.e. in $(0, \infty) \times (0, \infty)$. That is to say, we have

$$cxf^2(x) = dyg^2(y) = \text{constant},$$

a.e. in $(0, \infty) \times (0, \infty)$. Thus

$$\int_0^\infty f^2(x)dx = \infty,$$

which contradicts the assumption $0 < \int_0^\infty f^2(x)dx < \infty$. Hence, the inequality (2.2) takes the form of strict inequality.

Assume that the constant A in the inequality (2.1) is not the best possible, then there exists a positive number K with $K < A$ and $a > 0$, such that

$$\int_a^\infty\int_0^\infty \frac{|\log x-\log y|^\gamma}{\alpha x+\beta y+\min\{x,y\}} f(x)g(y)dxdy < K\left(\int_a^\infty f^2(x)dx\right)^{1/2}\left(\int_a^\infty g^2(y)dy\right)^{1/2}. \tag{2.3}$$

For $0<\varepsilon<1$, setting

$$f_\varepsilon(x)=\begin{cases} x^{-\frac{\varepsilon+1}{2}}, \text{ for } x\in[b,\infty),\\ 0, \qquad \text{for}(0,b).\end{cases} \quad g_\varepsilon(y)=\begin{cases} y^{-\frac{\varepsilon+1}{2}}, \text{ for } x\in[b,\infty),\\ 0, \qquad \text{for}(0,b).\end{cases}$$

Then

$$K\left(\int_a^\infty f_\varepsilon^2(x)dx\right)^{1/2}\left(\int_a^\infty g_\varepsilon^2(y)dy\right)^{1/2} = K\frac{1}{\varepsilon a^\sigma}.$$

Let $y=tx$, we get

$$\begin{aligned} &\int_a^\infty\int_0^\infty \frac{|\log x-\log y|^\gamma}{\alpha x+\beta y+\min\{x,y\}} f_\varepsilon(x)g_\varepsilon(y)dxdy \\ &= \int_a^\infty\int_b^\infty \frac{|\log x-\log y|^\gamma}{\alpha x+\beta y+\min\{x,y\}} x^{-\frac{\varepsilon+1}{2}}y^{-\frac{\varepsilon+1}{2}}dxdy \\ &= \int_a^\infty\int_{b/x}^\infty \frac{|\log t|^\gamma}{\alpha+t\beta+\min\{1,t\}} x^{-(\varepsilon+1)}t^{-\frac{\varepsilon+1}{2}}dtdx. \end{aligned} \tag{2.4}$$

Letting $b\to 0^+$, by (2.3) and Fatou's lemma, we have

$$\begin{aligned} &\int_a^\infty\int_0^\infty \frac{|\log t|^\gamma}{\alpha+t\beta+\min\{1,t\}} x^{-(\varepsilon+1)}t^{-\frac{\varepsilon+1}{2}}dtdx \\ &= \frac{1}{\varepsilon a^\varepsilon}\int_0^\infty \frac{|\log t|^\gamma}{\alpha+t\beta+\min\{1,t\}} t^{-\frac{\varepsilon+1}{2}}dt \le K\frac{1}{\varepsilon a^\varepsilon}, \end{aligned}$$

which yields

$$\lim_{\varepsilon\to 0^+}\int_0^\infty \frac{|\log t|^\gamma}{\alpha+t\beta+\min\{1,t\}} t^{-\frac{\varepsilon+1}{2}}dt = A\le K.$$

The contradiction implies the constant A is the best possible. ∎

Theorem 2. *Suppose $f\ge 0$ and $0<\int_0^\infty f^2(x)dx<\infty$. Then*

$$\int_0^\infty\left(\int_0^\infty \frac{|\log x-\log y|^\gamma}{\alpha x+\beta y+\min\{x,y\}} f(x)dx\right)^2 dy < A^2\int_0^\infty f^2(x)dx. \tag{2.5}$$

Proof. Let

$$g(y) = \int_0^\infty \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \min\{x, y\}} f(x)dx,$$

then by (2.2), we have

$$(2.6)\qquad \begin{aligned} 0 < \int_0^\infty g^2(y)dy &= \int_0^\infty \left(\int_0^\infty \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \min\{x, y\}} f(x)dx \right)^2 dy \\ &= \int_0^\infty \int_0^\infty \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \min\{x, y\}} f(x)g(y)dxdy \\ &\leq A \left(\int_0^\infty f^2(x)dx \right)^{1/2} \left(\int_0^\infty g^2(y)dy \right)^{1/2}, \end{aligned}$$

which yields

$$(2.7)\qquad 0 < \int_0^\infty g^2(y)dy \leq A^2 \int_0^\infty f^2(x)dx < \infty.$$

By (2.1), both (2.6) and (2.7) take the form of strict inequality, so we have the inequality (2.5). On the other hand, suppose that (2.5) is valid. By Hölder's inequality, we get

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \min\{x, y\}} f(x)g(y)dxdy \\ &\qquad = \int_0^\infty \left(\int_0^\infty \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \min\{x, y\}} f(x)dx \right) g(y)dy \\ &\qquad < A \left(\int_0^\infty f^2(x)dx \right)^{1/2} \left(\int_0^\infty g^2(y)dy \right)^{1/2} \end{aligned}$$

which is the inequality (2.1). ∎

Remark 1. If we take $\gamma = \alpha = \beta = 1$, then the inequality (1.3) can be induced by the inequality (2.1).

3. Several special inequalities

In this section, by choosing different γ, α, β, we establish several special inequalities. In what follows, assume that (1.1) is satisfied.

(1) If $\gamma = 0, \alpha = \beta = 1$, then

$$(3.1)\qquad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x + y + \min\{x, y\}} dxdy < A \left(\int_0^\infty f^2(x)dx \right)^{1/2} \left(\int_0^\infty g^2(y)dy \right)^{1/2},$$

where

$$A = 4\int_0^1 \frac{1}{2t^2+1}dt = 2\sqrt{2}\arctan(\sqrt{2}).$$

(2) If $\gamma = 2$, $\alpha = \beta = 1$, then

$$\int_0^\infty\int_0^\infty \frac{|\log x - \log y|^2}{x+y+\min\{x,y\}} f(x)g(y)dxdy < A\left(\int_0^\infty f^2(x)dx\right)^{1/2}\left(\int_0^\infty g^2(y)dy\right)^{1/2}, \tag{3.2}$$

where

$$A = 16\int_0^1 \frac{|\log t|^2}{2t^2+1}dt = 30.24955\ldots.$$

References

[1] G. H. Hardy, J. E. Littlewood, G. Polya, *Inequalities*, Cambridge University Press, 1934.

[2] B. He, Y. Qian, Y. Li, *On analogues of Hilbert's inequalities*, Comm. Math. Anal. 4 (2) (2008), 47–53.

[3] Y. Li, J. Wu, B. He, *A new Hilbert-type integral inequality and the equivalent form*, Internat. J. Math. Math. Sci. 2006, Art. ID 45378, 6 pp.

[4] D. S. Mitrinović, J. Pečarić, A. M. Fink, *Inequalities Involving Functions and their Integrals and Derivatives*, Kluwer Academic Publishers, Boston, 1991.

[5] D. S. Mitrinović, J. Pečarić, A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.

[6] W. T. Sulaiman, *Four inequalities similar to Hardy-Hilbert's integral inequality*, JIPAM. J. Inequal. Pure Appl. Math. 2006, 7(2), Article 76, 8 pp.

[7] B. Yang, *On the norm of an integral operator and applications*, J. Math. Anal. Appl. 321 (2006), 182–192.

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