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George A. Anastassiou

# POINCARÉ AND SOBOLEV TYPE INEQUALITIES FOR WIDDER DERIVATIVES

**Abstract.** Various $L_p$ form Poincaré and Sobolev type inequalities, forward and reverse, are given involving Widder derivative ([1]).

## 1. Background

The following come from [1].

Let $f, u_0, u_1, \ldots, u_n \in C^{n+1}([a,b])$, $n \geq 0$, and the Wronskians

$$W_i(x) := W[u_0(x), u_1(x), \ldots, u_i(x)] \tag{1}$$

$$:= \begin{vmatrix} u_0(x) & u_1(x) & \ldots & u_i(x) \\ u_0'(x) & u_1'(x) & \ldots & u_i'(x) \\ \vdots & & & \\ u_0^{(i)}(x) & u_1^{(i)}(x) & \ldots & u_i^{(i)}(x) \end{vmatrix},$$

$i = 0, 1, \ldots, n$. Here $W_0(x) = u_0(x)$. Assume $W_i(x) > 0$ over $[a,b]$, $i = 0, 1, \ldots, n$. For $i \geq 0$, the differential operator of order $i$ (Widder derivative):

$$L_i f(x) := \frac{W[u_0(x), u_1(x), \ldots, u_{i-1}(x), f(x)]}{W_{i-1}(x)}, \tag{2}$$

$i = 1, \ldots, n+1$; $L_0 f(x) := f(x)$, $\forall x \in [a,b]$.

Consider also

$$g_i(x,t) := \frac{1}{W_i(t)} \begin{vmatrix} u_0(t) & u_1(t) & \ldots & u_i(t) \\ u_0'(t) & u_1'(t) & \ldots & u_i'(t) \\ \vdots & & & \\ u_0^{(i-1)}(t) & u_1^{(i-1)}(t) & \ldots & u_i^{(i-1)}(t) \\ u_0(x) & u_1(x) & \ldots & u_i(x) \end{vmatrix}, \tag{3}$$

$i = 1, 2, \ldots, n$; $g_0(x,t) := \frac{u_0(x)}{u_0(t)}$, $\forall x, t \in [a,b]$.

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**Example.** ([1]) Sets of the form $\{u_0, u_1, \ldots, u_n\}$ are $\{1, x, x^2, \ldots, x^n\}$, $\left\{1, \sin x, -\cos x, -\sin 2x, \cos 2x, \ldots, (-1)^{n-1}\sin nx, (-1)^n \cos nx\right\}$, etc.

We also mention the generalized Widder-Taylor's formula, see [1].

**Theorem 1.** *Let the functions* $f, u_0, u_1, \ldots, u_n \in C^{n+1}([a,b])$, *and the Wronskians* $W_0(x), W_1(x), \ldots, W_n(x) > 0$ *on* $[a,b]$, $x \in [a,b]$. *Then for* $t \in [a,b]$ *we have*

$$f(x) = f(t)\frac{u_0(x)}{u_0(t)} + L_1 f(t)\, g_1(x,t) + \ldots + L_n f(t)\, g_n(x,t) + R_n(x), \tag{4}$$

*where*

$$R_n(x) := \int_t^x g_n(x,s)\, L_{n+1} f(s)\, ds. \tag{5}$$

E.g. ([1]) one could take $u_0(x) = c > 0$. If $u_i(x) = x^i$, $i = 0, 1, \ldots, n$, defined on $[a,b]$, then

$$L_i f(t) = f^{(i)}(t) \text{ and } g_i(x,t) = \frac{(x-t)^i}{i!}, \quad t \in [a,b]. \tag{6}$$

We need

**Corollary 2.** (to Theorem 1) *Additionally assume that for fixed* $x_0 \in [a,b]$ *we have* $L_i f(x_0) = 0$, $i = 0, 1, \ldots, n$. *Then*

$$f(x) = \int_{x_0}^x g_n(x,t)\, L_{n+1} f(t)\, dt, \quad \forall x \in [a,b]. \tag{7}$$

**Corollary 3.** (to Theorem 1) *Let* $f, u_0 \in C^1([a,b])$, $u_0(x) > 0$, *for all* $x \in [a,b]$. *Then*

$$f(x) = f(t)\frac{u_0(x)}{u_0(t)} + u_0(x)\int_t^x \frac{L_1 f(s)}{u_0(s)} ds, \quad \forall x, t \in [a,b], \tag{8}$$

*where*

$$L_1 f(s) = \frac{W[u_0(s), f(s)]}{u_0(s)} = \frac{u_0(s) f'(s) - u_0'(s) f(s)}{u_0(s)}. \tag{9}$$

We need to make

**Remark 4.** We define (see [1])

$$\phi_0(x) := W_0(x), \quad \phi_1(x) := \frac{W_1(x)}{(W_0(x))^2}, \ldots,$$

in general

$$\phi_k(x) := \frac{W_k(x)\, W_{k-2}(x)}{(W_{k-1}(x))^2}, \quad k = 2, 3, \ldots, n. \tag{10}$$

The functions $\phi_i(x)$ are positive on $[a,b]$. According to [1] we get, for any $x, x_0 \in [a,b]$ that

$$(11)\qquad g_n(x,x_0) = \frac{\phi_0(x)}{\phi_0(x_0)\dots\phi_n(x_0)} \int_{x_0}^{x} \phi_1(x_1) \int_{x_0}^{x_1} \dots \int_{x_0}^{x_{n-2}} \phi_{n-1}(x_{n-1}) \int_{x_0}^{x_{n-1}} \phi_n(x_n)\,dx_1 dx_2 \dots dx_n$$

$$= \frac{1}{\phi_0(x_0)\dots\phi_n(x_0)} \int_{x_0}^{x} \phi_0(s)\dots\phi_n(s)\,g_{n-1}(x,s)\,ds.$$

We get that $g_n(x,x) = 0$, all $x \in [a,b]$, and $g_n(x,x_0) > 0$, $x > x_0$, $x, x_0 \in [a,b]$, any $n \in \mathbb{N}$. Also $g_0(x,x_0) > 0$ for any $x, x_0 \in [a,b]$.

By (11) we notice that

$$(12)\qquad \begin{aligned} g_n(x,t) &< 0, \quad x < t, \quad n \text{ odd},\\ g_n(x,t) &> 0, \quad x < t, \quad n \text{ even}, \end{aligned}$$

where $x, t \in [a,b]$.

In the next we work under the terms and assumptions of Theorem 1 and Corollary 2, and the rest of the above conclusions.

## 2. Results

We give the following weighted Neumann-Poincaré type inequality.

**Theorem 5.** *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\nu > 0$. Consider $f, u_0 \in C^1([a,b])$, $u_0 > 0$ on $[a,b]$.*

*Then*

$$(13)\qquad 1)\ \left\| \frac{f}{u_0} - \frac{1}{b-a}\int_a^b \frac{f(t)}{u_0(t)}dt \right\|_{L_\nu(a,b)} \le (b-a)^{\left(\frac{1}{p}+\frac{1}{\nu}\right)} \left\| \frac{L_1 f}{u_0} \right\|_{L_q(a,b)}.$$

*When $\nu = q$ we get*

$$(14)\qquad 2)\ \left\| \frac{f}{u_0} - \frac{1}{b-a}\int_a^b \frac{f(t)}{u_0(t)}dt \right\|_{L_q(a,b)} \le (b-a) \left\| \frac{L_1 f}{u_0} \right\|_{L_q(a,b)}.$$

*When $\nu = p = q = 2$ we have*

$$(15)\qquad 3)\ \left\| \frac{f}{u_0} - \frac{1}{b-a}\int_a^b \frac{f(t)}{u_0(t)}dt \right\|_{L_2(a,b)} \le (b-a) \left\| \frac{L_1 f}{u_0} \right\|_{L_2(a,b)}.$$

*Equivalently,*

$$(15^*)\qquad \int_a^b \left( \frac{f(x)}{u_0(x)} - \frac{1}{b-a}\int_a^b \frac{f(t)}{u_0(t)}dt \right)^2 dx \le (b-a)^2 \int_a^b \left( \frac{L_1 f(x)}{u_0(x)} \right)^2 dx.$$

**Proof.** Let $A = \frac{1}{b-a}\int_a^b \frac{f(t)}{u_0(t)}dt$.

By the intermediate value theorem there exists $c \in [a,b]$ with $\frac{f(c)}{u_0(c)} = A$. Thus by (8) we get

$$\frac{f(x)}{u_0(x)} - \frac{f(c)}{u_0(c)} = \int_c^x \frac{L_1 f(s)}{u_0(s)} ds, \tag{16}$$

i.e.

$$\frac{f(x)}{u_0(x)} - A = \int_c^x \frac{L_1 f(s)}{u_0(s)} ds, \quad \forall x \in [a,b]. \tag{17}$$

Therefore

$$\begin{aligned} \left|\frac{f(x)}{u_0(x)} - A\right| &\le \left|\int_c^x \frac{|L_1 f(s)|}{u_0(s)} ds\right| \le \int_a^b \frac{|L_1 f(s)|}{u_0(s)} ds \\ &\le (b-a)^{1/p}\left(\int_a^b \left(\frac{|L_1 f(s)|}{u_0(s)}\right)^q ds\right)^{1/q}, \quad \forall x \in [a,b]. \end{aligned} \tag{18}$$

Hence

$$\left|\frac{f(x)}{u_0(x)} - A\right|^\nu \le (b-a)^{\frac{\nu}{p}} \left\|\frac{L_1 f}{u_0}\right\|_{L_q(a,b)}^\nu, \quad \forall x \in [a,b]. \tag{19}$$

Consequently we obtain

$$\int_a^b \left|\frac{f(x)}{u_0(x)} - A\right|^\nu dx \le (b-a)^{\left(\frac{\nu}{p}+1\right)} \left\|\frac{L_1 f}{u_0}\right\|_{L_q(a,b)}^\nu, \tag{20}$$

proving the claim. ∎

**Corollary 6.** (to Theorem 5) *Let* $\int_a^b \frac{f(t)}{u_0(t)}dt = 0$. *Then*

$$1) \quad \left\|\frac{f}{u_0}\right\|_{L_\nu(a,b)} \le (b-a)^{\left(\frac{1}{p}+\frac{1}{\nu}\right)} \left\|\frac{L_1 f}{u_0}\right\|_{L_q(a,b)}, \tag{21}$$

$$2) \quad \left\|\frac{f}{u_0}\right\|_{L_q(a,b)} \le (b-a) \left\|\frac{L_1 f}{u_0}\right\|_{L_q(a,b)}, \tag{22}$$

$$3) \quad \left\|\frac{f}{u_0}\right\|_{L_2(a,b)} \le (b-a) \left\|\frac{L_1 f}{u_0}\right\|_{L_2(a,b)}, \tag{23}$$

*equivalently,*

$$\int_a^b \left(\frac{f(x)}{u_0(x)}\right)^2 dx \le (b-a)^2 \int_a^b \left(\frac{L_1 f(x)}{u_0(x)}\right)^2 dx. \tag{23*}$$

It follows the related result.

**Proposition 7.** *Let $\nu > 0$ and consider $f, u_0 \in C^1([a,b])$, $u_0 > 0$ on $[a,b]$.*

*Then*

$$(24) \qquad 1) \left\| \frac{f}{u_0} - \frac{1}{b-a} \int_a^b \frac{f(t)}{u_0(t)} dt \right\|_{L_\nu(a,b)} \le (b-a)^{1/\nu} \left\| \frac{L_1 f}{u_0} \right\|_{L_1(a,b)}.$$

*When $\nu = 1$ we have*

$$(25) \qquad 2) \left\| \frac{f}{u_0} - \frac{1}{b-a} \int_a^b \frac{f(t)}{u_0(t)} dt \right\|_{L_1(a,b)} \le (b-a) \left\| \frac{L_1 f}{u_0} \right\|_{L_1(a,b)}.$$

**Proof.** By (18) we have

$$(26) \qquad \left| \frac{f(x)}{u_0(x)} - \frac{1}{b-a} \int_a^b \frac{f(t)}{u_0(t)} dt \right|^\nu \le \left\| \frac{L_1 f}{u_0} \right\|_{L_1(a,b)}^\nu, \quad \forall x \in [a,b].$$

Hence

$$(27) \qquad \int_a^b \left| \frac{f(x)}{u_0(x)} - \frac{1}{b-a} \int_a^b \frac{f(t)}{u_0(t)} dt \right|^\nu dx \le (b-a) \left\| \frac{L_1 f}{u_0} \right\|_{L_1(a,b)}^\nu,$$

proving the claim. ■

We continue with generalized Dirichlet-Poincaré type inequalities.

**Theorem 8.** *Let $f, u_0, u_1, \ldots, u_n \in C^{n+1}([a,b])$, $n \in \mathbb{Z}_+$; $W_0, W_1, \ldots, W_n > 0$ on $[a,b]$, and for fixed $x_0 \in [a,b]$ assume that $L_i f(x_0) = 0$, $i = 0, 1, \ldots, n$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\nu > 0$.*

*Then*

$$(28) \qquad 1) \; \|f\|_{L_\nu(x_0,b)} \le \left( \int_{x_0}^b \left( \int_{x_0}^x (g_n(x,t))^p dt \right)^{\nu/p} dx \right)^{1/\nu} \|L_{n+1} f\|_{L_q(x_0,b)}.$$

$$(29) \qquad 2) \; \|f\|_{L_q(x_0,b)} \le \left( \int_{x_0}^b \left( \int_{x_0}^x (g_n(x,t))^p dt \right)^{q/p} dx \right)^{1/q} \|L_{n+1} f\|_{L_q(x_0,b)}.$$

*And*

$$(30) \qquad 3) \; \|f\|_{L_2(x_0,b)} \le \left( \int_{x_0}^b \left( \int_{x_0}^x (g_n(x,t))^2 dt \right) dx \right)^{1/2} \|L_{n+1} f\|_{L_2(x_0,b)}.$$

**Proof.** By (7) we have

$$(31) \qquad |f(x)| = \left| \int_{x_0}^x g_n(x,t) L_{n+1} f(t)\, dt \right| \le \int_{x_0}^x g_n(x,t) \left| L_{n+1} f(t) \right| dt$$

$$\leq \left( \int_{x_0}^{x} (g_n(x,t))^p dt \right)^{1/p} \left( \int_{x_0}^{x} \left| L_{n+1} f \left( t \right) \right|^q dt \right)^{1/q}$$

$$\leq \left( \int_{x_0}^{x} (g_n(x,t))^p dt \right)^{1/p} \|L_{n+1} f\|_{L_q(x_0,b)}, \quad \forall x \in [x_0, b].$$

That is

$$|f(x)|^\nu \leq \left( \int_{x_0}^{x} (g_n(x,t))^p dt \right)^{\nu/p} \|L_{n+1} f\|^\nu_{L_q(x_0,b)}, \tag{32}$$

and

$$\int_{x_0}^{b} |f(x)|^\nu dx \leq \left( \int_{x_0}^{b} \left( \int_{x_0}^{x} (g_n(x,t))^p dt \right)^{\nu/p} dx \right) \|L_{n+1} f\|^\nu_{L_q(x_0,b)}, \tag{33}$$

proving the claim. ∎

We give

**Theorem 9.** *Same assumptions as in Theorem 8.*
*Then*

$$1) \ \|f\|_{L_\nu(a,x_0)} \leq \left( \int_{a}^{x_0} \left( \int_{x}^{x_0} |g_n(x,t)|^p dt \right)^{\nu/p} dx \right)^{1/\nu} \|L_{n+1} f\|_{L_q(a,x_0)}. \tag{34}$$

$$2) \ \|f\|_{L_q(a,x_0)} \leq \left( \int_{a}^{x_0} \left( \int_{x}^{x_0} |g_n(x,t)|^p dt \right)^{q/p} dx \right)^{1/q} \|L_{n+1} f\|_{L_q(a,x_0)}. \tag{35}$$

*And*

$$3) \ \|f\|_{L_2(a,x_0)} \leq \left( \int_{a}^{x_0} \left( \int_{x}^{x_0} (g_n(x,t))^2 dt \right) dx \right)^{1/2} \|L_{n+1} f\|_{L_2(a,x_0)}. \tag{36}$$

**Proof.** By (7) we have

$$|f(x)| = \left| \int_{x_0}^{x} g_n(x,t) L_{n+1} f(t)\, dt \right| = \left| \int_{x}^{x_0} g_n(x,t) L_{n+1} f(t)\, dt \right| \tag{37}$$

$$\leq \int_{x}^{x_0} |g_n(x,t)| \, |L_{n+1} f(t)| \, dt$$

$$\leq \left( \int_{x}^{x_0} |g_n(x,t)|^p \, dt \right)^{1/p} \left( \int_{x}^{x_0} |L_{n+1} f(t)|^q dt \right)^{1/q}$$

$$\leq \left( \int_{x}^{x_0} |g_n(x,t)|^p \, dt \right)^{1/p} \|L_{n+1} f\|_{L_q(a,x_0)}.$$

That is

$$|f(x)|^{\nu} \leq \left( \int_x^{x_0} |g_n(x,t)|^p \, dt \right)^{\nu/p} \|L_{n+1}f\|^{\nu}_{L_q(a,x_0)}, \tag{38}$$

and

$$\int_a^{x_0} |f(x)|^{\nu} \, dx \leq \left( \int_a^{x_0} \left( \int_x^{x_0} |g_n(x,t)|^p \, dt \right)^{\nu/p} dx \right) \|L_{n+1}f\|^{\nu}_{L_q(a,x_0)}, \tag{39}$$

proving the claim. ∎

We continue with

**Proposition 10.** *Let* $f, u_0, u_1, \dots, u_n \in C^{n+1}([a,b])$, $n \in \mathbb{Z}_+$; $W_0, W_1, \dots, W_n > 0$ *on* $[a,b]$, *and for fixed* $x_0 \in [a,b]$ *assume that* $L_i f(x_0) = 0$, $i = 0, 1, \dots, n$; $\nu > 0$.

*Then*

$$1) \ \|f\|_{L_{\nu}(x_0,b)} \leq \left( \int_{x_0}^{b} \left( \int_{x_0}^{x} g_n(x,t)dt \right)^{\nu} dx \right)^{1/\nu} \|L_{n+1}f\|_{L_{\infty}(x_0,b)}. \tag{40}$$

$$2) \ \|f\|_{L_1(x_0,b)} \leq \left( \int_{x_0}^{b} \left( \int_{x_0}^{x} g_n(x,t)dt \right) dx \right) \|L_{n+1}f\|_{L_{\infty}(x_0,b)}. \tag{41}$$

**Proof.** By (7) we have

$$|f(x)| \leq \int_{x_0}^{x} g_n(x,t) \, |L_{n+1}f(t)| \, dt \leq \left( \int_{x_0}^{x} g_n(x,t)dt \right) \|L_{n+1}f\|_{\infty,[x_0,b]}. \tag{42}$$

That is

$$|f(x)|^{\nu} \leq \left( \int_{x_0}^{x} g_n(x,t)dt \right)^{\nu} \|L_{n+1}f\|^{\nu}_{\infty,[x_0,b]}, \tag{43}$$

and

$$\int_{x_0}^{b} |f(x)|^{\nu} \, dx \leq \left( \int_{x_0}^{b} \left( \int_{x_0}^{x} g_n(x,t)dt \right)^{\nu} dx \right) \|L_{n+1}f\|^{\nu}_{\infty,[x_0,b]}, \tag{44}$$

proving the claim. ∎

We give

**Proposition 11.** *All as in Proposition* 10.

*Then*

$$1) \ \|f\|_{L_{\nu}(a,x_0)} \leq \left( \int_a^{x_0} \left( \int_x^{x_0} |g_n(x,t)| \, dt \right)^{\nu} dx \right)^{1/\nu} \|L_{n+1}f\|_{L_{\infty}(a,x_0)}. \tag{45}$$

$$2) \ \|f\|_{L_1(a,x_0)} \leq \left( \int_a^{x_0} \left( \int_x^{x_0} |g_n(x,t)| \, dt \right) dx \right) \|L_{n+1}f\|_{L_{\infty}(a,x_0)}. \tag{46}$$

**Proof.** By (7) we have
(47)
$$|f(x)| \leq \int_x^{x_0} |g_n(x,t)| \, |L_{n+1}f(t)| \, dt \leq \left( \int_x^{x_0} |g_n(x,t)| \, dt \right) \|L_{n+1}f\|_{\infty,[a,x_0]} \,.$$

That is

$$|f(x)|^{\nu} \leq \left( \int_x^{x_0} |g_n(x,t)| \, dt \right)^{\nu} \|L_{n+1}f\|^{\nu}_{\infty,[a,x_0]} \,, \tag{48}$$

and

$$\int_a^{x_0} |f(x)|^{\nu} \, dx \leq \left( \int_a^{x_0} \left( \int_x^{x_0} |g_n(x,t)| \, dt \right)^{\nu} dx \right) \|L_{n+1}f\|^{\nu}_{\infty,[a,x_0]} \,, \tag{49}$$

proving the claim. ■

We continue with reverse Dirichlet-Poincaré type inequalities.

**Theorem 12.** *Let $f, u_0, u_1, \ldots, u_n \in C^{n+1}([a,b])$, $n \in \mathbb{Z}_+$; $W_0, W_1, \ldots, W_n > 0$ on $[a,b]$, and for a fixed $a \leq x_0 < b$ assume that $L_i f(x_0) = 0$, $i = 0, 1, \ldots, n$. Let $0 < p < 1$, $q < 0 : \frac{1}{p} + \frac{1}{q} = 1$, $\nu > 0$. Further suppose that $L_{n+1}f$ is of fixed sign and nowhere zero on $[x_0, b]$.*

*Then*

$$1) \;\; \|f\|_{L_\nu(x_0,b)} \geq \left( \int_{x_0}^{b} \left( \int_{x_0}^{x} (g_n(x,t))^p dt \right)^{\nu/p} dx \right)^{1/\nu} \|L_{n+1}f\|_{L_q(x_0,b)} \,. \tag{50}$$

$$2) \;\; \|f\|_{L_p(x_0,b)} \geq \left( \int_{x_0}^{b} \left( \int_{x_0}^{x} (g_n(x,t))^p dt \right) dx \right)^{1/p} \|L_{n+1}f\|_{L_q(x_0,b)} \,. \tag{51}$$

$$3) \;\; \|f\|_{L_1(x_0,b)} \geq \left( \int_{x_0}^{b} \left( \int_{x_0}^{x} (g_n(x,t))^p dt \right)^{1/p} dx \right) \|L_{n+1}f\|_{L_q(x_0,b)} \,. \tag{52}$$

**Proof.** Here we have by (7) and assumption that

$$|f(x)| = \int_{x_0}^{x} g_n(x,t) \, |L_{n+1}f(t)| \, dt, \tag{53}$$

$\forall x \in [x_0, b]$.

Hence by reverse Hölder's inequality we obtain

$$\begin{aligned} |f(x)| &\geq \left( \int_{x_0}^{x} (g_n(x,t))^p dt \right)^{1/p} \left( \int_{x_0}^{x} |L_{n+1}f(t)|^q dt \right)^{1/q} \\ &\geq \left( \int_{x_0}^{x} (g_n(x,t))^p dt \right)^{1/p} \left( \int_{x_0}^{b} |L_{n+1}f(t)|^q \, dt \right)^{1/q}, \end{aligned} \tag{54}$$

true for all $x \in (x_0, b]$.

That is

$$|f(x)|^{\nu} \geq \left( \int_{x_0}^{x} (g_n(x,t))^p dt \right)^{\nu/p} \|L_{n+1}f\|^{\nu}_{L_q(x_0,b)}, \tag{55}$$

true for all $x \in [x_0, b]$,

and

$$\int_{x_0}^{b} |f(x)|^{\nu} dx \geq \left( \int_{x_0}^{b} \left( \int_{x_0}^{x} (g_n(x,t))^p dt \right)^{\nu/p} dx \right) \|L_{n+1}f\|^{\nu}_{L_q(x_0,b)}, \tag{56}$$

proving the claim. ■

We give

**THEOREM 13.** *Let $f, u_0, u_1, \ldots, u_n \in C^{n+1}([a,b])$, $n$ is odd; $W_0, W_1, \ldots, W_n > 0$ on $[a,b]$, and for a fixed $a < x_0 \leq b$ assume that $L_i f(x_0) = 0$, $i = 0, 1, \ldots, n$. Let $0 < p < 1$, $q < 0 : \frac{1}{p} + \frac{1}{q} = 1$, $\nu > 0$. Further suppose that $L_{n+1}f$ is of fixed sign and nowhere zero on $[a, x_0]$.*

*Then*

$$\text{1)}\quad \|f\|_{L_\nu(a,x_0)} \geq \left( \int_{a}^{x_0} \left( \int_{x}^{x_0} (-g_n(x,t))^p dt \right)^{\nu/p} dx \right)^{1/\nu} \|L_{n+1}f\|_{L_q(a,x_0)}. \tag{57}$$

$$\text{2)}\quad \|f\|_{L_p(a,x_0)} \geq \left( \int_{a}^{x_0} \left( \int_{x}^{x_0} (-g_n(x,t))^p dt \right) dx \right)^{1/p} \|L_{n+1}f\|_{L_q(a,x_0)}. \tag{58}$$

$$\text{3)}\quad \|f\|_{L_1(a,x_0)} \geq \left( \int_{a}^{x_0} \left( \int_{x}^{x_0} (-g_n(x,t))^p dt \right)^{1/p} dx \right) \|L_{n+1}f\|_{L_q(a,x_0)}. \tag{59}$$

**Proof.** Here by (7) and assumption we have

$$\begin{aligned} |f(x)| &= \left| \int_{x_0}^{x} g_n(x,t) L_{n+1}f(t)\, dt \right| = \left| \int_{x}^{x_0} g_n(x,t) L_{n+1}f(t)\, dt \right| \\ &\overset{(12)}{=} \left| \int_{x}^{x_0} (-g_n(x,t))\, L_{n+1}f(t)\, dt \right| = \int_{x}^{x_0} (-g_n(x,t))\, |L_{n+1}f(t)|\, dt. \end{aligned} \tag{60}$$

So by reverse Hölder's inequality we obtain

$$\begin{aligned} |f(x)| &\geq \left( \int_{x}^{x_0} (-g_n(x,t))^p dt \right)^{1/p} \left( \int_{x}^{x_0} |L_{n+1}f(t)|^q dt \right)^{1/q} \\ &\geq \left( \int_{x}^{x_0} (-g_n(x,t))^p dt \right)^{1/p} \|L_{n+1}f\|_{L_q(a,x_0)}, \end{aligned} \tag{61}$$

true for all $x \in [a, x_0)$,

and

$$|f(x)|^{\nu} \geq \left(\int_{x}^{x_0} (-g_n(x,t))^p dt\right)^{\nu/p} \|L_{n+1}f\|^{\nu}_{L_q(a,x_0)}, \tag{62}$$

true for all $x \in [a, x_0]$.

Thus

$$\int_{a}^{x_0} |f(x)|^{\nu} dx \geq \left(\int_{a}^{x_0}\left(\int_{x}^{x_0} (-g_n(x,t))^p dt\right)^{\nu/p} dx\right) \|L_{n+1}f\|^{\nu}_{L_q(a,x_0)}, \tag{63}$$

proving the claim. ∎

We add

**Theorem 14.** *Now $n$ is even, the rest as in Theorem 13.*

*Then*

$$1)\ \|f\|_{L_{\nu}(a,x_0)} \geq \left(\int_{a}^{x_0}\left(\int_{x}^{x_0} (g_n(x,t))^p dt\right)^{\nu/p} dx\right)^{1/\nu} \|L_{n+1}f\|_{L_q(a,x_0)}. \tag{64}$$

$$2)\ \|f\|_{L_p(a,x_0)} \geq \left(\int_{a}^{x_0}\left(\int_{x}^{x_0} (g_n(x,t))^p dt\right) dx\right)^{1/p} \|L_{n+1}f\|_{L_q(a,x_0)}. \tag{65}$$

$$3)\ \|f\|_{L_1(a,x_0)} \geq \left(\int_{a}^{x_0}\left(\int_{x}^{x_0} (g_n(x,t))^p dt\right)^{1/p} dx\right) \|L_{n+1}f\|_{L_q(a,x_0)}. \tag{66}$$

**Proof.** Similar to Theorem 13. ∎

We continue with Sobolev type inequalities.

**Theorem 15.** *Same assumptions as in Theorem 8. Call*

$$M_{\nu,1} := \max_{0\leq j\leq n}\left\{\left(\int_{x_0}^{b}\left(\int_{x_0}^{x} (g_j(x,t))^p dt\right)^{\nu/p} dx\right)^{1/\nu}\right\}, \quad \nu > 0. \tag{67}$$

*Then*

$$1)\ \|f\|_{L_{\nu}(x_0,b)} \leq \left(\frac{M_{\nu,1}}{n+1}\right)\left(\sum_{j=0}^{n} \|L_{j+1}f\|_{L_q(x_0,b)}\right), \tag{68}$$

$$2)\ \|f\|_{L_q(x_0,b)} \leq \left(\frac{M_{q,1}}{n+1}\right)\left(\sum_{j=0}^{n} \|L_{j+1}f\|_{L_q(x_0,b)}\right), \tag{69}$$

*and when $\nu = p = q = 2$ we get*

$$3)\ \|f\|_{L_2(x_0,b)} \leq \left(\frac{M_{2,1}}{n+1}\right)\left(\sum_{j=0}^{n} \|L_{j+1}f\|_{L_2(x_0,b)}\right), \tag{70}$$

*where*

$$M_{2,1} := \max_{0\le j\le n}\left\{\left(\int_{x_0}^{b}\left(\int_{x_0}^{x}(g_j(x,t))^2dt\right)dx\right)^{1/2}\right\}. \tag{71}$$

**Proof.** The assumptions of Theorem 8 are fulfilled for all $j=0,1,\dots,n$. Thus by (28) we get

$$\begin{aligned}\|f\|_{L_\nu(x_0,b)} &\le \left(\int_{x_0}^{b}\left(\int_{x_0}^{x}(g_j(x,t))^p dt\right)^{\nu/p}dx\right)^{1/\nu}\|L_{j+1}f\|_{L_q(x_0,b)}\\ &\le M_{\nu,1}\|L_{j+1}f\|_{L_q(x_0,b)},\end{aligned} \tag{72}$$

for all $j=0,1,\dots,n$.

From (72) by addition we get

$$(n+1)\|f\|_{L_\nu(x_0,b)} \le M_{\nu,1}\left(\sum_{j=0}^{n}\|L_{j+1}f\|_{L_q(x_0,b)}\right), \tag{73}$$

proving the claim. ∎

We continue with

**THEOREM 16.** *Same assumptions as in Theorem* 8. *Call*

$$M_{\nu,2} := \max_{0\le j\le n}\left\{\left(\int_{a}^{x_0}\left(\int_{x}^{x_0}|g_j(x,t)|^p dt\right)^{\nu/p}dx\right)^{1/\nu}\right\}, \tag{74}$$

$\nu>0$.

*Then*

$$1)\ \|f\|_{L_\nu(a,x_0)} \le \left(\frac{M_{\nu,2}}{n+1}\right)\left(\sum_{j=0}^{n}\|L_{j+1}f\|_{L_q(a,x_0)}\right), \tag{75}$$

$$2)\ \|f\|_{L_q(a,x_0)} \le \left(\frac{M_{q,2}}{n+1}\right)\left(\sum_{j=0}^{n}\|L_{j+1}f\|_{L_q(a,x_0)}\right), \tag{76}$$

*and when* $\nu=p=q=2$ *we get*

$$3)\ \|f\|_{L_2(a,x_0)} \le \left(\frac{M_{2,2}}{n+1}\right)\left(\sum_{j=0}^{n}\|L_{j+1}f\|_{L_2(a,x_0)}\right), \tag{77}$$

*where*

$$M_{2,2} := \max_{0\le j\le n}\left\{\left(\int_{a}^{x_0}\left(\int_{x}^{x_0}(g_j(x,t))^2dt\right)dx\right)^{1/2}\right\}. \tag{78}$$

**Proof.** Similar to Theorem 15, based on Theorem 9. ∎

We continue with $L_\infty$ results.

**PROPOSITION 17.** *All as in Proposition* 10. *Call*

$$K_{\nu,1} := \max_{0\le j\le n}\left\{\left(\int_{x_0}^{b}\left(\int_{x_0}^{x} g_j(x,t)dt\right)^{\nu} dx\right)^{1/\nu}\right\}, \qquad \nu > 0.$$

*Then*

$$(79) \qquad 1)\ \|f\|_{L_\nu(x_0,b)} \le \left(\frac{K_{\nu,1}}{n+1}\right)\left(\sum_{j=0}^{n}\|L_{j+1}f\|_{L_\infty(x_0,b)}\right),$$

*and*

$$(80) \qquad 2)\ \|f\|_{L_1(x_0,b)} \le \left(\frac{K_{1,1}}{n+1}\right)\left(\sum_{j=0}^{n}\|L_{j+1}f\|_{L_\infty(x_0,b)}\right).$$

**Proof.** Similar to Theorem 15, based on Proposition 10. ∎

**PROPOSITION 18.** *All as in Proposition* 10. *Call*

$$(81) \qquad K_{\nu,2} := \max_{0\le j\le n}\left\{\left(\int_{a}^{x_0}\left(\int_{x}^{x_0} |g_n(x,t)|\,dt\right)^{\nu} dx\right)^{1/\nu}\right\}, \quad \nu > 0.$$

*Then*

$$(82) \qquad 1)\ \|f\|_{L_\nu(a,x_0)} \le \left(\frac{K_{\nu,2}}{n+1}\right)\left(\sum_{j=0}^{n}\|L_{j+1}f\|_{L_\infty(a,x_0)}\right),$$

*and*

$$(83) \qquad 2)\ \|f\|_{L_1(a,x_0)} \le \left(\frac{K_{1,2}}{n+1}\right)\left(\sum_{j=0}^{n}\|L_{j+1}f\|_{L_\infty(a,x_0)}\right).$$

**Proof.** Based on Proposition 11. ∎

We continue with reverse Sobolev type inequalities.

**THEOREM 19.** *Assume here that $L_{j+1}f$ is of fixed sign and nowhere zero on $[x_0,b]$, for $j = 0,1,\ldots,n$. The rest are supposed as in Theorem* 12. *Call*

$$(84) \qquad S_{\nu,1} := \min_{0\le j\le n}\left\{\left(\int_{x_0}^{b}\left(\int_{x_0}^{x} (g_j(x,t))^p dt\right)^{\nu/p} dx\right)^{1/\nu}\right\}, \quad \nu > 0.$$

*Then*

$$(85) \qquad 1)\ \|f\|_{L_\nu(x_0,b)} \ge \left(\frac{S_{\nu,1}}{n+1}\right)\left(\sum_{j=0}^{n}\|L_{j+1}f\|_{L_q(x_0,b)}\right),$$

$$(86) \qquad 2)\ \|f\|_{L_p(x_0,b)} \ge \left(\frac{S_{p,1}}{n+1}\right)\left(\sum_{j=0}^{n}\|L_{j+1}f\|_{L_q(x_0,b)}\right),$$

*and*

$$(87)\qquad 3)\ \|f\|_{L_1(x_0,b)} \geq \left(\frac{S_{1,1}}{n+1}\right)\left(\sum_{j=0}^{n}\|L_{j+1}f\|_{L_q(x_0,b)}\right).$$

**Proof.** The assumptions of Theorem 12 are fulfilled for all $j = 0, 1, \ldots, n$. Thus by (50) we get

$$(88)\qquad \begin{aligned}\|f\|_{L_\nu(x_0,b)} &\geq \left(\int_{x_0}^{b}\left(\int_{x_0}^{x}(g_j(x,t))^p dt\right)^{\nu/p} dx\right)^{1/\nu}\|L_{j+1}f\|_{L_q(x_0,b)}\\ &\geq S_{\nu,1}\|L_{j+1}f\|_{L_q(x_0,b)},\end{aligned}$$

for all $j = 0, 1, \ldots, n$.

From (88) by addition we get

$$(89)\qquad (n+1)\|f\|_{L_\nu(x_0,b)} \geq S_{\nu,1}\left(\sum_{j=0}^{n}\|L_{j+1}f\|_{L_q(x_0,b)}\right),$$

proving the claim. ∎

We continue with

**THEOREM 20.** *Assume all as in Theorem* 13. *Here* $n = 2k+1$, $k \in \mathbb{Z}_+$; $\nu > 0$. *Further suppose that* $L_{j+1}f$ *is of fixed sign and nowhere zero on* $[a, x_0]$ *for all odds* $j \in [1, n]$. *Call*

$$(90)\qquad S_{\nu,2} := \min_{odd\ j\in[1,n]}\left\{\left(\int_{a}^{x_0}\left(\int_{x}^{x_0}(-g_j(x,t))^p dt\right)^{\nu/p} dx\right)^{1/\nu}\right\}.$$

*Then*

$$(91)\qquad 1)\ \|f\|_{L_\nu(a,x_0)} \geq \left(\frac{S_{\nu,2}}{k+1}\right)\left(\sum_{\substack{j\in[1,n]\\ odd}}\|L_{j+1}f\|_{L_q(a,x_0)}\right),$$

$$(92)\qquad 2)\ \|f\|_{L_p(a,x_0)} \geq \left(\frac{S_{p,2}}{k+1}\right)\left(\sum_{\substack{j\in[1,n]\\ odd}}\|L_{j+1}f\|_{L_q(a,x_0)}\right),$$

*and*

$$(93)\qquad 3)\ \|f\|_{L_1(a,x_0)} \geq \left(\frac{S_{1,2}}{k+1}\right)\left(\sum_{\substack{j\in[1,n]\\ odd}}\|L_{j+1}f\|_{L_q(a,x_0)}\right).$$

**Proof.** As in Theorem 19, based on Theorem 13. Since $n = 2k+1$, $k \in \mathbb{Z}_+$, there are $(k+1)$ odd numbers in $[1, n]$, so we apply (57) $(k+1)$ times. ∎

We finish with

**Theorem 21.** *Assume all as in Theorem 14. Here $n = 2k$, $k \in \mathbb{Z}_+$; $\nu > 0$. Further suppose that $L_{j+1}f$ is of fixed sign and nowhere zero on $[a, x_0]$ for all evens $j \in [0, n]$. Call*

$$S_{\nu,3} := \min_{\text{even } j \in [0,n]} \left\{ \left( \int_a^{x_0} \left( \int_x^{x_0} (g_j(x,t))^p dt \right)^{\nu/p} dx \right)^{1/\nu} \right\}. \tag{94}$$

*Then*

$$1) \ \|f\|_{L_\nu(a,x_0)} \geq \left( \frac{S_{\nu,3}}{k+1} \right) \left( \sum_{\substack{j \in [0,n] \\ \text{even}}} \|L_{j+1}f\|_{L_q(a,x_0)} \right), \tag{95}$$

$$2) \ \|f\|_{L_p(a,x_0)} \geq \left( \frac{S_{p,3}}{k+1} \right) \left( \sum_{\substack{j \in [0,n] \\ \text{even}}} \|L_{j+1}f\|_{L_q(a,x_0)} \right), \tag{96}$$

*and*

$$3) \ \|f\|_{L_1(a,x_0)} \geq \left( \frac{S_{1,3}}{k+1} \right) \left( \sum_{\substack{j \in [0,n] \\ \text{even}}} \|L_{j+1}f\|_{L_q(a,x_0)} \right). \tag{97}$$

**Proof.** As in Theorem 19, based on Theorem 14. Since $n = 2k$, $k \in \mathbb{Z}_+$, there are $(k+1)$ even numbers in $[0, n]$, so we apply (64) $(k+1)$ times. ∎

## References

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DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF MEMPHIS
MEMPHIS, TN 38152 U.S.A.
E-mail: ganastss@memphis.edu

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