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CONTRIBUTIONS TO THE ERDŐS' CONJECTURE ON ARITHMETIC PROGRESSIONS

Abstract. The most famous unsolved is Erdős' conjecture that every set $A \subset \mathbb{N}$ such that $\sum_{a \in A} a^{-1} = \infty$ contains arbitrarily long arithmetic progressions. In this paper a new aspects of this one are presented.

1. The first contribution

The most famous unsolved is Erdős' conjecture that every set $A \subset \mathbb{N}$ such that $\sum_{a \in A} a^{-1} = \infty$ contains arbitrarily long arithmetic progressions. Erdős has estimated the solution as worth 3000$ (see [2,3]). In this paper we do not present a solution of this great Erdős' problem. We will touch only a few new aspects of this one and clarify it a little further.

In view of that, in [4] W. Sierpiński proved the following two interesting lemmas:

Lemma 1. *Let $\{u_n\}$ be any sequence of positive integers. Then there exists a sequence $\{v_n\}_{n=1}^{\infty} \subset \mathbb{N}$ such that $\lim_{n\to\infty} u_n/v_n = 0$ and the set $\mathbb{N}\setminus\{v_n\}_{n=1}^{\infty}$ contains no infinite arithmetic progression.*

Lemma 2. *Let $\{u_n\}_{n=1}^{\infty}$ be an increasing sequence of positive integers. Then there exists a sequence $\{v_n\}_{n=1}^{\infty}$ of positive integers increasing faster than the sequence $\{u_n\}_{n=1}^{\infty}$ (in the sense: $\lim_{n\to\infty} u_n/v_n = 0$), which contains finite arithmetic progressions of any length.*

The above Lemmas became now completed with the following relevant result.

Lemma 3. *Let $\{u_n\}_{n=1}^{\infty}$ be an increasing sequence of positive integers such that $\sum \frac{1}{u_n} = \infty$. Then there exists an increasing sequence of positive integers*

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$\{v_n\}_{n=1}^{\infty}$ satisfying the following conditions: $\sum \frac{1}{v_n} = \infty$, $\lim \frac{u_n}{v_n} = 0$ and the sequence $\{v_n\}_{n=1}^{\infty}$ contains arithmetic progressions of any length.

Proof. Let $\{k_n\}_{n=1}^{\infty}$ and $\{l_n\}_{n=1}^{\infty}$ be two auxiliary sequences of positive integers that meet the following conditions:

(a) $$k_{n+1} - k_n > l_n > k_1;$$

(b) $$l_{n+1} > l_n^2;$$

(c) $$\sum_{r=1+k_n}^{k_{n+1}-l_n} \frac{1}{u_r} \geq l_n;$$

(d) $$u_{k_{n+1}-i} \leq l_n \left(u_{k_{n+1}} - i\, l_n\right) \qquad \text{for } i = 0, 1, \ldots, l_n - 1.$$

As far as condition (d) is concerned, it should be observed that for every $i \in \{1, \ldots, k_{n+1}\}$, we have:

$$u_{k_{n+1}-i} \leq u_{k_{n+1}} - i.$$

Now, the sequence $\{v_n\}_{n=1}^{\infty}$ shall be determined in the following manner:

$$v_{k_{n+1}-i} = l_n^2 \left(u_{k_{n+1}} - i\, l_n\right) \qquad \text{for } i = 0, 1, \ldots, l_n - 1,$$

and

$$v_{k_{n+1}-i} = l_n\, u_{k_{n+1}-i} \qquad \text{for } i = l_n, l_n + 1, \ldots, k_{n+1} - k_n - 1,$$

and $n \in \mathbb{N}$.

Furthermore, we set $v_i = i$ for $i = 1, 2, \ldots, k_1$. It should be noted that

1) in view of (d) and by the definition of v_n we have:

$$\frac{u_{k_{n+1}-i}}{v_{k_{n+1}-i}} \leq \frac{1}{l_n} \quad \text{for } i = 0, 1, \ldots, k_{n+1} - k_n - 1;$$

2) the subsequence $\{v_{k_{n+1}-n+1}, v_{k_{n+1}-n+2}, \ldots, v_{k_{n+1}}\}$ is an arithmetic progression with the difference term equal to $-l_n^3$;

3) $\sum\limits_{r=1+k_n}^{k_{n+1}-l_n} \frac{1}{v_r} = \sum\limits_{r=1+k_n}^{k_{n+1}-l_n} \frac{1}{l_n\, u_r} \geq 1$ for each $n \in \mathbb{N}$, which in view of (c) follows;

4) condition (d) also guarantees that $v_{k_{n+1}-l_n+1} > v_{k_{n+1}-l_n}$, i.e., the sequence $\{v_n :\ k_n + 1 \leq n \leq k_{n+1}\}$ is increasing;

5) in view of (a) and (b) the sequence $\{v_n\}_{n=1}^{\infty}$ is increasing.

Consequently, it follows from 1) that $\lim\limits_{n\to\infty} u_n/v_n = 0$; whereas from 3) that $\sum\limits_{n=1}^{\infty} \frac{1}{v_n} = \infty$. Next, the condition 2) means that the sequence $\{v_n\}_{n=1}^{\infty}$ contains arithmetical progressions of any length. ■

2. The second contribution

Let δ denote the family of all nonincreasing sequences $\{a_n\}_{n=1}^{\infty}$ of positive reals such that

$$\lim_{n\to\infty} a_n = 0 \qquad \text{and} \qquad \sum a_n = \infty.$$

Let $\{p_n\}_{n=1}^{\infty}$ denote the increasing sequence of all prime numbers. In [7] the following two results are proven:

Theorem 4. *For a given sequence $\{a_n\} \in \delta$ there exists a sequence $\{b_n\} \in \delta$ such that $\sum \min\{a_n, b_n\} < \infty$ iff $\liminf n\, a_n = 0$.*

Theorem 5. *Let $\{a_n\}, \{\alpha_n\} \in \delta$. If $\liminf n\, a_n = 0$ and $\liminf a_n \alpha_n^{-1} = 0$ then there exists a sequence $\{b_n\} \in \delta$ such that*

$$\sum \min\{\alpha_n, b_n\} = +\infty \qquad \textit{and} \qquad \sum \min\{a_n, b_n\} < \infty.$$

Moreover, we need also the following two known facts:

Theorem 6. ([5]) *We have*

$$\lim_{n\to\infty} \frac{p_n}{n \log n} = 1.$$

Theorem 7. *If $\{x_n\}_{n=1}^{\infty}$ is nondecreasing sequence of positive integers then*

$$\sum \frac{1}{x_n} = \infty \qquad \Longleftrightarrow \qquad \sum \frac{1}{n + x_n} = \infty.$$

Using all above theorems we get the following two corollaries, which are our second contribution to Erdős' problem.

Theorem 8. *By Theorems 4, 6 and 7 there exists an increasing sequence of positive integers $\{b_n\}_{n=1}^{\infty}$ such that*

$$\sum \frac{1}{b_n} = \infty \qquad \textit{and} \qquad \sum \min\left\{\frac{1}{b_n}, \frac{1}{p_n}\right\} < \infty.$$

We note, that by Ex. 1 of [7] this sequence $\{b_n\}_{n=1}^{\infty}$ can be obtained effectively!

Theorem 9. *Let $\alpha \in (0,1]$. By Theorems 5, 6 and 7 there exists an increasing sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers such that*

$$\sum \min\left\{\frac{1}{c_n}, \frac{1}{p_n}\right\} = \infty \qquad \textit{and} \qquad \sum \min\left\{\frac{1}{c_n}, \frac{1}{p_n\,(\log n)^{\alpha}}\right\} < \infty.$$

Remark 10. The last two theorems are very attractive in comparison with T. Tao and B. Green beautiful result (see [1, 6]) that there exist arbitrarily long arithmetic progressions of prime numbers.

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