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SUMS OF PRODUCTS OF GENERALIZED FIBONACCI AND LUCAS NUMBERS

Abstract. In this paper we obtain explicit formulae for sums of products of a fixed number of consecutive generalized Fibonacci and Lucas numbers. These formulae are related to the recent work of Belbachir and Bencherif. We eliminate all restrictions about the initial values and the form of the recurrence relation. In fact, we consider six different groups of three sums that include alternating sums and sums in which terms are multiplied by binomial coefficients and by natural numbers. The proofs are direct and use the formula for the sum of the geometric series.

1. Introduction

Let p and $q \neq 0$ be complex numbers. The generalized Fibonacci and Lucas sequences $\{U_n\} = \{U_n(p, q)\}$ and $\{V_n\} = \{V_n(p, q)\}$ are defined by

$$U_0 = 0, \quad U_1 = 1, \quad U_n = p\,U_{n-1} - q\,U_{n-2} \qquad (n \geq 2),$$

and

$$V_0 = 2, \quad V_1 = p, \quad V_n = p\,V_{n-1} - q\,V_{n-2} \qquad (n \geq 2).$$

The numbers U_n and V_n have been studied by Lucas [3] (see also [2]).

2. Sums of products of Fibonacci and Lucas numbers

We first want to find the formulae for the sums

$$\Psi_1 = \sum_{j=0}^{n} U_{a+bj}(p, q)\, U_{c+dj}(p, q),$$

$$\Psi_2 = \sum_{j=0}^{n} U_{a+bj}(p, q)\, V_{c+dj}(p, q),$$

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$$\Psi_3 = \sum_{j=0}^{n} V_{a+b\,j}(p,\,q)\,V_{c+d\,j}(p,\,q),$$

when $n \geq 0$, $a \geq 0$, $c \geq 0$, $b > 0$ and $d > 0$ are integers.

In [1] Belbachir and Bencherif have found explicit expressions for these sums (and for the related alternating sums) only in the special case when $q = \pm 1$ and $b = d = 2$. The main goal in this paper is to completely eliminate these assumptions and to treat some other similar sums. In the end, we consider altogether eighteen sums that are grouped by three in six classes. Once we discovered the formulae for the sums Ψ_1, Ψ_2 and Ψ_3 (the first class) and much simpler sums Ψ_4, Ψ_5 and Ψ_6 (the second class in which the terms are multiplied by binomial coefficients $\binom{n}{j}$), the remarkable feature is that in other classes of sums essentially the same formulae hold.

Since this paper contains more than two hundred claims we can only prove a few that can serve the reader as examples in checking the truth of the others. We thank the referee for useful comments that improved our results and their presentation.

Let α and β be the roots of $x^2 - p\,x + q = 0$. Then $\alpha = \frac{p+\Delta}{2}$ and $\beta = \frac{p-\Delta}{2}$, where $\Delta = \sqrt{p^2 - 4\,q}$. Moreover, $\alpha - \beta = \Delta$, $\alpha + \beta = p$, $\alpha\,\beta = q$ and the Binet forms of U_n and V_n are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad V_n = \alpha^n + \beta^n,$$

if $\alpha \neq \beta$, and

$$\widetilde{U}_n = n\,\alpha^{n-1}, \qquad \widetilde{V}_n = 2\,\alpha^n,$$

if $\alpha = \beta$.

Let $E = \alpha^{b+d}$, $F = \alpha^b\,\beta^d$, $G = \alpha^d\,\beta^b$ and $H = \beta^{b+d}$. Let $e = \alpha^{a+c}$, $f = \alpha^a\,\beta^c$, $g = \alpha^c\,\beta^a$ and $h = \beta^{a+c}$. When $E \neq 1$, for any integer $n \geq 0$, let $E_n = \frac{E^{n+1}-1}{E-1}$. We similarly define F_n, G_n and H_n. On the other hand, when $\alpha^b \neq \beta^b$, for any integer $n \geq 0$, let $b_n = \frac{\alpha^{b(n+1)} - \beta^{b(n+1)}}{\alpha^{b\,n}(\alpha^b - \beta^b)}$ and $b_n^* = \frac{\alpha^{b(n+1)} - \beta^{b(n+1)}}{\beta^{b\,n}(\alpha^b - \beta^b)}$. We similarly define d_n and d_n^*.

Theorem 1. (a) *When $\Delta = 0$ and $E = 1$, then*

$$\Psi_1 = \frac{e\,(n+1)\,\left[6\,a\,c + 3\,n(a\,d + b\,c) + n(2\,n+1)b\,d\right]}{6\,\alpha^2}.$$

(b) *When $\Delta = 0$ and $E \neq 1$, then $\Psi_1 = \frac{e[M\,a\,c + N(a\,d+b\,c) + P\,b\,d]}{\alpha^2(E-1)^3}$, with*

$$M = (E-1)^2(E^{n+1}-1), \;\; N = E(E-1)\left[n\,E^{n+1} - (n+1)\,E^n + 1\right],$$

$$P = E\left[n^2\,E^{n+2} - (2\,n^2 + 2\,n - 1)E^{n+1} + (n+1)^2\,E^n - E - 1\right].$$

Proof of (a). Since $\Delta = 0$ and $E = \alpha^{b+d} = 1$, we see that the product $\tilde{U}_{a+bj}(p,q)\,\tilde{U}_{c+dj}(p,q)$ is equal to

$$(a+bj)\,\alpha^{a+bj-1}\,(c+dj)\,\alpha^{c+dj-1} = \frac{e}{\alpha^2}\left[a\,c + j\,(a\,d + b\,c) + j^2\,b\,d\right].$$

From $\sum_{j=0}^{n} 1 = n+1$, $\sum_{j=0}^{n} j = \frac{n(n+1)}{2}$, and $\sum_{j=0}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$, it follows that Ψ_1 has the above value. ∎

Proof of (b). Since $\Delta = 0$, the product $\tilde{U}_{a+bj}(p,q)\,\tilde{U}_{c+dj}(p,q)$ is

$$(a+bj)\,\alpha^{a+bj-1}\,(c+dj)\,\alpha^{c+dj-1} = \frac{e\,E^j}{\alpha^2}\left[a\,c + j\,(a\,d + b\,c) + j^2\,b\,d\right].$$

From $\sum_{j=0}^{n} E^j = E_n$, $\sum_{j=0}^{n} j\,E^j = \frac{N}{(E-1)^3}$, and $\sum_{j=0}^{n} j^2\,E^j = \frac{P}{(E-1)^3}$, it follows that Ψ_1 has the above value. ∎

The following theorem covers for the sum Ψ_1 the cases when $\Delta \neq 0$. It uses Table 1 below that should be read as follows. The symbols ■ and □ in column E mean $E \neq 1$ and $E = 1$. In column b they mean $\alpha^b \neq \beta^b$ and $\alpha^b = \beta^b$. In columns F, G, H and d they have analogous meanings. The symbol ⊠ is a conditional □. How it works becomes clear from the following interpretation of the third subcase or row that should be read as follows: When $(\Delta \neq 0)$, $E = 1$ and $\alpha^b = \beta^b$, then $G = 1$ and $H = F$ and for $F \neq 1$ the product $\Delta^2\,\Psi_1$ is equal to $(n+1)\,(e-g) + F_n\,(h-f)$.

Theorem 2. *When $\Delta \neq 0$, then Table 1 gives the value of $\Delta^2\,\Psi_1$.*

Proof of row 1. When $\Delta \neq 0$, the product $U_{a+bj}(p,q)\,U_{c+dj}(p,q)$ is

$$\left(\frac{\alpha^{a+bj} - \beta^{a+bj}}{\Delta}\right)\cdot\left(\frac{\alpha^{c+dj} - \beta^{c+dj}}{\Delta}\right) = \frac{e\,E^j}{\Delta^2} - \frac{f\,F^j}{\Delta^2} - \frac{g\,G^j}{\Delta^2} + \frac{h\,H^j}{\Delta^2}.$$

From $\sum_{j=0}^{n} E^j = E_n$, we get $\Delta^2\,\Psi_1 = e\,E_n - f\,F_n - g\,G_n + h\,H_n$. ∎

Proof of row 2. When $\Delta \neq 0$ and $E = \alpha^{b+d} = 1$, we get

$$U_{a+bj}(p,q)\,U_{c+dj}(p,q) = \frac{e}{\Delta^2} - \frac{fF^j}{\Delta^2} - \frac{g}{\Delta^2}\left(\frac{\beta^b}{\alpha^b}\right)^j + \frac{hH^j}{\Delta^2}.$$

From $\sum_{j=0}^{n} 1 = (n+1)$, $\sum_{j=0}^{n} F^j = F_n$ and $\sum_{j=0}^{n}\left(\frac{\beta^b}{\alpha^b}\right)^j = b_n$ (for $\alpha^b \neq \beta^b$), it follows that $\Delta^2\,\Psi_1 = e\,(n+1) - f\,F_n - g\,b_n + h\,H_n$. ∎

Proof of row 3. When $\Delta \neq 0$, $E = \alpha^{b+d} = 1$ and $\alpha^b = \beta^b$, then

$$G = \beta^b\,\alpha^d = \alpha^b\,\alpha^d = E = 1$$

and $H = \beta^b\,\beta^d = \alpha^b\,\beta^d = F$. Hence,

$$U_{a+bj}(p,q)\,U_{c+dj}(p,q) = \frac{e-g}{\Delta^2} + \frac{(h-f)F^j}{\Delta^2}.$$

	E	F	G	H	b	d	$\Delta^2 \Psi_1$
1	$\blacksquare$	$\blacksquare$	$\blacksquare$	$\blacksquare$			$E_n e - F_n f - G_n g + H_n h$
2	$\square$	$\blacksquare$		$\blacksquare$	$\blacksquare$		$(n+1) e - F_n f - b_n g + H_n h$
3	$\square$	$\blacksquare$	$\boxtimes$	F	$\square$		$(n+1)(e-g) + F_n(h-f)$
4	$\square$		$\blacksquare$	$\blacksquare$		$\blacksquare$	$(n+1) e - d_n f - G_n g + H_n h$
5	$\square$	$\boxtimes$	$\blacksquare$	G		$\square$	$(n+1)(e-f) + G_n(h-g)$
6	$\square$	$\square$				$\boxtimes$	(see 5)
7	$\square$		$\square$		$\boxtimes$		(see 3)
8	$\square$		$\square$			$\blacksquare$	$(n+1)(e-g) + d_n (h-f)$
9	$\square$			$\square$		$\blacksquare$	$(n+1)(e+h) - d_n f - d_n^* g$
10	$\blacksquare$	$\square$	$\blacksquare$		$\blacksquare$		$E_n e - (n+1) f - G_n g + b_n h$
11	$\blacksquare$	$\square$	E	$\boxtimes$	$\square$		$E_n (e-g) + (n+1)(h-f)$
12		$\square$	$\blacksquare$	$\blacksquare$		$\blacksquare$	$d_n^* e - (n+1) f - G_n g + H_n h$
13	$\boxtimes$	$\square$	$\blacksquare$	G		$\square$	$(n+1)(e-f) + G_n (h-g)$
14		$\square$	$\square$			$\blacksquare$	$d_n^* e - (n+1)(f+g) + d_n h$
15		$\square$		$\square$	$\boxtimes$		(see 11)
16		$\square$		$\square$		$\blacksquare$	$d_n^* (e-g) + (n+1)(h-f)$
17		$\blacksquare$	$\square$	$\blacksquare$	$\blacksquare$		$b_n^* e - F_n f - (n+1) g + H_n h$
18	$\boxtimes$	$\blacksquare$	$\square$	F	$\square$		$(n+1)(e-g) + F_n (h-f)$
19	$\blacksquare$	$\blacksquare$	$\square$			$\blacksquare$	$E_n e - F_n f - (n+1) g + d_n h$
20	$\blacksquare$	E	$\square$	$\boxtimes$		$\square$	$E_n(e-f) + (n+1)(h-g)$
21			$\square$	$\square$	$\blacksquare$	$\boxtimes$	$b_n^* (e-f) + (n+1)(h-g)$
22	$\blacksquare$		$\blacksquare$	$\square$	$\blacksquare$		$E_n e - b_n^* f - G_n g + (n+1) h$
23	$\blacksquare$	$\boxtimes$	E	$\square$	$\square$		$E_n (e-g) + (n+1)(h-f)$
24	$\blacksquare$	$\blacksquare$		$\square$		$\blacksquare$	$E_n e - F_n f - d_n^* g + (n+1) h$
25	$\blacksquare$	E	$\boxtimes$	$\square$		$\square$	$E_n (e-f) + (n+1)(h-g)$

Table 1. The product $\Delta^2 \Psi_1$ when $\Delta \neq 0$.

From $\sum_{j=0}^{n} 1 = (n+1)$ and $\sum_{j=0}^{n} F^j = F_n$ (for $F \neq 1$, of course), it follows that the product $\Delta^2 \Psi_1$ is equal to $(e-g)(n+1) + (h-f) F_n$. $\blacksquare$

The missing case in the Table 1 after the third row is clearly when $E = 1$, $\alpha^b = \beta^b$ and $F = 1$. However, it is easy to see that this situation can not happen since $\Delta \neq 0$, $b > 0$ and $d > 0$. The similar statement holds for all other subcases missing from the Table 1.

Notice that $\alpha^n = \frac{V_n + \Delta U_n}{2}$ and $\beta^n = \frac{V_n - \Delta U_n}{2}$ for $\Delta \neq 0$ and $\alpha^n = \beta^n = \frac{\widetilde{U}_{n+1}}{n+1} = \frac{\widetilde{V}_n}{2}$ for $\Delta = 0$. Hence, it is clear that each of the above expressions for the sum Ψ_1 could be transformed into an expression in Lucas numbers U_n and V_n (or $\widetilde{U}_n$ and $\widetilde{V}_n$). In most cases these formulae are more complicated then the ones given above. This applies also to other sums that we consider in this paper.

	$\Delta\,\Psi_2$
1	$E_n\,e + F_n\,f - G_n\,g - H_n\,h$
2	$(n+1)\,e + F_n\,f - b_n\,g - H_n\,h$
3	$(n+1)(e-g) + F_n(f-h)$
4	$(n+1)\,e + d_n\,f - G_n\,g - H_n\,h$
5	$(n+1)(e+f) - G_n(g+h)$
8	$(n+1)(e-g) + d_n\,(f-h)$
9	$(n+1)(e-h) + d_n\,f - d_n^*\,g$
10	$E_n\,e + (n+1)\,f - G_n\,g - b_n\,h$
11	$E_n\,(e-g) + (n+1)(f-h)$
12	$d_n^*\,e + (n+1)\,f - G_n\,g - H_n\,h$
13	$(n+1)(e+f) - G_n\,(g+h)$
14	$d_n^*\,e + (n+1)(f-g) - d_n\,h$
16	$d_n^*(e-g) + (n+1)(f-h)$
17	$b_n^*\,e + F_n\,f - (n+1)\,g - H_n\,h$
18	$(n+1)\,(e-g) + F_n\,(f-h)$
19	$E_n\,e + F_n\,f - (n+1)\,g - d_n\,h$
20	$E_n(e+f) - (n+1)(g+h)$
21	$b_n^*\,(e+f) - (n+1)(g+h)$
22	$E_n\,e + b_n^*\,f - G_n\,g - (n+1)\,h$
23	$E_n\,(e-g) + (n+1)(f-h)$
24	$E_n\,e + F_n\,f - d_n^*\,g - (n+1)\,h$
25	$E_n\,(e+f) - (n+1)(g+h)$

Table 2. The product $\Delta\,\Psi_2$ when $\Delta \neq 0$.

Next we do the same for the sum Ψ_2. Of course, the first is the simpler case when $\Delta = 0$.

Theorem 3. (a) *When $\Delta = 0$ and $E = 1$, then*

$$\Psi_2 = \frac{e\,(n+1)\,[\,2\,a + n\,b\,]}{\alpha}.$$

(b) *When $\Delta = 0$ and $E \neq 1$, then*

$$\Psi_2 = \frac{2\,e}{\alpha}\left[E_n\,a + \frac{E\left(n\,E^{n+1} - (n+1)E^n + 1\right)\,b}{(E-1)^2}\right].$$

The following theorem is rather similar to Theorem 2 and covers for the sum Ψ_2 the cases when $\Delta \neq 0$. Its Table 2 above has the same columns 2–7 as in the Table 1 so that we shall give only the first and the last column with rows 6, 7 and 15 omitted.

Theorem 4. *When $\Delta \neq 0$, then Table 2 gives the value of $\Delta\,\Psi_2$.*

Somewhat simpler is the third sum Ψ_3 that we treat now in much the same way. We begin with two cases when $\Delta = 0$.

Theorem 5. (a) *When $\Delta = 0$ and $E = 1$, then $\Psi_3 = 4\,(n+1)\,e$.*
(b) *When $\Delta = 0$ and $E \neq 1$, then $\Psi_3 = 4\,e\,E_n$.*

The following theorem considers for the sum Ψ_3 the cases when $\Delta \neq 0$. Its Table 3 below is again reduced to the first and the last column because the other columns and the missing rows agree with those of Table 1.

Theorem 6. *When $\Delta \neq 0$, then Table 3 gives the value of Ψ_3.*

3. Sums with binomial coefficients

In this section we consider the sums

$$\Psi_4 = \sum_{j=0}^{n} \binom{n}{j} U_{a+b\,j}(p,\,q)\,U_{c+d\,j}(p,\,q),$$

$$\Psi_5 = \sum_{j=0}^{n} \binom{n}{j} U_{a+b\,j}(p,\,q)\,V_{c+d\,j}(p,\,q),$$

$$\Psi_6 = \sum_{j=0}^{n} \binom{n}{j} V_{a+b\,j}(p,\,q)\,V_{c+d\,j}(p,\,q),$$

when $n \geq 0$, $a \geq 0$, $c \geq 0$, $b > 0$ and $d > 0$ are integers.

	Ψ_3
1	$E_n\, e + F_n\, f + G_n\, g + H_n\, h$
2	$(n+1)\, e + F_n\, f + b_n\, g + H_n\, h$
3	$(n+1)(e+g) + F_n(f+h)$
4	$(n+1)\, e + d_n\, f + G_n\, g + H_n\, h$
5	$(n+1)(e+f) + G_n(g+h)$
8	$(n+1)(e+g) + d_n\,(f+h)$
9	$(n+1)(e+h) + d_n\, f + d_n^*\, g$
10	$E_n\, e + (n+1)\, f + G_n\, g + b_n\, h$
11	$E_n\,(e+g) + (n+1)(f+h)$
12	$d_n^*\, e + (n+1)\, f + G_n\, g + H_n\, h$
13	$(n+1)(e+f) + G_n\,(g+h)$
14	$d_n^*\, e + (n+1)(f+g) + d_n\, h$
16	$d_n^*(e+g) + (n+1)(f+h)$
17	$b_n^*\, e + F_n\, f + (n+1)\, g + H_n\, h$
18	$(n+1)(e+g) + F_n\,(f+h)$
19	$E_n\, e + F_n\, f + (n+1)\, g + d_n\, h$
20	$E_n(e+f) + (n+1)(g+h)$
21	$b_n^*(e+f) + (n+1)(g+h)$
22	$E_n\, e + b_n^*\, f + G_n\, g + (n+1)\, h$
23	$E_n\,(e+g) + (n+1)(f+h)$
24	$E_n\, e + F_n\, f + d_n^*\, g + (n+1)\, h$
25	$E_n\,(e+f) + (n+1)(g+h)$

Table 3. The sum Ψ_3 when $\Delta \neq 0$.

Theorem 7. (a) *When $\Delta = 0$, then*

$$\Psi_4 = \begin{cases} \frac{e\,a\,c}{\alpha^2}, & \text{if } n = 0, \\ \frac{e[(E+1)a\,c + E(a\,d + b\,c + b\,d)]}{\alpha^2}, & \text{if } n = 1, \\ \frac{e(E+1)^{n-2}\left[(E+1)^2\,a\,c + n\,E(E+1)(a\,d+b\,c) + n\,E(n\,E+1)\,b\,d\right]}{\alpha^2}, & \text{if } n \geq 2, \end{cases}$$

$$\Psi_5 = \begin{cases} \frac{2\,e\,a}{\alpha}, & \text{if } n = 0, \\ \frac{2\,e\,(E+1)^{n-1}[(E+1)a + n\,E\,b]}{\alpha}, & \text{if } n \geq 1. \end{cases}$$

(b) *When* $\Delta \neq 0$, *then*

$$\Psi_4 = \frac{(E+1)^n e - (F+1)^n f - (G+1)^n g + (H+1)^n h}{\Delta^2},$$

$$\Psi_5 = \frac{(E+1)^n e + (F+1)^n f - (G+1)^n g - (H+1)^n h}{\Delta}.$$

(c) The sum Ψ_6 *is equal to*

$$(E+1)^n e + (F+1)^n f + (G+1)^n g + (H+1)^n h.$$

Proof of (c). Since

$$\binom{n}{j} V_{a+bj}(p,\, q)\, V_{c+dj}(p,\, q) = \binom{n}{j} \left(e\, E^j + f\, F^j + g\, G^j + h\, H^j\right),$$

from $\sum_{j=0}^{n} \binom{n}{j} E^j = (E+1)^n$, it follows that Ψ_6 indeed has the above value. ∎

4. The improved alternating sums

In this section we consider the sums obtained from the sums Ψ_1–Ψ_6 by multiplication of their terms with the powers of a fixed complex number k. When $k = -1$ we obtain the familiar alternating sums. More precisely, we study the sums

$$\Psi_7 = \sum_{j=0}^{n} k^j\, U_{a+bj}(p,\, q)\, U_{c+dj}(p,\, q),$$

$$\Psi_8 = \sum_{j=0}^{n} k^j\, U_{a+bj}(p,\, q)\, V_{c+dj}(p,\, q),$$

$$\Psi_9 = \sum_{j=0}^{n} k^j\, V_{a+bj}(p,\, q)\, V_{c+dj}(p,\, q),$$

$$\Psi_{10} = \sum_{j=0}^{n} k^j \binom{n}{j} U_{a+bj}(p,\, q)\, U_{c+dj}(p,\, q),$$

$$\Psi_{11} = \sum_{j=0}^{n} k^j \binom{n}{j} U_{a+bj}(p,\, q)\, V_{c+dj}(p,\, q),$$

$$\Psi_{12} = \sum_{j=0}^{n} k^j \binom{n}{j} V_{a+bj}(p,\, q)\, V_{c+dj}(p,\, q),$$

when $n \geq 0$, $a \geq 0$, $c \geq 0$, $b > 0$ and $d > 0$ are integers.

Let $E = k\,\alpha^{b+d}$, $F = k\,\alpha^b\,\beta^d$, $G = k\,\alpha^d\,\beta^b$ and $H = k\,\beta^{b+d}$. When $E \neq 1$, for any integer $n \geq 0$, let $E_n = \frac{E^{n+1}-1}{E-1}$. We similarly define F_n, G_n and H_n.

In this section we can assume that $k \neq 1$ and $k \neq 0$ because the case when $k = 1$ was treated earlier while for $k = 0$ all sums are equal to zero.

With this new meaning of the symbols E, F, G and H we have the following result.

THEOREM 8. (a) *The values given in Theorems* 1 *and* 2, 3 *and* 4, *and* 5 *and* 6 *express the sums* Ψ_7, Ψ_8 *and* Ψ_9, *respectively. In particular, when* $\Delta \neq 0$, *then Tables* 1, 2 *and* 3 *give the values of* $\Delta^2 \Psi_7$, $\Delta \Psi_8$ *and* Ψ_9.

(b) *The values given in Theorem* 7 *for the sums* Ψ_4, Ψ_5 *and* Ψ_6 *express also the sums* Ψ_{10}, Ψ_{11} *and* Ψ_{12}.

Proof of (b) for Ψ_{12}. Since

$$k^j \binom{n}{j} V_{a+bj}(p, q)\, V_{c+dj}(p, q) = \binom{n}{j} \left(e\, E^j + f\, F^j + g\, G^j + h\, H^j \right),$$

from $\sum_{j=0}^{n} \binom{n}{j} E^j = (E+1)^n$, it follows that Ψ_{12} indeed has the same expression as the sum Ψ_6. ∎

5. Terms multiplied by natural numbers

In this section we study the sums

$$\Psi_{13} = \sum_{j=0}^{n} k^j\,(j+1)\, U_{a+bj}(p, q)\, U_{c+dj}(p, q),$$

$$\Psi_{14} = \sum_{j=0}^{n} k^j\,(j+1)\, U_{a+bj}(p, q)\, V_{c+dj}(p, q),$$

$$\Psi_{15} = \sum_{j=0}^{n} k^j\,(j+1)\, V_{a+bj}(p, q)\, V_{c+dj}(p, q),$$

$$\Psi_{16} = \sum_{j=0}^{n} k^j\,(j+1) \binom{n}{j} U_{a+bj}(p, q)\, U_{c+dj}(p, q),$$

$$\Psi_{17} = \sum_{j=0}^{n} k^j\,(j+1) \binom{n}{j} U_{a+bj}(p, q)\, V_{c+dj}(p, q),$$

$$\Psi_{18} = \sum_{j=0}^{n} k^j\,(j+1) \binom{n}{j} V_{a+bj}(p, q)\, V_{c+dj}(p, q),$$

when $n \geq 0$, $a \geq 0$, $c \geq 0$, $b > 0$ and $d > 0$ are integers.

Let $E = k\,\alpha^{b+d}$, $F = k\,\alpha^b\,\beta^d$, $G = k\,\alpha^d\,\beta^b$ and $H = k\,\beta^{b+d}$. Let $e = \alpha^{a+c}$, $f = \alpha^a\,\beta^c$, $g = \alpha^c\,\beta^a$ and $h = \beta^{a+c}$. When $E \neq 1$, for any integer $n \geq 0$, let

$E_n = \frac{(n+1)E^{n+2} - (n+2)E^{n+1} + 1}{(E-1)^2}$. We similarly define F_n, G_n and H_n. On the other hand, when $\alpha^b \neq \beta^b$, for any integer $n \geq 0$, let

$$b_n = \frac{\alpha^{b(n+2)} + (n+1)\beta^{b(n+2)} - (n+2)\alpha^b\,\beta^{n+1}}{\alpha^{b\,n}(\alpha^b - \beta^b)^2}$$

and

$$b_n^* = \frac{\beta^{b(n+2)} + (n+1)\alpha^{b(n+2)} - (n+2)\beta^b\,\alpha^{n+1}}{\beta^{b\,n}(\alpha^b - \beta^b)^2}.$$

We similarly define d_n and d_n^*.

Theorem 9. (a) *When $\Delta = 0$ and $E = 1$, then*

$$\Psi_{13} = \frac{e\,(n+1)(n+2)\,[6\,a\,c + 4\,n\,(a\,d + b\,c) + n\,(3\,n+1)\,b\,d]}{12\,\alpha^2},$$

$$\Psi_{14} = \frac{2\,e\,(n+1)(n+2)\,[3\,a + 2\,n\,b]}{6\,\alpha},$$

and

$$\Psi_{15} = 2\,e\,(n+1)(n+2).$$

(b) *When $\Delta = 0$ and $E \neq 1$, then*

$$\Psi_{13} = \frac{e}{\alpha^2}\left[E_n\,a\,c + \frac{E}{(E-1)^3}\,M\,(a\,d + b\,c) + \frac{E}{(E-1)^4}\,N\,b\,d\right],$$

where M and N are polynomials $n\,(n+1)\,E^{n+2} - 2\,n\,(n+2)\,E^{n+1} + (n+1)(n+2)\,E^n - 2$ *and* $n^2\,(n+1)\,E^{n+3} - n\,(3\,n^2 + 6\,n - 1)\,E^{n+2} + (n+2)(3\,n^2 + 3\,n - 2)\,E^{n+1} - (n+2)(n+1)^2\,E^n + 4\,E + 2$,

$$\Psi_{14} = \frac{2\,e}{\alpha}\left[E_n\,a + \frac{E}{(E-1)^3}\,M\,b\right],$$

and

$$\Psi_{15} = 4\,e\,E_n.$$

Proof of (b) for Ψ_{14}. Since $\Delta = 0$, the product

$$k^j\,(j+1)\,\tilde{U}_{a+b\,j}(p,\,q)\,\tilde{V}_{c+d\,j}(p,\,q)$$

is

$$2\,k^j\,(j+1)(a+b\,j)\,\alpha^{a+b\,j-1}\,\alpha^{c+d\,j} = \frac{2\,e\,E^j}{\alpha}\,[a\,(j+1) + j\,(j+1)\,b]\,.$$

From $\sum_{j=0}^{n}(j+1)\,E^j = E_n$ and $\sum_{j=0}^{n} j\,(j+1)\,E^j = \frac{E\,M}{(E-1)^3}$, it follows that Ψ_{14} has the above value. ∎

Theorem 10. *When $\Delta \neq 0$, then Tables 1, 2 and 3 give the values of $\Delta^2\,\Psi_{13}$, $\Delta\,\Psi_{14}$ and Ψ_{15}.*

Proof of row 1 in Table 1 for Ψ_{13}. When $\Delta \neq 0$, we have

$$k^j\,(j+1)\,U_{a+b\,j}(p,\,q)\,U_{c+d\,j}(p,\,q) = k^j\,(j+1)\left(\frac{\alpha^{a+b\,j}-\beta^{a+b\,j}}{\Delta}\right)\cdot\left(\frac{\alpha^{c+d\,j}-\beta^{c+d\,j}}{\Delta}\right) = (j+1)\left(\frac{e\,E^j}{\Delta^2}-\frac{f\,F^j}{\Delta^2}-\frac{g\,G^j}{\Delta^2}+\frac{h\,H^j}{\Delta^2}\right).$$

From $\sum_{j=0}^{n}(j+1)E^j = E_n$, we get $\Delta^2\,\Psi_{13} = e\,E_n - f F_n - g\,G_n + hH_n$. ∎

For any integer $n \geq 0$, let $E_n^* = (n+1)\,E+1$, $E_n^{**} = E_n^*\,(E+1)^{n-1}$. We define F_n^*, G_n^*, H_n^*, F_n^{**}, G_n^{**} and H_n^{**} similarly.

THEOREM 11. (a) *When $\Delta = 0$, then*

$$\Psi_{16} = \begin{cases} \frac{e\,a\,c}{\alpha^2}, & \text{if } n=0,\\ \frac{e[(2\,E+1)a\,c+2\,E(a\,d+b\,c+b\,d)]}{\alpha^2}, & \text{if } n=1,\\ \frac{e[(E+1)(3\,E+1)a\,c+2\,E\,(3\,E+2)(a\,d+b\,c)+4\,E(3\,E+1)\,b\,d)]}{\alpha^2}, & \text{if } n=2,\\ \frac{e(E+1)^{n-3}\left[\,E_n^*\,(E+1)^2\,a\,c+n\,E\,(E+1)(E_n^*+1)\,(a\,d+b\,c)+R\,b\,d\right]}{\alpha^2}, & \text{if } n\geq 3, \end{cases}$$

where $R = n\,E(n(n+1)E^2+4\,n\,E+2)$,

$$\Psi_{17} = \begin{cases} \frac{2\,e\,a}{\alpha}, & \text{if } n=0,\\ \frac{2\,e\left[(2\,E+1)\,a+2\,E\,b\right]}{\alpha}, & \text{if } n=1,\\ \frac{2\,e\,(E+1)^{n-2}\left[(E+1)\,E_n^*\,a+n\,E\,(E_n^*+1)\,b\right]}{\alpha}, & \text{if } n\geq 2. \end{cases}$$

(b) *When $\Delta \neq 0$, then*

$$\Psi_{16} = \frac{E_n^{**}\,e - F_n^{**}\,f - G_n^{**}\,g + H_n^{**}\,h}{\Delta^2},$$

$$\Psi_{17} = \frac{E_n^{**}\,e + F_n^{**}\,f - G_n^{**}\,g - H_n^{**}\,h}{\Delta}.$$

(c) *The sum Ψ_{18} is equal to $E_n^{**}\,e + F_n^{**}\,f + G_n^{**}\,g + H_n^{**}\,h$.*

Proof of (c). Since

$$k^j\,(j+1)\binom{n}{j}\,V_{a+b\,j}(p,\,q)\,V_{c+d\,j}(p,\,q) = (j+1)\binom{n}{j}\,\left(e\,E^j + f\,F^j + g\,G^j + h\,H^j\right),$$

from $\sum_{j=0}^{n}(j+1)\binom{n}{j}E^j = E_n^{**}$, it follows that Ψ_{18} indeed has the above value. ∎

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