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QUANTIFIERS ON LATTICES WITH AN ANTITONE INVOLUTION

Abstract. Quantifiers on lattices with an antitone involution are considered and it is proved that the poset of existential quantifiers is antiisomorphic to the poset of relatively complete sublattices.

So-called monadic algebras are often used as an algebraic axiomatization of predicate calculus, see e. g. [1] – [4]. In fact, every one of these predicate logics is a lattice with respect to the induced order. Monadic algebras were investigated in connection with lattices representing many-valued predicate calculus by Rutledge ([4]), MV-algebras (which are the algebraic counterpart of many-valued logics) by Di Nola and Grigolia ([2]), basic algebras (which are generalizations of MV-algebras) by Chajda and Kolařík ([1]) and residuated lattices by Rachůnek and Švrček ([3]). Existential quantifiers were already characterized by means of algebraic methods (using closure operators and relatively complete subalgebras) for MV-algebras, pseudo MV-algebras and a number of other algebras used in the axiomatization of both classical and non-classical (in particular many-valued) logics (including basic algebras). Since all these algebras are bounded lattices the natural question arises if a general and unified approach can be developed. The aim of the present paper is to show that quantifiers may be characterized for all possible logics having a bounded lattice as their underlying structure.

Definition 1. A *lattice with an antitone involution* is an algebra $(L, \vee, \wedge, \neg)$ of type $(2, 2, 1)$ such that $(L, \vee, \wedge)$ is a lattice and

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$$\neg(x \vee y) = \neg x \wedge \neg y$$
$$\neg(x \wedge y) = \neg x \vee \neg y$$
$$\neg\neg x = x$$

holds for all $x, y \in L$. ($\neg$ is assumed to bind stronger than $\vee$ and $\wedge$.)

In the following let $\mathcal{L} = (L, \vee, \wedge, \neg)$ denote an arbitrary, non-empty but fixed lattice with an antitone involution.

The concept of an existential quantifier on $\mathcal{L}$ can be introduced in the following way:

Definition 2. We call $\exists : L \to L$ an *existential quantifier* on $\mathcal{L}$ if for all $x, y \in L$

(i) $x \leq \exists x$,
(ii) $\exists(x \vee y) = \exists x \vee \exists y$ and
(iii) $\exists(\neg\exists x) = \neg\exists x$.

Let $\mathbf{E}(\mathcal{L})$ denote the set of all existential quantifiers on $\mathcal{L}$. For all $\exists_1, \exists_2 \in \mathbf{E}(\mathcal{L})$ we define $\exists_1 \leq \exists_2$ if and only if $\exists_1 x \leq \exists_2 x$ for all $x \in L$.

Remark 3. $(\mathbf{E}(\mathcal{L}), \leq)$ is a poset.

Remark 4. The first two axioms are motivated by the tautologies

(i') $P(x) \to \exists x P(x)$
(ii') $\exists x(P(x) \vee Q(x)) \leftrightarrow (\exists x P(x) \vee \exists x Q(x))$

of first order predicate logic. The third axiom is common in classical predicate calculus and hence we will use it here.

Remark 5. The axioms (i) – (iii) are independent. This can be seen as follows: The constant function with value 0 on $(\{-1, 0, 1\}, \leq)$ satisfies all axioms except (i). The function on $(\{-3, -2, -1, \ldots, 3\}, \leq)$ mapping -2 to 3, 0 to 1 and 2 to 3 and fixing the other elements satisfies all axioms except (ii). Finally, the constant function with value 1 on $(\{-1, 1\}, \leq)$ satisfies all axioms except (iii). In all of these examples the antitone involution is given by the mapping $x \mapsto -x$.

Remark 6. A mapping $\exists : L \to L$ satisfying (i), (ii) and $\exists\exists x = \exists x$ need not be an existential quantifier as can be seen by the following example: Let $\mathcal{L} = (\{0, a, b, 1\}, \vee, \wedge, \neg)$ be the four-element Boolean algebra (i. e. $\neg a = b$

and $\neg 0 = 1$) and define $\exists : L \to L$ by $\exists 0 := 0$, $\exists a := a$ and $\exists x := 1$ otherwise. Then $\exists$ satisfies (i), (ii) and $\exists\exists x = \exists x$, but $\exists \notin \mathbf{E}(\mathcal{L})$ since $\exists(\neg\exists a) = \exists(\neg a) = \exists b = 1 \neq b = \neg a = \neg\exists a$.

Remark 7. An existential quantifier need not be a lattice endomorphism as can be seen by the following example: Let $\mathcal{L} = (\{0, a, b, 1\}, \vee, \wedge, \neg)$ be the four-element Boolean algebra (i. e. $\neg a = b$ and $\neg 0 = 1$) and define $\exists : L \to L$ by $\exists 0 := 0$ and $\exists x := 1$ otherwise. Then $\exists \in \mathbf{E}(\mathcal{L})$ but $\exists$ is not a lattice endomorphism since $\exists(a \wedge b) = \exists 0 = 0 \neq 1 = 1 \wedge 1 = \exists a \wedge \exists b$.

Lemma 8. *Let $\exists \in \mathbf{E}(\mathcal{L})$. Then for all $x, y \in L$ it holds:*

(i) $x \leq \exists y$ *if and only if* $\exists x \leq \exists y$.

(ii) $x \leq y$ *implies* $\exists x \leq \exists y$.

(iii) $\neg\exists x \leq \exists(\neg x)$

(iv) $\exists\exists x = \exists x$

(v) $\exists(x \vee \exists y) = \exists x \vee \exists y$

(vi) *If $\mathcal{L}$ is bounded then* $\exists 0 = 0$ *and* $\exists 1 = 1$.

Proof.

(iv): $\exists\exists x = \exists(\neg\neg\exists x) = \exists(\neg\exists(\neg\exists x)) = \neg\exists(\neg\exists x) = \neg\neg\exists x = \exists x$

(ii): If $x \leq y$ then $\exists x \leq \exists x \vee \exists y = \exists(x \vee y) = \exists y$.

(i): If $x \leq \exists y$ then $\exists x \leq \exists\exists y = \exists y$ and if $\exists x \leq \exists y$ then $x \leq \exists x \leq \exists y$.

(iii): $\neg\exists x \leq \neg x \leq \exists(\neg x)$

(v) : $\exists(x \vee \exists y) \leq \exists(\exists x \vee \exists y) = \exists\exists(x \vee y) = \exists(x \vee y) = \exists x \vee \exists y = \exists(x \vee y) \leq \exists(x \vee \exists y)$

(vi): If $\mathcal{L}$ is bounded then $1 \leq \exists 1$ and hence $\exists 1 = 1$ and $\exists 0 = \exists(\neg 1) = \exists(\neg\exists 1) = \neg\exists 1 = \neg 1 = 0$. ∎

In classical predicate calculus the universal quantifier can be introduced by means of the existential one via the rule

$$\forall := \neg\exists\neg.$$

Proceeding in the same way we also obtain a universal quantifier. The following corollary shows that this universal quantifier has many natural properties usually accepted in classical predicate calculus.

Corollary 9. *Let $\exists \in \mathbf{E}(\mathcal{L})$ and put $\forall := \neg\exists\neg$. Then for all $x, y \in L$ it holds:*

(i) $\forall x \leq x$

(ii) $\forall(x \wedge y) = \forall x \wedge \forall y$

(iii) $\forall(\neg\forall x) = \neg\forall x$

(iv) $x \geq \forall y$ *if and only if* $\forall x \geq \forall y$

(v) $x \leq y$ *implies* $\forall x \leq \forall y$.

(vi) $\neg\forall x \geq \forall(\neg x)$

(vii) $\forall\forall x = \forall x$

(viii) $\forall(x \wedge \forall y) = \forall x \wedge \forall y$

(ix) *If* $\mathcal{L}$ *is bounded then* $\forall 0 = 0$ *and* $\forall 1 = 1$.

Definition 10. A sublattice $(M, \vee, \wedge)$ of $\mathcal{L}$ is called *relatively complete* if for every $x \in L$ the poset $(\{y \in M \mid y \geq x\}, \leq)$ has a smallest element. Let $\mathbf{R}(\mathcal{L})$ denote the set of all relatively complete sublattices of $\mathcal{L}$. For every $(M, \vee, \wedge) \in \mathbf{R}(\mathcal{L})$ let $\exists_M$ denote the mapping from L to L assigning to each element x of L the smallest element of $(\{y \in M \mid y \geq x\}, \leq)$.

Remark 11. The empty sublattice of $\mathcal{L}$ is not relatively complete. $(\mathbf{R}(\mathcal{L}), \subseteq)$ is a poset with greatest element L. If $\mathcal{L}$ is bounded then $\{0, 1\}$ is the smallest element of $\mathbf{R}(\mathcal{L})$.

Definition 12. A poset $(P, \leq)$ is said to satisfy the *descending chain condition* (DDC, for short) if in $(P, \leq)$ there does not exist an infinite strictly descending sequence $a_1 > a_2 > a_3 > \ldots$ of elements. Let Sub$\mathcal{L}$ denote the set of all sublattices of $\mathcal{L}$.

Proposition 13. *Every sublattice* $(M, \vee, \wedge)$ *of* $\mathcal{L}$ *satisfying the DCC and having the property that to every* $x \in L$ *there exists an element* y *of* M *with* $y \geq x$ *is relatively complete.*

Proof. Let $\mathcal{M} = (M, \vee, \wedge)$ be a sublattice of $\mathcal{L}$ satisfying the DCC and having the property that to every $x \in L$ there exists an element y of M with $y \geq x$. Assume $\mathcal{M} \notin \mathbf{R}(\mathcal{L})$. Then there exists an $a \in L$ such that $A := \{x \in M \mid x \geq a\}$ does not have a smallest element. By our assumption, $A \neq \emptyset$. Let a_1 be an element of A. Since a_1 is not the smallest element of A there exists an element b_1 of A with $b_1 \not\geq a_1$. Put $a_2 := a_1 \wedge b_1$. Since $a_1, b_1 \in A$ and $\mathcal{M} \in$ Sub$\mathcal{L}$, $a_2 \in A$ and because of $b_1 \not\geq a_1$ we have $a_2 < a_1$. Since a_2 is not the smallest element of A there exists an element b_2 of A with $b_2 \not\geq a_2$. Put $a_3 := a_2 \wedge b_2$. Since $a_2, b_2 \in A$ and $\mathcal{M} \in$ Sub$\mathcal{L}$, $a_3 \in A$ and because of $b_2 \not\geq a_2$, $a_3 < a_2$. Together we have $a_1, a_2, a_3 \in M$ and

$a_1 > a_2 > a_3$. Going on in this way we would obtain an infinite descending chain in $(M, \leq)$ contradicting the DCC. Hence $\mathcal{M} \in \mathbf{R}(\mathcal{L})$. ∎

Definition 14. For bounded $\mathcal{L}$ let $\mathrm{Sub}_{01}\mathcal{L}$ denote the set of all sublattices of $\mathcal{L}$ containing 0 and 1.

Corollary 15. *If $\mathcal{L}$ is bounded and satisfies the DCC then $\mathbf{R}(\mathcal{L}) = \mathrm{Sub}_{01}\mathcal{L}$.*

Theorem 16. *The mappings $\exists \mapsto \exists(L)$ and $M \mapsto \exists_M$ are mutually inverse antiisomorphisms between $(\mathbf{E}(\mathcal{L}), \leq)$ and $(\mathbf{R}(\mathcal{L}), \subseteq)$.*

Proof. Let $a, b \in L$. First assume $\exists \in \mathbf{E}(\mathcal{L})$. Then $\exists a \vee \exists b = \exists(a \vee b) \in \exists(L)$ and $\neg\exists a = \exists(\neg\exists a) \in \exists(L)$ and hence $\exists(L)$ is closed with respect to $\vee$ and $\neg$. According to the de Morgan laws, $\exists(L)$ is also closed with respect to $\wedge$ and hence a sublattice of $\mathcal{L}$. $\exists a$ is the smallest element of $\{x \in \exists(L) \mid x \geq a\}$ since $\exists b \geq a$ implies $\exists b = \exists\exists b \geq \exists a$. Therefore $\exists(L) \in \mathbf{R}(\mathcal{L})$. Moreover, since both $\exists_{\exists(L)}a$ and $\exists a$ are the smallest element of $\{x \in \exists(L) \mid x \geq a\}$, $\exists_{\exists(L)}a = \exists a$, i. e. $\exists_{\exists(L)} = \exists$.
Conversely, let $M \in \mathbf{R}(\mathcal{L})$. Then $\exists_M a$ is the smallest element of $\{x \in M \mid x \geq a\}$. Hence $a \leq \exists_M a$. Obviously, $\exists_M(a \vee b) \geq \exists_M a, \exists_M b$ and therefore $\exists_M(a \vee b) \geq \exists_M a \vee \exists_M b$. On the other hand, $\exists_M a \vee \exists_M b \geq a \vee b$ and therefore $\exists_M a \vee \exists_M b \geq \exists_M(a \vee b)$. This shows $\exists_M(a \vee b) = \exists_M a \vee \exists_M b$. Since $\exists_M a \in M \in \mathrm{Sub}\mathcal{L}$, $\neg\exists_M a \in M$ and hence $\exists_M(\neg\exists_M a) = \neg\exists_M a$. Together we obtain $\exists_M \in \mathbf{E}(\mathcal{L})$. Obviously, $\exists_M(L) \subseteq M$. If $a \in M$ then $a = \exists_M a \in \exists_M(L)$, i. e. $M \subseteq \exists_M(L)$. This shows $M = \exists_M(L)$. If $\exists_1, \exists_2 \in \mathbf{E}(\mathcal{L})$ and $\exists_1 \leq \exists_2$ then $\exists_2 a \leq \exists_1\exists_2 a \leq \exists_2\exists_2 a = \exists_2 a$ and hence $\exists_2 = \exists_1\exists_2$ which implies $\exists_2(L) = \exists_1\exists_2(L) \subseteq \exists_1(L)$. Finally, if $M_1, M_2 \in \mathbf{R}(\mathcal{L})$ and $M_1 \subseteq M_2$ then $\exists_{M_1} a \geq \exists_{M_2} a$ and hence $\exists_{M_1} \geq \exists_{M_2}$. ∎

Corollary 17. *$\exists_L$ (which is the identical mapping on L) is the smallest element of $(\mathbf{E}(\mathcal{L}), \leq)$ and if $\mathcal{L}$ is bounded, $\exists_{\{0,1\}}$ (which maps 0 to 0 and $L \setminus \{0\}$ to 1) is the greatest element of $(\mathbf{E}(\mathcal{L}), \leq)$.*

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