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CLUSTER SETS AND RELATED PROPERTIES OF MULTIFUNCTIONS

Abstract. In this paper we present some types of cluster sets of multifunction. Using these concepts we relate properties of cluster sets to some generalized continuity properties, minimality of multifunctions and closedness of its graphs.

Throughout this work, (X,τ) and (Y,τ^*) denote topological spaces in which no separation axioms are assumed unless explicitly stated. For a subset A of a topological space we denote by $Cl(A)$ and $Int(A)$ the closure and the interior of A, respectively.

We will use $F:(X,\tau)\to(Y,\tau^*)$ to denote that F is a correspondence which assigns to each element x of X a nonempty subset $F(x)$ of Y. Such a mapping we will call a multifunction. In this regard, we will keep the notation $f:(X,\tau)\to(Y,\tau^*)$ for a single-valued function from (X,τ) to (Y,τ^*).

A single-valued function $f:(X,\tau)\to(Y,\tau^*)$ is called a selection for $F:(X,\tau)\to(Y,\tau^*)$ if $f(x)\in F(x)$ for every $x\in X$. A multifunction $H:(X,\tau)\to(Y,\tau^*)$ is called a multi-valued selection for $F:(X,\tau)\to(Y,\tau^*)$ if $H(x)\subset F(x)$ for every $x\in X$.

The upper and lower inverse images of a set $B\subset Y$ under F will be denoted by $F^+(B)$ and $F^-(B)$, respectively; that is $F^+(B)=\{x\in X: F(x)\subset W\}$ and $F^-(B)=\{x\in X: F(x)\cap W\neq\emptyset\}$. For $A\subset X$ its image under F is the set $F(A)=\bigcup\{F(x)\subset Y: x\in X\}$. The graph of F, denoted by $Gr(F)$ is the set $\{(x,y)\in X\times Y: y\in F(x)\}$. Notice that $Gr(H)\subset Gr(F)$ if and only if $H(x)\subset F(x)$ for every $x\in X$. In this case we will simply write that $H\subset F$.

Definition 1. For a multifunction $F:(X,\tau)\to(Y,\tau^*)$ and $x\in X$, we define the following types of cluster sets:

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(a) u.α.$C_F(x) = \{y \in Y : x \in Int(Cl(Int(F^+(W))))$ for any open set W with $y \in W\}$;
(b) u.q.$C_F(x) = \{y \in Y : x \in Cl(Int(F^+(W)))$ for any open set W with $y \in W\}$;
(c) u.p.$C_F(x) = \{y \in Y : x \in Int(Cl(F^+(W)))$ for any open set W with $y \in W\}$;
(d) u.β.$C_F(x) = \{y \in Y : x \in Cl(Int(Cl(F^+(W))))$ for any open set W with $y \in W\}$;
(e) u.$C_F(x) = \{y \in Y : x \in Cl(F^+(W))$ for any open set W with $y \in W\}$;
(f) l.c.$C_F(x) = \{y \in Y : x \in Int(F^-(W))$ for any open set W with $y \in W\}$;
(g) l.α.$C_F(x) = \{y \in Y : x \in Int(Cl(Int(F^-(W))))$ for any open set W with $y \in W\}$;
(h) l.q.$C_F(x) = \{y \in Y : x \in Cl(Int(F^-(W)))$ for any open set W with $y \in W\}$;
(i) l.p.$C_F(x) = \{y \in Y : x \in Int(Cl(F^-(W)))$ for any open set W with $y \in W\}$;
(j) l.β.$C_F(x) = \{y \in Y : x \in Cl(Int(Cl(F^-(W))))$ for any open set W with $y \in W\}$;
(k) l.$C_F(x) = \{y \in Y : x \in Cl(F^-(W))$ for any open set W with $y \in W\}$.

REMARK 2. Hrycay [8, Definition 3.1, Lemma 3.2] introduced the following classical type of cluster set: $\bigcap\{Cl(F(U)) : \text{open } U \subset X \text{ with } x \in U\}$. It is easy to see that this set is equal to $l.C_F(x)$. It is well known [8, Theorem 3.3] that, F has closed graph if and only if $F(x) = l.C_F(x)$ for any $x \in X$.

The following definition was introduced in [9]:

Let $\mathcal{A}$ be a non-empty family of non-empty subset of X. A point $y \in Y$ is $\mathcal{A}$-cluster point of multifunction $F : (X, \tau) \to (Y, \tau^*)$ at $x \in X$ ($y \in \mathcal{A}_F(x)$) if for any open sets U, V with $x \in U$, $y \in V$, there is a set $K \in \mathcal{A}$, $K \subset U$, such that for any $x' \in K$ we have $F(x') \cap V \neq \emptyset$.

It is easy to see that in the case when $\mathcal{A}$ is the family of all non-empty open sets $A \subset X$, then $\mathcal{A}_F(x) = l.q.C_F(x)$. This equality also holds when $\mathcal{A}$ is the family of all non-empty α-open sets $A \subset X$ [22] (resp. semi-open sets $A \subset X$ [15]) whose characteristic property is $A \subset Int(Cl(Int(A)))$ (resp. $A \subset Cl(Int(A))$).

If $\mathcal{A}$ is the family of all non-empty pre-open sets $A \subset X$ [19] (resp. β-open sets $A \subset X$ [1]) whose characteristic property is $A \subset Int(Cl(A))$ (resp. $A \subset Cl(Int(Cl(A)))$), then $\mathcal{A}_F(x) = l.\beta.C_F(x)$. In this case important is that the set $A \cap Int(Cl(Int(A)))$ (resp. $A \cap Cl(Int(A))$) is pre-open (resp. β-open) for every $A \subset X$.

REMARK 3. If a single-valued function $f : (X, \tau) \to (Y, \tau^*)$ is given, then under the natural interpretation of $f(x)$ as one point set $F(x) = \{f(x)\}$, Definition 1. gives the following types of cluster sets for single valued functions: $\alpha.C_f(x) = u.\alpha.C_F(x) = l.\alpha.C_F(x)$, $q.C_f(x) = u.q.C_F(x) = l.q.C_F(x)$, $p.C_f(x) = u.p.C_F(x) = l.p.C_F(x)$, $\beta.C_f(x) = u.\beta.C_F(x) = l.\beta.C_F(x)$ and $C_f(x) = u.C_F(x) = l.C_F(x)$.

Clearly, $C_f(x)$ is the cluster set which was investigated in [2] and [37] and $q.C_f(x)$ was investigated in [17] and [35].

Definition 4. A multifunction $F : (X,\tau) \to (Y,\tau^*)$ is said to be

(a) upper semi continuous (briefly u.s.c.) (resp. lower semi continuous (briefly l.s.c.)) at a point $x \in X$ if for each open set $V \subset Y$ such that $x \in F^+(V)$ (resp. $x \in F^-(V)$), $x \in Int(F^+(V))$ (resp. $x \in Int(F^-(V))$). F is said to be u.s.c. (resp. l.s.c.) if it is such at any point [13], [23];

(b) upper α-continuous (briefly u.α.c.) (resp. lower α-continuous (briefly l.α.c.)) at a point $x \in X$ if for each open set $V \subset Y$ such that $x \in F^+(V)$ (resp. $x \in F^-(V)$), $x \in Int(Cl(Int(F^+(V))))$ (resp. $x \in Int(Cl(Int(F^-(V)))))$. F is said to be u.$\alpha$.c. (resp. l.$\alpha$.c.) if it is such at any point [21];

(c) upper pre-continuous (briefly u.p.c.) (resp. lower pre-continuous (briefly l.p.c.)) at a point $x \in X$ if for each open set $V \subset Y$ such that $x \in F^+(V)$ (resp. $x \in F^-(V)$), $x \in Int(Cl(F^+(V)))$ (resp. $x \in Int(Cl(F^-(V)))$). F is said to be u.p.c. (resp. l.p.c.) if it is such at any point [24];

(d) upper quasi continuous (briefly u.q.c.) (resp. lower quasi continuous (briefly l.q.c.)) at a point $x \in X$ if for each open set $V \subset Y$ such that $x \in F^+(V)$ (resp. $x \in F^-(V)$), $x \in Cl(Int(F^+(V)))$ (resp. $x \in Cl(Int(F^-(V)))$). F is said to be u.q.c. (resp. l.q.c.) if it is such at any point [25];

(e) upper β-continuous (briefly u.β.c.) (resp. lower β-continuous (briefly l.β.c.)) at a point $x \in X$ if for each open set $V \subset Y$ such that $x \in F^+(V)$ (resp. $x \in F^-(V)$), $x \in Cl(Int(Cl(F^+(V))))$ (resp. $x \in Cl(Int(Cl(F^-(V)))))$. F is said to be u.$\beta$.c. (resp. l.$\beta$.c.) if it is such at any point [26];

(f) minimal at a point $x \in X$ if for each open set $V \subset Y$ such that $x \in F^-(V)$, $x \in Cl(Int(F^+(V)))$. F is said to be minimal if it is such at any point [14], [20];

(g) α-minimal (resp. p-minimal, β-minimal) at a point $x \in X$ if for each open set $V \subset Y$ such that $x \in F^-(V)$, $x \in Int(Cl(Int(F^+(V))))$ (resp. $x \in Int(Cl(F^+(V)))$, $x \in Cl(Int(Cl(F^+(V))))$). F is said to be α-minimal (resp. p-minimal, β-minimal) if it is such at any point [31];

(h) Cl-minimal at a point $x \in X$ if for each open set $V \subset Y$ such that $x \in F^-(V)$, $x \in Cl(F^+(V))$.

REMARK 5. If a single-valued function $f : (X, \tau) \to (Y, \tau^*)$ is treated as a multifunction $F : X \to Y$ given by $F(x) = \{f(x)\}$ then, in the above definition, the conditions in (b) and α-minimality (resp. (c) and p-minimality, (d) and minimality, (e) and β-minimality) are eqivalent to the α-continuity [18] (resp. pre-continuity [19], quasi-continuity [15, 12], β-continuity [1]) of f.

A multifunction $F : (X, \tau) \to (Y, \tau^*)$ is said to be quasi-continuous [3], [27] (resp. α-continuous [34], [29], pre-continuous [34], [28], β-continuous [34], [30]) at a point $x \in X$ if for any open sets W, $V \subset Y$ such that $F(x) \subset W$ and $F(x) \cap V \neq \emptyset$ the following holds $x \in Cl(Int(F^+(W) \cap F^-(V)))$ (resp. $x \in Int(Cl(Int(F^+(W) \cap F^-(V))))$, $x \in Int(Cl(F^+(W) \cap F^-(V)))$, $x \in Cl(Int(Cl(F^+(W) \cap F^-(V))))$).

REMARK 6. If a single-valued function $f : (X, \tau) \to (Y, \tau^*)$ is treated as a multifunction $F : (X, \tau) \to (Y, \tau^*)$ given by $F(x) = \{f(x)\}$, then the quasi-continuity (resp. α-continuity, pre-continuity, β-continuity) of F is equivalent to the quasi continuity (resp. α-continuity, pre-continuity, β-continuity) of f.

In [16, Theorem 1.] it was proved that a multifunction F is minimal if and only if any selection f of F is quasi-continuous. We can extend this result in the following way:

THEOREM 7. *For a multifunction $F : (X, \tau) \to (Y, \tau^*)$ and every point $x \in X$, the following statements are equivalent:*

(a) *F is minimal (resp. α-minimal, p-minimal, β-minimal) at x;*
(b) *any multi-valued selection H of F is quasi-continuous (resp. α-continuous, pre-continuous, β-continuous) at x;*
(c) *any selection h of F is quasi-continuous (resp. α-continuous, pre-continuous, β-continuous) at x;*
(d) *$F(x) \subset u.q.C_H(x)$ (resp. $F(x) \subset u.\alpha.C_H(x)$, $F(x) \subset u.p.C_H(x)$, $F(x) \subset u.\beta.C_H(x)$) for any multivalued selection H of F;*
(e) *$F(x) \subset q.C_h(x)$ (resp. $F(x) \subset \alpha.C_h(x)$, $F(x) \subset p.C_h(x)$, $F(x) \subset \beta.C_h(x)$) for any single-valued selection h of F.*

Proof. (a) implies (b): Let H be a multi-valued selection of F, let $x \in X$ and let V, W be open sets such that $H(x) \subset W$ and $H(x) \cap V \neq \emptyset$. Then $F(x) \cap W \cap V \neq \emptyset$ and by the minimality (resp. α-minimality, p-minimality, β-minimality) of F we have $x \in Cl(Int(F^+(W \cap V)))$ (resp. $x \in Int(Cl(Int(F^+(W \cap V))))$, $x \in Int(Cl(F^+(W \cap V)))$, $x \in Cl(Int(Cl(F^+(W \cap V))))$). Since $F^+(W \cap V) \subset H^+(W) \cap H^-(V)$, then $x \in Cl(Int(H^+(W) \cap H^-(V)))$ (resp. $x \in Int(Cl(Int(H^+(W) \cap H^-(V))))$, $x \in Int(Cl(H^+(W) \cap H^-(V)))$, $x \in Cl(Int(Cl(H^+(W) \cap H^-(V))))$) and the proof of (b) is finished.

The implication (b)$\Rightarrow$(c) is an immediate consequence of Remark 6.

(c) implies (a): Suppose that F is not minimal (resp. not α-minimal, not p-minimal, not β-minimal) at some $x \in X$. Then $x \notin Cl(Int(F^+(V)))$ (resp. $x \notin Int(Cl(Int(F^+(V))))$, $x \notin Int(Cl(F^+(V)))$, $x \notin Cl(Int(Cl(F^+(V)))))$ for some open set V such that $F(x) \cap V \neq \emptyset$. Define a selection h (resp. f, s, w) of F as follows:

$h(x) \in F(x) \cap V$;
$h(p) \in F(p) \setminus V$ for all $p \in (X \setminus F^+(V)) \cap Int(Cl(X \setminus F^+(V)))$;
$h(p) \in F(p)$ elsewhere;

(resp.

$f(x) \in F(x) \cap V$;
$f(p) \in F(p) \setminus V$ for all $p \in (X \setminus F^+(V)) \cap Cl(Int(Cl(X \setminus F^+(V))))$;
$f(p) \in F(p)$ elsewhere;,
$s(x) \in F(x) \cap V$;
$s(p) \in F(p) \setminus V$ for all $p \in (X \setminus F^+(V)) \cap Cl(Int(X \setminus F^+(V)))$;
$s(p) \in F(p)$ elsewhere;,
$w(x) \in F(x) \cap V$;
$w(p) \in F(p) \setminus V$ for all $p \in (X \setminus F^+(V)) \cap Int(Cl(Int(X \setminus F^+(V))))$;
$w(p) \in F(p)$ elsewhere).

It is suficient to prove that h is not quasi-continuous (resp. f is not α-continuous, s is not pre-continuous, w is not β-continuous) at x.

It is obvious that $h(x)$ (resp. $f(x)$, $s(x)$, $w(x)$) belongs to V. It is easy to show, that $x \notin Cl(Int(h^{-1}(V)))$ (resp. $x \notin Int(Cl(Int(f^{-1}(V))))$, $x \notin Int(Cl(s^{-1}(V)))$, $x \notin Cl(Int(Cl(w^{-1}(V)))))$. Indeed, let us suppose the contrary: $x \in Cl(Int(h^{-1}(V)))$ (resp. $x \in Int(Cl(Int(f^{-1}(V))))$, $x \in Int(Cl(s^{-1}(V)))$, $x \in Cl(Int(Cl(w^{-1}(V)))))$. Then $x \in Cl(Int(h^{-1}(V))) \cap Int(Cl(X \setminus F^+(V)))$ (resp. $x \in Int(Cl(Int(f^{-1}(V)))) \cap Cl(Int(Cl(X \setminus F^+(V))))$, $x \in Int(Cl(s^{-1}(V))) \cap Cl(Int(X \setminus F^+(V)))$, $x \in Cl(Int(Cl(w^{-1}(V)))) \cap Int(Cl(Int(X \setminus F^+(V)))))$. Then there exists a point p such that $p \in (X \setminus F^+(V)) \cap Int(Cl(X \setminus F^+(V)))$ and $h(p) \in V$ (resp. $p \in (X \setminus F^+(V)) \cap Cl(Int(Cl(X \setminus F^+(V))))$ and $f(p) \in V$, $p \in (X \setminus F^+(V)) \cap Cl(Int(X \setminus F^+(V)))$ and $s(p) \in V$, $p \in (X \setminus F^+(V)) \cap Int(Cl(Int(X \setminus F^+(V))))$ and $w(p) \in V$, but it is impossible.

Assume that (a) is satisfied and let H be a multi-valued selection of F, then $u.q.C_F(x) \subset u.q.C_H(x)$, $u.\alpha.C_F(x) \subset u.\alpha.C_H(x)$, $u.p.C_F(x) \subset u.p.C_H(x)$, $u.\beta.C_F(x) \subset u.\beta.C_H(x)$, so the minimality (resp. α-minimality, p-minimality, β-minimality) of F at x implies the condition (d).

The implication (d) $\Rightarrow$ (e) is a consequence of Remark 3.

Assume that (e) is satisfied and let h be a selection of F. Then $h(x) \in q.C_h(x)$ (resp. $h(x) \in \alpha.C_h(x)$, $h(x) \in p.C_h(x)$, $h(x) \in \beta.C_h(x)$) and therefore, $x \in Cl(Int(h^{-1}(V)))$ (resp. $x \in Int(Cl(Int(h^{-1}(V))))$, $x \in Int(Cl(h^{-1}(V)))$, $x \in Cl(Int(Cl(h^{-1}(V)))))$ for each open set containing

$h(x)$ that means the quasi-continuity (resp. α-continuity, pre-continuity, β-continuity) of h at x and proves (c). The proof is complete.

COROLLARY 8. *A single-valued function $f : (X,\tau) \to (Y,\tau^*)$ is α-continuous (resp. quasi-continuous, pre-continuous, β-continuous) at a point $x \in X$ if and only if $f(x) \in \alpha.C_f(x)$ (resp. $f(x) \in q.C_f(x)$* [35, Proposition 2]*, $f(x) \in p.C_f(x)$, $f(x) \in \beta.C_f(x)$).*

THEOREM 9. *Let (Y,τ^*) be a Hausdorff topological space and let $H : (X,\tau) \to (Y,\tau^*)$ be a l.p.c. (resp. l.α.c., l.s.c.) at a point $x \in X$ multi-valued selection of a minimal (resp. β-minimal, Cl-minimal) multifunction $F : (X,\tau) \to (Y,\tau^*)$, then $F(x) = H(x)$. Moreover, if (Y,τ^*) is regular, then F is u.s.c. at x.*

Proof. Suppose that $F(x) \not\subset H(x)$, let $y \in F(x) \setminus H(x)$ and let $z \in H(x)$. Since (Y,τ^*) is Hausdorff, then there exist disjoint open sets V and W such that $y \in V$ and $z \in W$. Since H is l.p.c. (resp. l.α.c., l.s.c.) at x, we have $x \in Int(Cl(H^-(W)))$ (resp. $x \in Int(Cl(Int(H^-(W))))$, $x \in Int(H^-(W))$) and, by the minimality (resp. β-minimality, Cl-minimality) of F, $x \in Cl(Int(F^+(V)))$ (resp. $x \in Cl(Int(Cl(F^+(V))))$, $x \in Cl(F^+(V))$). So, by $H \subset F$ we have $x \in Cl(Int(H^+(V)))$ (resp. $x \in Cl(Int(Cl(H^+(V))))$, $x \in Cl(H^+(V))$) and consequently $Int(Cl(H^-(W))) \cap Cl(Int(H^+(V))) \neq \emptyset$ (resp. $Int(Cl(Int(H^-(W)))) \cap Cl(Int(Cl(H^+(V)))) \neq \emptyset$, $Int(H^-(W)) \cap Cl(H^+(V)) \neq \emptyset$) which implies $H^-(W) \cap H^+(V) \neq \emptyset$ and gives a contradiction.

For the second part of the proof suppose that F is not u.s.c. at x. Then there exists an open set V_0 such that $F(x) \subset V_0$ and $x \in Cl(X \setminus F^+(V_0))$. Let $y^* \in H(x)$ and let W_0 be an open set such that $y^* \in W_0 \subset Cl(W_0) \subset V_0$. Since H is l.p.c. (resp. l.α.c., l.s.c.) at a point x, we have $x \in Int(Cl(H^-(W_0)))$ (resp. $x \in Int(Cl(Int(H^-(W_0))))$, $x \in Int(H^-(W_0))$). So there is an $p \in Int(Cl(H^-(W_0)))$ (resp. $p \in Int(Cl(Int(H^-(W_0))))$, $p \in Int(H^-(W_0))$) satisfying $F(p) \not\subset V_0$. This gives $F(p) \cap (Y \setminus Cl(W_0)) \neq \emptyset$ and, by the minimality (resp. β-minimality, Cl-minimality) of F, $p \in Cl(Int(F^+(Y \setminus Cl(W_0))))$ (resp. $p \in Cl(Int(Cl(F^+(Y \setminus Cl(W_0)))))$, $p \in Cl(F^+(Y \setminus Cl(W_0))))$. Since $H \subset F$, $p \in Cl(Int(H^+(Y \setminus Cl(W_0))))$ (resp. $p \in Cl(Int(Cl(H^+(Y \setminus Cl(W_0)))))$, $p \in Cl(H^+(Y \setminus Cl(W_0))))$ and consequently, $Int(Cl(H^-(W_0))) \cap Cl(Int(H^+(Y \setminus Cl(W_0)))) \neq \emptyset$ (resp. $Int(Cl(Int(H^-(W_0)))) \cap Cl(Int(Cl(H^+(Y \setminus Cl(W_0))))) \neq \emptyset$, $Int(H^-(W_0)) \cap Cl(H^+(Y \setminus Cl(W_0))) \neq \emptyset$) which gives $H^-(W_0) \cap H^+(Y \setminus Cl(W_0)) \neq \emptyset$. This is a contradiction and the proof is complete.

COROLLARY 10. *Let (Y,τ^*) be a Hausdorff topological space and let $f : (X,\tau) \to (Y,\tau^*)$ be a selection of a multifunction $F : (X,\tau) \to (Y,\tau^*)$. If* *one of the following conditions is satisfied, then* $F(x) = \{f(x)\}$. *Moreover, if* (Y, τ^*) *is regular, then* F *is continuous at* x.

(a) f *is pre-continuous at* x *and* F *is minimal;*
(b) f *is* α*-continuous at* x *and* F *is* β*-minimal;*
(c) f *is continuous at* x *and* F *is Cl-minimal.*

COROLLARY 11. ([16, Theorem 2.]) *Let* Y *be a Hausdorff topological space. If* f *is a selection of minimal multifunction* F *and* f *is continuous at a point* $x \in X$, *then* $F(x) = \{f(x)\}$. *Moreover, if* Y *is regular, then* F *is continuous at* x.

In [11] and [4] the following sets were used to investigate certain forms of lower semicontinuity of multifunctions F from a topological space (X, τ) to a metric space (Y, d): $F_\varepsilon(x) = \{y \in Y$: there exists an open set $U \subset X$ containing x such that $d(y, F(p)) < \varepsilon$ for all $p \in U\}$ and $F_0(x) = \bigcap\{F_\varepsilon(x) : \varepsilon > 0\}$, where $d(y, F(p)) = inf\,\{d(y, z) : z \in F(p)\}$.

It is easy to show that $F_0(x) = l.c.C_F(x)$.

F is called almost lower semicontinuous at x, if

$$F_\varepsilon(x) \neq \emptyset \text{ for all } \varepsilon > 0. \text{ [6]}$$

It is easy to see that the property described above is equivalent to the following:

$$x \in \bigcup\{Int(F^-(B(y, \varepsilon))) : y \in Y\} \quad \text{for all } \varepsilon > 0, \tag{1}$$

where $B(y, \varepsilon)$ denotes the open ball with center at y and radius ε.

The following concept was introduced in [38]:

A multifunction $F : (X, \tau) \to (Y, d)$ is quasi-lower semicontinuous at $x \in X$, if for every $\varepsilon > 0$ there exist a point $y \in F(x)$ and an open set U such that $x \in U$ and $F(p) \cap B(y, \varepsilon) \neq \emptyset$ for all $p \in U$.

It is obvious, that this condition is equivalent to the following:

$$x \in \bigcup\{Int(F^-(B(y, \varepsilon))) : y \in F(x)\} \quad \text{for all } \varepsilon > 0. \tag{2}$$

In [11] it was proved that the conditions (1) and (2) are equivalent.

However, it is easy to see that the following two conditions are not in general equivalent

$$x \in \bigcup\{Int(F^-(V)) : V \in \mathcal{V}\} \quad \text{for any open covering } \mathcal{V} \text{ of } Y, \tag{3}$$

$$x \in \bigcup\{Int(F^-(V)) : V \in \mathcal{V}\} \quad \text{for any open covering } \mathcal{V} \text{ of } F(x), \tag{4}$$

In [10], the term "almost lower semicontinuous" was applied to the multifunctions $F : (X, \tau) \to (Y, \tau^*)$ satisfying the condition (4). According to (2), we can name this type of continuity by quasi-lower semi continuity (briefly Q-l.s.c.). Of course, the conditions (2) and (4) in general are not equivalent.

If a single-valued function $f : (X,\tau) \to (Y,\tau^*)$ is treated as a multifunction $F : (X,\tau) \to (Y,\tau^*)$ given by $F(x) = \{f(x)\}$, then the condition (4) is equivalent to the continuity but the condition (3) is equivalent to the T_1-continuity [7] of f at x given by the following property:

$$x \in \bigcup \{Int(f^{-1}(V)) : V \in \mathcal{V}\} \quad \text{for any open covering } \mathcal{V} \text{ of } (Y,\tau^*).$$

So, it is possible to treat the condition (3) as some generalization of T_1-continuity into the multifunctions.

A multifunction $F : (X,\tau) \to (Y,\tau^*)$ is said to be u-t-cliquish (resp. l-t-cliquish) [9] at a point $x \in X$ if $x \in Cl(\bigcup \{Int(F^+(V)) : V \in \mathcal{V}\}))$ (resp. $x \in Cl(\bigcup \{Int(F^-(V)) : V \in \mathcal{V}\})$) for any open covering $\mathcal{V}$ of (Y,τ^*).

Such conditions are some generalizations of the concept of T_1-cliquishness defined as follows [32]:

A single-valued function $f : (X,\tau) \to (Y,\tau^*)$ is said to be T_1-cliqiush at a point $x \in X$ if $x \in Cl(\bigcup \{Int(f^{-1}(V)) : V \in \mathcal{V}\})$ for any open covering $\mathcal{V}$ of (Y,τ^*).

The following forms of cliquishness were investigated in [33]:

A single-valued function $f : (X,\tau) \to (Y,\tau^*)$ is said to be χ^1-α-cliquish (resp. χ^1-s-cliquish, pre χ^1-α-cliquish, pre χ^1-s-cliquish) at a point $x \in X$ if $x \in \bigcup\{Int(Cl(Int(f^{-1}(V)))) : V \in \mathcal{V}\}$ (resp. $x \in \bigcup\{Cl(Int(f^{-1}(V))) : V \in \mathcal{V}\}$, $x \in \bigcup\{Int(Cl(f^{-1}(V))) : V \in \mathcal{V}\}$, $x \in \bigcup\{Cl(Int(Cl(f^{-1}(V)))) : V \in \mathcal{V}\}$) for any open covering $\mathcal{V}$ of (Y,τ^*); (see [33, Definition 2.1., Proposition 2.3. (iii), Diagram 3.13]).

Definition 12. A multifunction $F : (X,\tau) \to (Y,\tau^*)$ is said to be

(a) u-t-α-cliquish (resp. u-t-q-cliquish, u-t-p-cliquish, u-t-β-cliquish) at a point $x \in X$ if $x \in \bigcup \{Int(Cl(Int(F^+(V)))) : V \in \mathcal{V}\}$ (resp. $x \in \bigcup \{Cl(Int(F^+(V))) : V \in \mathcal{V}\}$, $x \in \bigcup \{Int(Cl(F^+(V))) : V \in \mathcal{V}\}$, $x \in \bigcup \{Cl(Int(Cl(F^+(V)))) : V \in \mathcal{V}\}$) for any open covering $\mathcal{V}$ of (Y,τ^*);
(b) l-t-α-cliquish (resp. l-t-q-cliquish, l-t-p-cliquish, l-t-β-cliquish) at a point $x \in X$ if $x \in \bigcup \{Int(Cl(Int(F^-(V)))) : V \in \mathcal{V}\}$ (resp. $x \in \bigcup \{Cl(Int(F^-(V))) : V \in \mathcal{V}\}$, $x \in \bigcup \{Int(Cl(F^-(V))) : V \in \mathcal{V}\}$, $x \in \bigcup\{Cl(Int(Cl(F^-(V)))) : V \in \mathcal{V}\}$) for any open covering $\mathcal{V}$ of (Y,τ^*).

Proposition 13. *For any multifunction $F : (X,\tau) \to (Y,\tau^*)$, the following hold:*

(a) *F is u-t-α-cliquish (resp. u-t-q-cliquish, u-t-p-cliquish, u-t-β-cliquish) at a point $x \in X$ if and only if $u.\alpha.C_F(x) \neq \emptyset$ (resp. $u.q.C_F(x) \neq \emptyset$, $u.p.C_F(x) \neq \emptyset$, $u.\beta.C_F(x) \neq \emptyset$);*
(b) *F is l-t-α-cliquish (resp. l-t-q-cliquish, l-t-p-cliquish, l-t-β-cliquish, l-t-continuous) at a point $x \in X$ if and only if $l.\alpha.C_F(x) \neq \emptyset$ (resp. $l.q.C_F(x) \neq \emptyset$, $l.p.C_F(x) \neq \emptyset$, $l.\beta.C_F(x) \neq \emptyset$, $l.c.C_F(x) \neq \emptyset$).*

Proof. (a). Assume that F is u-t-α-cliquish (resp. u-t-q-cliquish, u-t-p-cliquish, u-t-β-cliquish) at a point $x \in X$ and let u.α.$C_F(x) = \emptyset$ (resp. u.q.$C_F(x) = \emptyset$, u.p.$C_F(x) = \emptyset$, u.β.$C_F(x) = \emptyset$). Then for every $y \in Y$ there exists an open set V_y with $y \in V_y$ such that $x \notin Int(Cl(Int(F^+(V_y))))$ (resp. $x \notin Cl(Int(F^+(V_y)))$, $x \notin Int(Cl(F^+(V_y)))$, $x \notin Cl(Int(Cl(F^+(V_y))))$). The family $\mathcal{V} = \{V_y : y \in Y\}$ forms an open covering of Y such that $x \notin \bigcup\{Int(Cl(Int(F^+(V)))) : V \in \mathcal{V}\}$ (resp. $x \notin \bigcup\{Cl(Int(F^+(V))) : V \in \mathcal{V}\}$, $x \notin \bigcup\{Int(Cl(F^+(V))) : V \in \mathcal{V}\}$, $x \notin \bigcup\{Cl(Int(Cl(F^+(V)))) : V \in \mathcal{V}\}$), that gives a contradiction.

The proof of the case (b) is analogous.

Corollary 14. *A single-valued function $f : (X, \tau) \to (Y, \tau^*)$ is t-α-cliquish (resp. t-q-cliquish, t-p-cliquish, t-β-cliquish) at a point $x \in X$ if and only if $\alpha.C_f(x) \neq \emptyset$ (resp. $q.C_f(x) \neq \emptyset$, $p.C_f(x) \neq \emptyset$, $\beta.C_f(x) \neq \emptyset$).*

For a given single-valued function f from a topological space (X, τ) to a metric space (Y, d) it is consider the following types of cliquishness:

f is cliquish [5], [36] (resp. q-cliquish [33, Definition 2.1., Proposition 2.3. (i)]) at a point $x \in X$ if $x \in Cl(\bigcup\{Int(f^{-1}(V)) : V \in \mathcal{V}_\varepsilon\})$ (resp. $x \in \bigcup\{Cl(Int(f^{-1}(V))) : V \in \mathcal{V}_\varepsilon\}$) for any $\varepsilon > 0$, where $\mathcal{V}_\varepsilon = \{B(y, \varepsilon) : y \in Y\}$.

Since t-q-cliquishness of $f : (X, \tau) \to (Y, d)$ implies its q-cliquishness, from Corollary 14 we have the following simple result:

Corollary 15. ([35, Proposition 3]) *For any single-valued function $f : (X, \tau) \to (Y, d)$, the condition $q.C_f(x) = \emptyset$ implies cliquishness of f at x.*

Now, we have the following definition based on the notion of quasi-lower semi continuity (property (4)).

Definition 16. A multifunction $F : (X, \tau) \to (Y, \tau^*)$ is said to be

(a) Q-u-α.cliquish (resp. Q-u-q.cliquish, Q-u-p.cliquish, Q-u-β.cliquish) at a point $x \in X$ if $x \in \bigcup\{Int(Cl(Int(F^+(V)))) : V \in \mathcal{V}\}$ (resp. $x \in \bigcup\{Cl(Int(F^+(V))) : V \in \mathcal{V}\}$, $x \in \bigcup\{Int(Cl(F^+(V))) : V \in \mathcal{V}\}$, $x \in \bigcup\{Cl(Int(Cl(F^+(V)))) : V \in \mathcal{V}\}$) for any open covering $\mathcal{V}$ of $F(x)$;

(b) Q-l-α.cliquish (resp. Q-l-q.cliquish, Q-l-p.cliquish, Q-l-β.cliquish) at a point $x \in X$ if $x \in \bigcup\{Int(Cl(Int(F^-(V)))) : V \in \mathcal{V}\}$ (resp. $x \in \bigcup\{Cl(Int(F^-(V))) : V \in \mathcal{V}\}$, $x \in \bigcup\{Int(Cl(F^-(V))) : V \in \mathcal{V}\}$, $x \in \bigcup\{Cl(Int(Cl(F^-(V)))) : V \in \mathcal{V}\}$) for any open covering $\mathcal{V}$ of $F(x)$.

Proposition 17. *For any multifunction $F : (X, \tau) \to (Y, \tau^*)$, the following hold:*

(a) *F is Q-u-α-cliquish (resp. Q-u-q-cliquish, Q-u-p-cliquish, Q-u-β-cliquish) at a point $x \in X$ if and only if $F(x) \cap u.\alpha.C_F(x) \neq \emptyset$ (resp. $F(x) \cap u.q.C_F(x) \neq \emptyset$, $F(x) \cap u.p.C_F(x) \neq \emptyset$, $F(x) \cap u.\beta.C_F(x) \neq \emptyset$);*

(b) *F is Q-l.s.c (resp. Q-l-α-cliquish, Q-l-q-cliquish, Q-l-p-cliquish, Q-l-β-cliquish) at a point $x \in X$ if and only if $F(x) \cap l.c.C_F(x) \neq \emptyset$ (resp. $F(x) \cap l.\alpha.C_F(x) \neq \emptyset$, $F(x) \cap l.q.C_F(x) \neq \emptyset$, $F(x) \cap l.p.C_F(x) \neq \emptyset$, $F(x) \cap l.\beta.C_F(x) \neq \emptyset$).*

Proof. (a). Assume that F is Q-u-α-cliquish (resp. Q-u-q-cliquish, Q-u-p-cliquish, Q-u-β-cliquish) at a point $x \in X$ and let $F(x) \cap u.\alpha.C_F(x) = \emptyset$ (resp. $F(x) \cap u.q.C_F(x) = \emptyset$, $F(x) \cap u.p.C_F(x) = \emptyset$, $F(x) \cap u.\beta.C_F(x) = \emptyset$). Then for every $z \in F(x)$ there exists an open set V_z such that $z \in V_z$ and $x \notin Int(Cl(Int(F^+(V_z))))$ (resp. $x \notin Cl(Int(F^+(V_z)))$, $x \notin Int(Cl(F^+(V_z)))$, $x \notin Cl(Int(Cl(F^+(V_z)))))$. The family $\mathcal{V} = \{V_z : z \in F(x)\}$ is an open covering of $F(x)$ satisfying $x \notin \bigcup\{Int(Cl(Int(F^+(V)))) : V \in \mathcal{V}\}$ (resp. $x \notin \bigcup\{Cl(Int(F^+(V))) : V \in \mathcal{V}\}$, $x \notin \bigcup\{Int(Cl(F^+(V))) : V \in \mathcal{V}\}$, $x \notin \bigcup\{Cl(Int(Cl(F^+(V)))) : V \in \mathcal{V}\}$),that gives a contradition.

Now, assume that $z \in u.\alpha.C_F(x)$ (resp. $z \in u.q.C_F(x)$, $z \in u.p.C_F(x)$, $z \in u.\beta.C_F(x)$) for some $z \in F(x)$. Then for any open covering $\mathcal{V}$ of $F(x)$ there exists an open set $V \in \mathcal{V}$ such that $z \in V$ and consequently, $x \in Int(Cl(Int(F^+(V))))$ (resp. $x \in Cl(Int(F^+(V)))$, $x \in Int(Cl(F^+(V)))$, $x \in Cl(Int(Cl(F^+(V)))))$. This implies that F is Q-u-α-cliquish (resp. Q-u-q-cliquish, Q-u-p-cliquish, Q-u-β-cliquish) at x.

The proof of the case (b) is analogous.

Remark 18. If F is u-t-α-cliquish (resp. u-t-q-cliquish, u-t-p-cliquish, u-t-β-cliquish, l-t-α-cliquish, l-t-q-cliquish, l-t-p-cliquish, l-t-β-cliquish) at every point $x \in X$, then by Proposition 13, $u.\alpha.C_F(x)$ (resp. $u.q.C_F(x)$, $u.p.C_F(x)$, $u.\beta.C_F(x)$, $l.\alpha.C_F(x)$, $l.q.C_F(x)$, $l.p.C_F(x)$, $u.\beta.C_F(x)$, $l.C_F(x)$) can be treated as the value of the multifunction $x \to u.\alpha.C_F(x)$ (resp. $x \to u.q.C_F(x)$, $x \to u.p.C_F(x)$, $x \to u.\beta.C_F(x)$, $x \to l.\alpha.C_F(x)$, $x \to l.q.C_F(x)$, $x \to l.p.C_F(x)$, $x \to l.\beta.C_F(x)$, $x \to l.C_F(x)$) at the point x.

Theorem 19. *Let F be a multifunction from a topological space (X, τ) to a regular topological space (Y, τ^*). Then the following statements hold:*

(a) *The multifunction $u.q.C_F$, $l.q.C_F$, $u.\beta.C_F$ and $l.\beta.C_F$ are minimal with closed graphs.*
(b) *The multifunction $u.p.C_F$, $l.p.C_F$, $u.\alpha.C_F$ and $l.\alpha.C_F$ are α-minimal with closed values.*

Proof. (a). Let V be an open subset of Y. At first we will show that

$$(5) \qquad (u.q.C_F)^-(V) \subset Cl(Int(l.\beta.C_F)^+(V)).$$

Let $x \in (u.q.C_F)^-(V)$ and suppose, on the contrary, $x \notin Cl(Int(l.\beta.C_F)^+(V))$. Then $u.q.C_F(x) \cap V \neq \emptyset$ and, by the regularity of Y, there is an open set W such that $u.q.C_F(x) \cap W \neq \emptyset$ and $Cl(W) \subset V$. Thus $x \in Cl(Int(F^+(W))) \cap$

$Int(Cl(X \setminus (l.\beta.C_F)^+(V))$ and, consequently there exists $p \in Int(F^+(W))$ such that $l.\beta.C_F(p) \cap (Y \setminus Cl(W)) \neq \emptyset$. We have $p \in Int(F^+(W)) \cap Cl(Int(Cl(F^-(Y \setminus Cl(W)))))$ which impies that $F^+(W) \cap F^-(Y \setminus Cl(W)) \neq \emptyset$ but this is impossible.

Now we show that

$$(l.\beta.C_F)^-(V) \subset Cl(Int(u.q.C_F)^+(V)). \tag{6}$$

If there exists $z \in (l.\beta.C_F)^-(V) \setminus Cl(Int(u.q.C_F)^+(V))$, then by the regularity of Y we have $z \in Cl(Int(Cl(F^-(W^*))))$ and $z \in Int(Cl(X \setminus (u.q.C_F)^+(V)))$ for some open set W^* such that $l.\beta.C_F(z) \cap W^* \neq \emptyset$ and $Cl(W^*) \subset V$. Consequently, there exists a point $s \in Int(Cl(F^-(W^*)))$ such that $u.q.C_F(s) \cap (Y \setminus Cl(W^*)) \neq \emptyset$. So $s \in Int(Cl(F^-(W^*))) \cap Cl(Int(F^+(Y \setminus Cl(W^*))))$ which gives a contradiction.

It is clear that $Cl(Int(l.\beta.C_F)^+(V)) \subset Cl(Int(l.\beta.C_F)^-(V))$. Thus, combining (5) and (6) we have $(u.q.C_F)^-(V) \subset Cl(Int(u.q.C_F)^+(V))$ which means that $u.q.C_F$ is minimal.

Analogously, since $Cl(Int(u.q.C_F)^+(V)) \subset Cl(Int(u.q.C_F)^-(V))$, then (6) and (5) simply $(l.\beta.C_F)^-(V) \subset Cl(Int(l.\beta.C_F^+(V)))$ which means that $l.\beta.C_F$ is minimal.

Now we show that

$$(l.q.C_F)^-(V) \subset Cl(Int(u.\beta.C_F)^+(V)), \tag{7}$$

and

$$(l.\beta.C_F)^-(V) \subset Cl(Int(l.q.C_F)^+(V)). \tag{8}$$

For this purpose we first assume that there exists $w \in (l.q.C_F)^-(V) \setminus Cl(Int(u.\beta.C_F)^+(V))$, then by the regularity of Y we have

$$w \in Cl(Int(F^-(V^*))) \qquad \text{and} \qquad w \in Int(Cl(X \setminus (u.\beta.C_F)^+(V)))$$

for some open set V^* such that $l.q.C_F(w) \cap V^* \neq \emptyset$ and $Cl(V^*) \subset V$. Then there exists a point $a \in Int(F^-(V^*))$ such that $u.\beta.C_F(a) \cap (Y \setminus Cl(V^*)) \neq \emptyset$ and consequently $a \in Int(F^-(V^*)) \cap Cl(Int(Cl(F^+(Y \setminus Cl(V^*)))))$. Thus $F^-(V^*) \cap F^+(Y \setminus Cl(V^*)) \neq \emptyset$ which is impossible, so condition (7) holds.

In the second part we assume that there exists a point $s \in (u.\beta.C_F)^-(V) \setminus Cl(Int(l.q.C_F)^+(V))$. By the regularity of Y, there exists an open set G such that $u.\beta.C_F(s) \cap G \neq \emptyset$ and $Cl(G) \subset V$. Then $s \in Cl(Int(Cl(F^+(G))))$ and $s \in Int(Cl(X \setminus l.q.C_F)^+(V))$. Then there exists a point $b \in Int(Cl(F^+(G)))$ such that $l.q.C_F(b) \cap (Y \setminus Cl(G)) \neq \emptyset$ and, consequently $b \in Int(Cl(F^+(G))) \cap Cl(Int(F^-(Y \setminus Cl(G))))$. Thus $F^+(G) \cap F^-(Y \setminus Cl(G)) \neq \emptyset$ which is impossible, so condition (8) holds.

It is clear that $Cl(Int(u.\beta.C_F)^+(V)) \subset Cl(Int(u.\beta.C_F)^-(V))$. So, by (8) and (7) we have $(l.q.C_F)^-(V) \subset Cl(Int(l.q.C_F)^+(V))$, so $l.q.C_F$ is minimal.

Since $Cl(Int(l.q.C_F)^+(V)) \subset Cl(Int(l.q.C_F)^-(V))$, combining (8) and (7) we have $(u.\beta.C_F)^-(V) \subset Cl(Int(u.\beta.C_F)^+(V))$, so $u.\beta.C_F$ is minimal.

Now, to prove the second part of (a), let $(s,z) \in Cl(Gr(u.q.C_F))$ (resp. $(s,z) \in Cl(Gr(l.q.C_F))$, $(s,z) \in Cl(Gr(u.\beta.C_F))$, $(s,z) \in Cl(Gr(l.\beta.C_F))$) and let (U,V) be a pair of open subsets such that $(s,z) \in U \times V$. Then there are $x \in U$ and $y \in V$ such that $y \in u.q.C_F(x)$ (resp. $y \in l.q.C_F(x)$, $y \in u.\beta.C_F(x)$, $y \in l.\beta.C_F(x)$) and consequently, $U \cap Int(F^+(V)) \neq \emptyset$ (resp. $U \cap Int(F^-(V)) \neq \emptyset$, $U \cap Int(Cl(F^+(V))) \neq \emptyset$, $U \cap Int(Cl(F^-(V))) \neq \emptyset$). So, $s \in Cl(Int(F^+(V)))$ (resp. $s \in Cl(Int(F^-(V)))$, $s \in Cl(Int(Cl(F^+(V))))$, $s \in Cl(Int(Cl(F^-(V))))$). This proves that $z \in u.q.C_F(s)$ (resp. $z \in l.q.C_F(s)$, $z \in u.\beta.C_F(s)$, $z \in l.\beta.C_F(s)$) and consequently, that $Gr(u.q.C_F)$ (resp. $Gr(l.q.C_F)$, $Gr(u.\beta.C_F)$, $Gr(l.\beta.C_F)$) is closed.

(b). Let V be an open subset of Y. We will show that

$$(u.p.C_F)^-(V) \subset Int(l.q.C_F)^+(V). \tag{9}$$

Suppose, on the contrary that there exists a point $x \in (u.p.C_F)^-(V) \setminus Int(l.q.C_F)^+(V)$. Then $x \in Cl(X \setminus (l.q.C_F)^+(V))$ and, by the regularity of Y, there exists an open set W such that $u.p.C_F(x) \cap W \neq \emptyset$ and $Cl(W) \subset V$. So $x \in Cl(X \setminus (l.q.C_F)^+(V)) \cap Int(Cl(F^+(W)))$ and, consequently there exists a point $p \in Int(Cl(F^+(W)))$ such that $l.q.C_F(p) \cap (Y \setminus Cl(W)) \neq \emptyset$. So $p \in Int(Cl(F^+(W))) \cap Cl(Int(F^-(Y \setminus Cl(W))))$ which gives a contradiction.

Since

$$Int(l.q.C_F)^+(V) \subset Int(l.q.C_F)^-(V)$$

and

$$(u.\beta.C_F)^+(V) \subset (u.p.C_F)^+(V),$$

by (7) and (9) we have $(u.p.C_F)^-(V) \subset Int(Cl(Int(u.p.C_F)^+(V)))$, so $u.p.C_F$ is α-minimal.

Now we show that

$$(u.\alpha.C_F)^-(V) \subset Int(l.\beta.C_F)^+(V). \tag{10}$$

If there exists $z \in (u.\alpha.C_F^-(V) \setminus Int(l.\beta.C_F)^+(V)$, then by the regularity of Y we have $z \in Int(Cl(Int(F^+(V^*)))) \cap Cl(X \setminus (l.\beta.C_F)^+(V))$ for some open set V^* such that $u.\alpha.C_F(z) \cap V^* \neq \emptyset$ and $Cl(V^*) \subset V$. Consequently, there exists a point $s \in Int(Cl(Int(F^+(V^*))))$ such that $l.\beta.C_F(s) \cap (Y \setminus Cl(V^*)) \neq \emptyset$. Thus $s \in Int(Cl(Int(F^+(V^*)))) \cap Cl(Int(Cl(F^-(Y \setminus Cl(V^*)))))$ which gives a contradiction.

It is clear that $Int(l.\beta.C_F)^+(V) \subset Int(l.\beta.C_F)^-(V)$ and $(u.q.C_F)^+(V) \subset (u.\alpha.C_F)^+(V)$. Thus (10) and (6) imply that

$$(u.\alpha.C_F)^-(V) \subset Int(Cl(Int(u.\alpha.C_F)^+(V))),$$

so $u.\alpha.C_F$ is α-minimal.

We will prove finally, that

(11) $$(l.p.C_F)^-(V) \subset Int(u.q.C_F)^+(V)$$

and

(12) $$(l.\alpha.C_F)^-(V) \subset Int(u.\beta.C_F)^+(V).$$

In the first case we assume that there exists a point $z \in (l.p.C_F)^-(V) \setminus Int(u.q.C_F)^+(V)$, then by the regularity of Y we have $z \in Int(Cl(F^-(W^*))) \cap Cl(X \setminus (u.q.C_F)^+(V))$ for some open set W^* such that $l.p.C_F(z) \cap W^* \neq \emptyset$ and $Cl(W^*) \subset V$. Then there exists a point $a \in Int(Cl(F^-(W^*)))$ such that $u.q.C_F(a) \cap (Y \setminus Cl(W^*)) \neq \emptyset$. Thus $a \in Int(Cl(F^-(W^*))) \cap Cl(Int(F^+(Y \setminus Cl(W^*))))$ which gives a contradiction.

In the second part we assume that there exists a point $w \in (l.\alpha.C_F)^-(V) \setminus Int(u.\beta.C_F)^+(V)$. By the regularity of Y, there exists an open set G^* such that $l.\alpha.C_F(w) \cap G^* \neq \emptyset$ and $Cl(G^*) \subset V$. Then $w \in Int(Cl(Int(F^-(G^*)))) \cap Cl(X \setminus (u.\beta.C_F)^+(V))$ and, consequently there exists a point $b \in Int(Cl(Int(F^-(G^*))))$ such that $u.\beta.C_F(b) \cap (Y \setminus Cl(G^*)) \neq \emptyset$. Thus $b \in Int(Cl(Int(F^-(G^*)))) \cap Cl(In(Cl(F^+(Y \setminus Cl(G^*)))))$ which gives a contradiction.

Since

$$Int(u.q.C_F)^+(V) \subset Int(u.q.C_F)^-(V)$$

and

$$(l.\beta.C_F)^+(V) \subset (l.p.C_F)^+(V),$$

by (11) and (5) we have $(l.p.C_F)^-(V) \subset Int(Cl(Int(l.p.C_F)^+(V)))$, so $l.p.C_F$ is α-minimal.

It is clear that $Int(u.\beta.C_F)^+(V) \subset Int(u.\beta.C_F)^-(V)$ and $(l.q.C_F)^+(V) \subset (l.\alpha.C_F)^+(V)$. Thus (12) and (8) imply that $(l.\alpha.C_F)^-(V) \subset Int(Cl(Int(l.\alpha.C_F)^+(V)))$, so $l.\alpha.C_F$ is α-minimal.

To prove the second part of (b) it is enough to see that $V \cap u.p.C_F(x) \neq \emptyset$ (resp. $V \cap l.p.C_F(x) \neq \emptyset$, $V \cap u.\alpha.C_F(x) \neq \emptyset$, $V \cap l.\alpha.C_F(x) \neq \emptyset$) implies $x \in Int(Cl(F^+(V)))$ (resp. $x \in Int(Cl(F^-(V)))$, $x \in Int(Cl(Int(F^+(V))))$, $x \in Int(Cl(Int(F^-(V))))$). So the proof of the theorem is complete.

Corollary 20. *Let f be a single-valued function from a topological space (X, τ) to a regular topological space (Y, τ^*). Then the following statements hold:*

(a) *The multifunction $q.C_f$* [16, Theorem 3] *and $\beta.C_f$ are minimal with closed graphs.*
(b) *The multifunction $p.C_f$ and $\alpha.C_f$ are α-minimal with closed values.*

Corollary 21. ([16, Theorem 5]) *Let (Y, τ^*) be a regular topological space. A multifunction $F : (X, \tau) \to (Y, \tau^*)$ is minimal with closed graph if and only* *if for any single-valued selection h of F the equality $F(x) = q.C_h(x)$ holds for each $x \in X$.*

Proof. Let F be minimal with closed graph and let $x \in X$. From Theorem 7 (e) we have $F(x) \subset q.C_h(x)$ and, since $q.C_h(x) \subset C_h(x) \subset l.C_F(x)$, from Remark 2, we have $F(x) \subset q.C_h(x)$. Conversely, if $F = q.C_h$, then F is minimal with closed graph by Corollary 20.

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