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WEAK FORMS OF OPEN AND CLOSED FUNCTIONS VIA b-θ-OPEN SETS

Abstract. In this paper, we introduce and study two new classes of functions called weakly b-θ-open functions and weakly b-θ-closed functions by using the notions of b-θ-open sets and b-θ-closure operator. The connections between these functions and other related functions are investigated.

1. Introduction

In 1996, Andrijević [3] introduced a new class of generalized open sets called b-open sets in a topological space. This class is a subset of the class of β-open sets [1]. Also the class of b-open sets is a superset of the class of semi-open sets [7] and the class of preopen sets [8]. In [4], the authors introduced and investigated θ-preopen functions and θ-preclosed functions by using pre-θ-interior and pre-θ-closure. In [5], weakly semi-θ-open functions and weakly semi-θ-closed functions are similarly investigated. The purpose of this paper is to introduce and investigate the notions of weakly b-θ-open functions and weakly b-θ-closed functions. Weak b-θ-openness (resp. weak b-θ-closedness) is a generalization of both θ-preopenness and weak semi-θ-openness (resp. θ-preclosedness and weak semi-θ-closedness). We investigate some of the fundamental properties of this class of functions. For the benefit of the reader, we recall some basic definitions and known results. Throughout the paper, X and Y (or (X,τ) and (Y,σ)) stand for topological spaces with no separation axioms assumed unless otherwise stated. Let A be a subset of X. The closure of A and the interior of A will be denoted by $Cl(A)$ and $Int(A)$, respectively.

Definition 1.1. A subset A of a space X is said to be b-open [3] if $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$.

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The complement of a b-open set is said to be b-closed. The intersection of all b-closed sets containing $A \subseteq X$ is called the b-closure of A and shall be denoted by $bCl(A)$. The union of all b-open sets of X contained in A is called the b-interior of A and is denoted by $bInt(A)$. A subset A is said to be b-regular if it is b-open and b-closed. The family of all b-open (resp. b-closed, b-regular) subsets of a space X is denoted by $BO(X)$ (resp. $BC(X)$, $BR(X)$) and the collection of all b-open subsets of X containing a fixed point x is denoted by $BO(X, x)$. The sets $BC(X, x)$ and $BR(X, x)$ are defined analogously.

Definition 1.2. A point $x \in X$ is called a b-θ-cluster [10] (resp. θ-cluster [14]) point of A if $bCl(U) \cap A \neq \phi$ (resp. $Cl(U) \cap A \neq \phi$) for every b-open (resp. open) set U of X containing x.

The set of all b-θ-cluster (resp. θ-cluster) points of A is called the b-θ-closure (resp. θ-closure) of A and is denoted by $bCl_\theta(A)$ (resp. $Cl_\theta(A)$). A subset A is said to be b-θ-closed (resp. θ-closed) if $bCl_\theta(A) = A$ (resp. $Cl_\theta(A) = A$). The complement of a b-θ-closed (resp. θ-closed) set is said to be b-θ-open (resp. θ-open). The b-θ-interior (resp. θ-interior) of A is defined by the union of all b-θ-open (resp. θ-open) sets contained in A and is denoted by $bInt_\theta(A)$ (resp. $Int_\theta(A)$). The family of all b-θ-open (resp. b-θ-closed) sets of a space X is denoted by $B\theta O(X, \tau)$ (resp. $B\theta C(X, \tau)$).

Definition 1.3. A subset A of a space X is said to be α-open [9] (resp. semi-open [7], preopen [8], β-open [1] or semi-preopen [2]) if $A \subseteq Int(Cl(Int(A)))$ (resp. $A \subseteq Cl(Int(A))$, $A \subseteq Int(Cl(A))$, $A \subseteq Cl(Int(Cl(A)))$).

The following basic properties of the b-closure are useful in the sequel:

Lemma 1.4. ([3]) *For a subset A of a space X, the following properties hold:*

(1) $bInt(A) = sInt(A) \cup pInt(A)$;
(2) $bCl(A) = sCl(A) \cap pCl(A)$;
(3) $bCl(X - A) = X - bInt(A)$;
(4) $x \in bCl(A)$ *if and only if* $A \cap U \neq \phi$ *for every* $U \in BO(X, x)$;
(5) $A \in BC(X)$ *if and only if* $A = bCl(A)$;
(6) $pInt(bCl(A)) = bCl(pInt(A))$.

Lemma 1.5. ([2]) *For a subset A of a space X, the following properties are hold:*

(1) $\alpha Int(A) = A \cap Int(Cl(Int(A)))$;
(2) $sInt(A) = A \cap Cl(Int(A))$;
(3) $pInt(A) = A \cap Int(Cl(A))$.

PROPOSITION 1.6. ([10]) *Let A and A_α ($\alpha \in \Lambda$) be any subsets of a space X. Then the following properties hold:*

(1) *if $A_\alpha \in B\theta O(X)$ for each $\alpha \in \Lambda$, then $\cup_{\alpha\in\Lambda} A_\alpha \in B\theta O(X)$;*
(2) *if A is b-closed, then $bInt(A) = bInt_\theta(A)$;*
(3) *$bCl_\theta(A)$ is b-θ-closed;*
(4) *$x \in bCl_\theta(A)$ if and only if $V \cap A \neq \phi$ for each $V \in BR(X,x)$;*
(5) *A is b-θ-open in X if and only if for each $x \in A$ there exists $V \in BR(X,x)$ such that $x \in V \subseteq A$.*

2. Characterizations of weakly b-θ-open functions

DEFINITION 2.1. A functions $f : X \to Y$ is said to be b-θ-open if for each open set U of X, $f(U)$ is b-θ-open.

DEFINITION 2.2. A functions $f : X \to Y$ is said to be weakly b-θ-open if $f(U) \subseteq bInt_\theta(f(Cl(U)))$ for each open set U of X.

Clearly, every b-θ-open function is also weakly b-θ-open, but the converse is not generally true.

EXAMPLE 2.3. Let $X = \{a,b,c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$. Then $BO(X) = (X,\tau)$. Let $f : X \to X$ be a function defined by $f(a) = c$, $f(b) = b$ and $f(c) = a$. Then f is a weakly b-θ-open function which is not b-θ-open, since for $U = \{a\}$, $f(U)$ is not b-θ-open in X.

THEOREM 2.4. *For a function $f : X \to Y$, the following conditions are equivalent:*

(1) *f is weakly b-θ-open;*
(2) *$f(Int_\theta(A)) \subseteq bInt_\theta(f(A))$ for every subset A of X;*
(3) *$Int_\theta(f^{-1}(B)) \subseteq f^{-1}(bInt_\theta(B))$ for every subset B of Y;*
(4) *$f^{-1}(bCl_\theta(B)) \subseteq Cl_\theta(f^{-1}(B))$ for every subset B of Y.*

Proof. (1)⇒(2): Let A be any subset of X and $x \in Int_\theta(A)$. Then, there exists an open set U such that $x \in U \subseteq Cl(U) \subseteq A$. Then, $f(x) \in f(U) \subseteq f(Cl(U)) \subseteq f(A)$. Since f is weakly b-θ-open, $f(U) \subseteq bInt_\theta(f(Cl(U))) \subseteq bInt_\theta(f(A))$. This implies that $f(x) \in bInt_\theta(f(A))$. This shows that $x \in f^{-1}(bInt_\theta(f(A)))$. Thus, $Int_\theta(A) \subseteq f^{-1}(bInt_\theta(f(A)))$, and so, $f(Int_\theta(A)) \subseteq bInt_\theta(f(A))$.

(2)⇒(3): Let B be any subset of Y. Then by (2), $f(Int_\theta(f^{-1}(B))) \subseteq bInt_\theta(f(f^{-1}(B))) \subseteq bInt_\theta(B)$. Therefore, $Int_\theta(f^{-1}(B)) \subseteq f^{-1}(bInt_\theta(B))$.

(3)⇒(4): Let B be any subset of Y. Using (3), we have

$$\begin{aligned} X - Cl_\theta(f^{-1}(B)) &= Int_\theta(X - f^{-1}(B)) \\ &= Int_\theta(f^{-1}(Y - B)) \\ &\subseteq f^{-1}(bInt_\theta(Y - B)) \\ &= f^{-1}(Y - bCl_\theta(B)) \\ &= X - f^{-1}(bCl_\theta(B)). \end{aligned}$$

Therefore, we obtain $f^{-1}(bCl_\theta(B)) \subseteq Cl_\theta(f^{-1}(B))$.

(4)⇒(1): Let V be any open set of X and $B = Y - f(Cl(V))$. By (4), $f^{-1}(bCl_\theta(Y - f(Cl(V)))) \subseteq Cl_\theta(f^{-1}(Y - f(Cl(V))))$. Therefore, we obtain $f^{-1}(Y - bInt_\theta(f(Cl(V)))) \subseteq Cl_\theta(X - f^{-1}(f(Cl(V)))) \subseteq Cl_\theta(X - Cl(V))$. Hence $V \subseteq Int_\theta(Cl(V)) \subseteq f^{-1}(bInt_\theta(f(Cl(V))))$ and $f(V) \subseteq bInt_\theta(f(Cl(V)))$. This shows that f is weakly b-θ-open. ∎

THEOREM 2.5. *For a function $f : X \to Y$, the following conditions are equivalent:*

(1) *f is weakly b-θ-open;*
(2) *For each $x \in X$ and each open subset U of X containing x, there exists a b-θ-open set V containing $f(x)$ such that $V \subseteq f(Cl(U))$.*

Proof. (1)⇒(2): Let $x \in X$ and U be an open set in X with $x \in U$. Since f is weakly b-θ-open, $f(x) \in f(U) \subseteq bInt_\theta(f(Cl(U)))$. Let $V = bInt_\theta(f(Cl(U)))$. Then V is b-θ-open and $f(x) \in V \subseteq f(Cl(U))$.

(2)⇒(1): Let U be an open set in X and let $y \in f(U)$. It follows from (2) that $V \subseteq f(Cl(U))$ for some b-θ-open set V in Y containing y. Hence, we have $y \in V \subseteq bInt_\theta(f(Cl(U)))$. This shows that $f(U) \subseteq bInt_\theta(f(Cl(U)))$. Thus f is weakly b-θ-open. ∎

THEOREM 2.6. *For a bijective function $f : X \to Y$, the following conditions are equivalent:*

(1) *f is weakly b-θ-open;*
(2) *$bCl_\theta(f(Int(F))) \subseteq f(F)$ for each closed set F in X;*
(3) *$bCl_\theta(f(U)) \subseteq f(Cl(U))$ for each open set U in X.*

Proof. (1)⇒(2): Let F be a closed set in X. Then we have $f(X - F) = Y - f(F) \subseteq bInt_\theta(f(Cl(X - F)))$, and so $Y - f(F) \subseteq Y - bCl_\theta(f(Int(F)))$. Hence $bCl_\theta(f(Int(F))) \subseteq f(F)$.

(2)⇒(3): Let U be an open set in X. Since $Cl(U)$ is a closed set and $U \subseteq Int(Cl(U))$, by (2) we have $bCl_\theta(f(U)) \subseteq bCl_\theta(f(Int(Cl(U)))) \subseteq f(Cl(U))$.

(3)⇒(1): Let V be an open set of X. Then, we have $Y - bInt_\theta(f(Cl(V)))$ $= bCl_\theta(Y - f(Cl(V))) = bCl_\theta(f(X - Cl(V))) \subseteq f(Cl(X - Cl(V))) =$

$f(X - Int(Cl(V))) \subseteq f(X - V) = Y - f(V)$. Therefore, we have $f(V) \subseteq bInt_\theta(f(Cl(V)))$ and hence f is weakly b-θ-open. ∎

The proof of the following theorem is straightforward and thus is omitted.

THEOREM 2.7. *For a function $f : X \to Y$, the following conditions are equivalent:*

(1) *f is weakly b-θ-open;*
(2) *$f(U) \subseteq bInt_\theta(f(Cl(U)))$ for each preopen set U of X;*
(3) *$f(U) \subseteq bInt_\theta(f(Cl(U)))$ for each α-open set U of X;*
(4) *$f(Int(Cl(U))) \subseteq bInt_\theta(f(Cl(U)))$ for each open set U of X;*
(5) *$f(Int(F)) \subseteq bInt_\theta(f(F))$ for each closed set F of X.*

3. Some properties of weakly b-θ-open functions

THEOREM 3.1. *Let X be a regular space. A function $f : (X, \tau) \to (Y, \sigma)$ is weakly b-θ-open if and only if f is b-θ-open.*

Proof. The sufficiency is clear. For the necessity, let W be a nonempty open subset of X. For each x in W, let U_x be an open set such that $x \in U_x \subseteq Cl(U_x) \subseteq W$. Hence we obtain that $W = \cup\{U_x : x \in W\} = \cup\{Cl(U_x) : x \in W\}$ and $f(W) = \cup\{f(U_x) : x \in W\} \subseteq \cup\{bInt_\theta(f(Cl(U_x))) : x \in W\} \subseteq bInt_\theta(f(\cup\{Cl(U_x) : x \in W\})) = bInt_\theta(f(W))$. Thus f is b-θ-open. ∎

Recall that a function $f : (X, \tau) \to (Y, \sigma)$ is said to be strongly continuous [6] if for every subset A of X, $f(Cl(A)) \subseteq f(A)$.

THEOREM 3.2. *If $f : (X, \tau) \to (Y, \sigma)$ is weakly b-θ-open and strongly continuous, then f is b-θ-open.*

Proof. Let U be an open subset of X. Since f is weakly b-θ-open, $f(U) \subseteq bInt_\theta(f(Cl(U)))$. However, because f is strongly continuous, $f(U) \subseteq bInt_\theta(f(U))$. Therefore $f(U)$ is b-θ-open. ∎

A function $f : (X, \tau) \to (Y, \sigma)$ is said to be contra b-θ-closed if $f(U)$ is a b-θ-open set of Y, for each closed set U in X.

THEOREM 3.3. *If $f : (X, \tau) \to (Y, \sigma)$ is a contra b-θ-closed function, then f is weakly b-θ-open.*

Proof. Let U be an open subset of X. Then, we have $f(U) \subseteq f(Cl(U)) = bInt_\theta(f(Cl(U)))$. ∎

EXAMPLE 3.4. The converse of Theorem 3.3 does not hold. Example 2.3 shows that a weakly b-θ-open function need not be contra b-θ-closed, since $f(\{b, c\}) = \{a, b\}$ is not b-θ-open in X.

DEFINITION 3.5. ([13]) A space X is said to be hyperconnected if every non-empty open subset of X is dense in X.

THEOREM 3.6. *If X is a hyperconnected space, then a function $f : X \to Y$ is weakly b-θ-open if and only if $f(X)$ is b-θ-open in Y.*

Proof. The sufficiency is clear. For the necessity observe that for any open subset U of X, $f(U) \subseteq f(X) = bInt_\theta(f(X)) = bInt_\theta(f(Cl(U)))$. Hence f is weakly b-open. ∎

DEFINITION 3.7. A function $f : X \to Y$ is called complementary weakly b-θ-open if for each open set U of X, $f(Fr(U))$ is b-θ-closed in Y, where $Fr(U)$ denotes the frontier of U.

It is clear that if A is closed, then $Int(A) \subseteq sInt(A) = bInt(A) = bInt_\theta(A)$.

THEOREM 3.8. *Let $B\theta O(X, \tau)$ be closed under finite intersection. If $f : (X, \tau) \to (Y, \sigma)$ is bijective weakly b-θ-open and complementary weakly b-θ-open, then f is b-θ-open.*

Proof. Let U be an open subset in X and $y \in f(U)$. Since f is weakly b-θ-open, by Theorem 2.5, for some $x \in U$ there exists a b-θ-open set V containing $f(x) = y$ such that $V \subseteq f(Cl(U))$. Now $Fr(U) = Cl(U) - U$ and thus $x \notin Fr(U)$. Hence $y \notin f(Fr(U))$ and therefore $y \in V \setminus f(Fr(U))$. Put $V_y = V - f(Fr(U))$. Then V_y is a b-θ-open set since f is complementary weakly b-θ-open. Furthermore, $y \in V_y$ and $V_y = V - f(Fr(U)) \subseteq f(Cl(U)) - f(Fr(U)) = f(Cl(U) - Fr(U)) = f(U)$. Therefore $f(U) = \cup\{V_y : V_y \in B\theta O(Y, \sigma), y \in f(U)\}$. Hence by Proposition 1.6, f is b-θ-open. ∎

DEFINITION 3.9. ([12]) A function $f : X \to Y$ is said to be almost open in the sense of Singal and Singal, written as $(a.o.S.)$ if the image of each regular open set U of X is an open set in Y.

DEFINITION 3.10. A function $f : X \to Y$ is said to be b-closed if for each closed set F of X, $f(F)$ is b-closed set in Y.

THEOREM 3.11. *If $f : (X, \tau) \to (Y, \sigma)$ is an a.o.S. and b-closed function, then it is a weakly b-θ-open function.*

Proof. Let U be an open set in X. Since f is $a.o.S.$ and $Int(Cl(U))$ is regular open, $f(Int(Cl(U)))$ is open in Y. Since f is b-closed, $f(U) \subseteq f(Int(Cl(U))) \subseteq Int(f(Cl(U))) \subseteq bInt(f(Cl(U))) = bInt_\theta(f(Cl(U)))$. This shows that f is weakly b-θ-open. ∎

DEFINITION 3.12. ([1]) A function $f : X \to Y$ is said to be β-open if the image of each open set U of X is a β-open set.

THEOREM 3.13. *If a function $f : X \to Y$ is weakly b-θ-open and precontinuous , then f is β-open.*

Proof. Let U be an open subset of X. Then by weak b-θ-openness of f, $f(U) \subseteq bInt_\theta(f(Cl(U)))$. Since f is precontinuous, $f(Cl(U)) \subseteq Cl(f(U))$. Hence we obtain that

$$\begin{aligned} f(U) &\subseteq bInt_\theta(f(Cl(U))) \\ &\subseteq bInt_\theta(Cl(f(U))) \\ &= bInt(Cl(f(U))) \\ &= sInt(Cl(f(U))) \cup pInt(Cl(f(U)))) \\ &\subseteq Cl(Int(Cl(f(U)))) \cup Int(Cl(f(U))) \\ &\subseteq Cl(Int(Cl(f(U)))) \end{aligned}$$

which shows that $f(U)$ is a β-open set in Y. Thus f is a β-open function. ∎

A topological space X is said to be b-θ-connected if it cannot be written as the union of two nonempty disjoint b-θ-open sets.

Theorem 3.14. *If $f : X \to Y$ is a weakly b-θ-open bijective function of a space X onto a b-θ-connected space Y, then X is connected.*

Proof. Suppose that X is not connected. Then there exist non-empty open sets U and V such that $U \cap V = \phi$ and $U \cup V = X$. Hence we have $f(U) \cap f(V) = \phi$ and $f(U) \cup f(V) = Y$. Since f is weakly b-θ-open, we have $f(U) \subseteq bInt_\theta(f(Cl(U)))$ and $f(V) \subseteq bInt_\theta(f(Cl(V)))$. Moreover U, V are open and also closed. We have $f(U) = bInt_\theta(f(U))$ and $f(V) = bInt_\theta(f(V))$. Hence, $f(U)$ and $f(V)$ are b-θ-open in Y. Thus, Y has been decomposed into two non-empty disjoint b-θ-open sets. This is contrary to the hypothesis that Y is a b-θ-connected space. Thus, X is connected. ∎

Theorem 3.15. *Let X be a regular space. Then for a function $f : X \to Y$, the following conditions are equivalent:*

(1) *f is weakly b-θ-open;*
(2) *For each θ-open set A in X, $f(A)$ is b-θ-open in Y;*
(3) *For any set B of Y and any θ-closed set A in X containing $f^{-1}(B)$, there exists a b-θ-closed set F in Y containing B such that $f^{-1}(F) \subseteq A$.*

Proof. (1)⇒(2): Let A be a θ-open set in X. Since X is regular, by Theorem 3.1, f is b-θ-open and A is open. Therefore $f(A)$ is b-θ-open in Y.

(2)⇒(3): Let B be any set in Y and A a θ-closed set in X such that $f^{-1}(B) \subseteq A$. Since $X - A$ is θ-open in X, by (2), $f(X - A)$ is b-θ-open in Y. Let $F = Y - f(X - A)$. Then F is b-θ-closed and $B \subseteq F$. Now $f^{-1}(F) = f^{-1}(Y - f(X - A)) = X - f^{-1}(f(X - A)) \subseteq A$.

(3)⇒(1): Let B be any set in Y. Let $A = Cl_\theta(f^{-1}(B))$. Since X is regular, A is a θ-closed set in X and $f^{-1}(B) \subseteq A$. Then there exists a b-θ-closed set F in Y containing B such that $f^{-1}(F) \subseteq A$. Since F is b-θ-closed,

$f^{-1}(bCl_\theta(B)) \subseteq f^{-1}(F) \subseteq A = Cl_\theta(f^{-1}(B))$. Therefore by Theorem 2.4, f is weakly b-θ-open. ∎

4. Weakly b-θ-closed functions

Definition 4.1. A functions $f : X \to Y$ is said to be b-θ-closed if for each closed set F of X, $f(F)$ is b-θ-closed.

Now, we define the generalized form of b-θ-closed functions.

Definition 4.2. A functions $f : X \to Y$ is said to be weakly b-θ-closed if $bCl_\theta(f(Int(F))) \subseteq f(F)$ for each closed set F of X.

Clearly, every b-θ-closed function is a weakly b-θ-closed function, but the converse is not generally true.

Example 4.3. Let $f : X \to Y$ be the function from Example 2.3. Then it is shown that f is a weakly b-θ-closed function which is not b-θ-closed, since $f(\{c, b\}) = \{a, b\}$ is not b-θ-closed in X.

Theorem 4.4. *For a function $f : X \to Y$, the following conditions are equivalent:*

(1) *f is weakly b-θ-closed;*
(2) *$bCl_\theta(f(U)) \subseteq f(Cl(U))$ for each open set U in X.*

Proof. (1)⇒(2): Let U be an open set in X. Since $Cl(U)$ is a closed set and $U \subseteq Int(Cl(U))$, we have $bCl_\theta(f(U)) \subseteq bCl_\theta(f(Int(Cl(U)))) \subseteq f(Cl(U))$.

(2)⇒(1): Let F be a closed set of X. Then, we have $bCl_\theta(f(Int(F))) \subseteq f(Cl(Int(F))) \subseteq f(Cl(F)) = f(F)$ and hence f is weakly b-θ-closed. ∎

Corollary 4.5. *A bijective function $f : X \to Y$ is weakly b-θ-open if and only if f is weakly b-θ-closed.*

Proof. This is an immediate consequence of Theorems 2.6 and 4.4. ∎

The proof of the following theorem is straightforward and thus is omitted.

Theorem 4.6. *For a function $f : X \to Y$, the following conditions are equivalent:*

(1) *f is weakly b-θ-closed;*
(2) *$bCl_\theta(f(Int(F))) \subseteq f(F)$ for each preclosed set F in X;*
(3) *$bCl_\theta(f(Int(F))) \subseteq f(F)$ for each α-closed set F in X;*
(4) *$bCl_\theta(f(Int(Cl(U)))) \subseteq f(Cl(U))$ for each subset U in X;*
(5) *$bCl_\theta(f(U)) \subseteq f(Cl(U))$ for each preopen set U in X.*

Theorem 4.7. *For a function $f : X \to Y$, the following conditions are equivalent:*

(1) *f is weakly b-θ-closed;*
(2) *$bCl_\theta(f(U)) \subseteq f(Cl(U))$ for each regular open set U in X;*
(3) *For each subset F in Y and each open set U in X with $f^{-1}(F) \subseteq U$, there exists a b-θ-open set A in Y with $F \subseteq A$ and $f^{-1}(A) \subseteq Cl(U)$;*
(4) *For each point y in Y and each open set U in X with $f^{-1}(y) \subseteq U$, there exists a b-θ-open set A in Y containing y and $f^{-1}(A) \subseteq Cl(U)$.*

Proof. (1)⇒(2): This is clear by Theorem 4.4.

(2)⇒(3): Let F be a subset of Y and U an open set in X with $f^{-1}(F) \subseteq U$. Then $f^{-1}(F) \cap Cl(X - Cl(U)) = \phi$ and consequently, $F \cap f(Cl(X - Cl(U))) = \phi$. Since $X - Cl(U)$ is regular open, $F \cap bCl_\theta(f(X - Cl(U))) = \phi$. Let $A = Y - bCl_\theta(f(X - Cl(U)))$. Then A is a b-θ-open set with $F \subseteq A$ and we have $f^{-1}(A) \subseteq X - f^{-1}(bCl_\theta(f(X - Cl(U)))) \subseteq X - f^{-1}f(X - Cl(U)) \subseteq Cl(U)$.

(3)⇒(4): This is obvious.

(4)⇒(1): Let F be closed in X and let $y \in Y - f(F)$. Since $f^{-1}(y) \subseteq X - F$, by (4) there exists a b-θ-open set A in Y with $y \in A$ and $f^{-1}(A) \subseteq Cl(X - F) = X - Int(F)$. Therefore $A \cap f(Int(F)) = \phi$, so that $y \notin bCl_\theta(f(Int(F)))$. Thus $bCl_\theta(f(Int(F))) \subseteq f(F)$. Hence f is weakly b-θ-closed. ∎

Theorem 4.8. *If $f : X \to Y$ is a bijective weakly b-θ-closed function, then for every subset F in Y and every open set U in X with $f^{-1}(F) \subseteq U$, there exists a b-θ-closed set B in Y such that $F \subseteq B$ and $f^{-1}(B) \subseteq Cl(U)$.*

Proof. Let F be a subset of Y and U be an open subset of X with $f^{-1}(F) \subseteq U$. Put $B = bCl_\theta(f(Int(Cl(U))))$. Then B is a b-θ-closed set in Y such that $F \subseteq B$, since $F \subseteq f(U) \subseteq f(Int(Cl(U))) \subseteq bCl_\theta(f(Int(Cl(U)))) = B$. Since f is weakly b-θ-closed, by Theorem 4.6, we have $f^{-1}(B) \subseteq Cl(U)$. ∎

Recall that, a set F in a space X is θ-compact [11] if for each cover Ω of F by open sets in X, there is a finite family $U_1, \cdots, U_n$ in Ω such that $F \subseteq Int(\cup\{Cl(U_i) : i = 1, \ldots, n\})$.

Theorem 4.9. *If $f : X \to Y$ is a weakly b-θ-closed function with θ-closed fibers, then $f(F)$ is b-θ-closed for each θ-compact set F in X.*

Proof. Let F be a θ-compact set and $y \in Y - f(F)$. Then $f^{-1}(y) \cap F = \phi$ and for each $x \in F$ there is an open set $U_x \subseteq X$ with $x \in U_x$ such that $Cl(U_x) \cap f^{-1}(y) = \phi$. Clearly $\Omega = \{U_x : x \in F\}$ is an open cover of F. Since F is θ-compact, there is a finite family $U_{x_1}, \cdots, U_{x_n}$ in Ω such that $F \subseteq Int(A)$, where $A = \cup\{Cl(U_{x_i}) : i = 1, \cdots, n\}$. Since f is weakly b-θ-closed, by Theorem 4.7, there exists a b-θ-open $B \subseteq Y$ with $f^{-1}(y) \subseteq f^{-1}(B) \subseteq Cl(X - A) = X - Int(A) \subseteq X - F$. Therefore $y \in B \subseteq Y -$

$f(F)$. By Proposition 1.6(1), $Y - f(F)$ is b-θ-open. This shows that $f(F)$ is b-θ-closed. ∎

A space (X, τ) is called b-T_2 if for any pair of distinct points x and y in X there exist b-open sets U and V in X containing x and y, respectively, such that $U \cap V = \phi$.

THEOREM 4.10. *If $f : X \to Y$ is a weakly b-θ-closed surjection and all pairs of disjoint fibers are strongly separated, then Y is b-T_2.*

Proof. Let y and z be two points in Y. Let U and V be open sets in X such that $f^{-1}(y) \in U$ and $f^{-1}(z) \in V$ with $Cl(U) \cap Cl(V) = \phi$. Since f is weakly b-θ-closed, by Theorem 4.7, there are b-θ-open sets F and B in Y such that $y \in F$ and $z \in B$, $f^{-1}(F) \subseteq Cl(U)$ and $f^{-1}(B) \subseteq Cl(V)$. Therefore $F \cap B = \phi$, because $Cl(U) \cap Cl(V) = \phi$ and f is surjective. Since every b-θ-open set is b-open. Then Y is b-T_2. ∎

A subset K of a space X is said to be b-closed relative to X [10] if for every cover $\{V_\alpha : \alpha \in \Lambda\}$ of K by b-open sets of X, there exists a finite subset Λ_0 of Λ such that $K \subseteq \cup\{bCl(V_\alpha) : \alpha \in \Lambda_0\}$.

THEOREM 4.11. *Let $f : X \to Y$ be a weakly b-θ-closed surjection with compact point inverses and K b-closed relative to Y, then $f^{-1}(K)$ is quasi H-closed relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Lambda\}$ be an open cover of $f^{-1}(K)$, where Λ is an index set. For each $y \in K$, $f^{-1}(y)$ is compact and $f^{-1}(y) \subseteq \cup_{\alpha\in\Lambda} U_\alpha$. There exists a finite subset $\Lambda(y)$ of Λ such that $f^{-1}(y) \subseteq \cup_{\alpha\in\Lambda(y)} U_\alpha$. Put $U(y) = \cup_{\alpha\in\Lambda(y)} U_\alpha$. Since f is b-θ-closed, by Theorem 4.7, there exists a b-θ-open set $V(y)$ containing y such that $f^{-1}(V(y)) \subseteq Cl(U(y))$. Since $V(y)$ is b-θ-open, by Proposition 1.6, there exists $V_0(y) \in BR(X, y)$ such that $y \in V_0(y) = bCl(V_0(y)) \subseteq V(y)$. Since the family $\{V_0(y) : y \in K\}$ is a cover of K by b-open sets of Y, there exist a finite number of points, say $y_1, y_2, \cdots, y_n$ of K such that $K \subseteq \cup_{i=1}^n bCl(V_0(y_i))$; hence $K \subseteq \cup_{i=1}^n V(y_i)$. Therefore, we obtain $f^{-1}(K) \subseteq \cup_{i=1}^n f^{-1}(V(y_i)) \subseteq \cup_{i=1}^n Cl(U(y_i)) = \cup_{i=1}^n \cup_{\alpha\in\Lambda(y_i)} Cl(U_\alpha)$. This shows that $f^{-1}(K)$ is quasi H-closed relative to X. ∎

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