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DIVISION POINT IN THE POINCARÉ UPPER HALF PLANE

Abstract. In this work, it is shown that the coordinates of the division point can be determined by the formula in the Poincaré upper half plane.

1. Introduction

The Poincaré upper half plane geometry has been introduced by Henri Poincaré. Let's denote this plane by H. The plane H is the upper half plane of the Euclidean analytical plane $\mathbb{R}^2$. Although the points in the plane H are same as the points in the upper half plane of the Euclidean analytical plane $\mathbb{R}^2$, the lines and the distance function between any two points are different. The lines in the plane H are defined by

$$_aL = \{(x,y) \in \mathbb{R}^2 \mid x = a, \quad y > 0, \quad a \in \mathbb{R}, \ a \ constant\} \qquad half\ lines$$

and

$$_cL_r = \{(x,y) \in \mathbb{R}^2 \mid (x-c)^2 + y^2 = r^2, \quad y > 0\ , \ c, r \in \mathbb{R}, \quad c, r\ constant, \ r > 0\} \qquad half\ circles$$

If $A = (x_1, y_1)$ and $B = (x_2, y_2)$ are any two points in H then the *Poincaré distance* between these points is given by

$$d_H(A,B) = \begin{cases} \left|\ln\ (y_2/y_1)\right|, & if\ x_1 = x_2 \\ \left|\ln\left(\dfrac{y_2\,(x_1 - c + r)}{y_1\,(x_2 - c + r)}\right)\right|, & if\ x_1 \neq x_2 \end{cases}$$

where

$$c = \left(y_2^2 - y_1^2 + x_2^2 - x_1^2\right)/2(x_2 - x_1),$$
$$r = \sqrt{(x_1 - c)^2 + y_1^2} = \sqrt{(x_2 - c)^2 + y_2^2}.$$

The geometry of the half plane H is a non-Euclidean, since it fails to satisfy the parallel postulate but satisfies all the remaining twelve axioms of

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the Euclidean plane geometry [2, 3, 4, 5]. In this half plane geometry, the lines and the function of distance are different, therefore, it seems interesting to study the Poincaré analogues of the topics that include the concept of distance in the Euclidean geometry. A few of such topics have been studied by some authors [1, 3–8]. In this work, it is shown that the coordinates of the division point can be determined by the formula in the Poincaré upper half plane.

2. Main results

2.1. Internally division point

THEOREM 1. (Law of Sines [5]) *In the hyperbolic triangle ABC let α, β, γ denote the angles at A, B, C, and a, b, c, denote the hyperbolic lengths of the sides opposite A, B, C, respectively, Then,*

$$\frac{\sin\alpha}{\sinh a}=\frac{\sin\beta}{\sinh b}=\frac{\sin\gamma}{\sinh c}.$$

THEOREM 2. *Let $A=(x_1,\ y_1)$ and $B=(x_2,\ y_2)$ be any two distinct points in the half plane* H. *If $C=(x,y)$ divides internally the line segment $[AB]$ in the k, then*

$$C(x,y)=\begin{cases}\left(x_1, e^{\left(k\ln y_2+\ln y_1\right)/(k+1)}\right), & \text{if } C \text{ is on a half line}\\ \left(c\pm\sqrt{r^2-(e^{arc\tanh(L/K)})^2}, e^{arc\tanh(L/K)}\right), & \\ & \text{if } C \text{ is on a half circle}\\ K=\cosh\left(\ln y_1\right)+\frac{k\sin\beta}{\sin\alpha}\cosh\left(\ln y_2\right), & \alpha \text{ is the angle } CBE\\ L=\sinh(\ln y_1)+\frac{k\sin\beta}{\sin\alpha}\sinh(\ln y_2)), & \text{and } \beta \text{ is the angle } ACD.\end{cases}$$

Proof. Case I: If C is any point on a half line ($x=x_1=x_2$), then (Fig. 1)

$$\frac{d_H(A,C)}{d_H(B,C)}=k,\qquad \frac{\left|\ln\frac{y}{y_1}\right|}{\left|\ln\frac{y_2}{y}\right|}=k,\qquad \left|\ln y-\ln y_1\right|=k\cdot\left|\ln y_2-\ln y\right|.$$

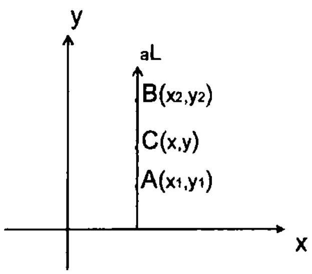

Fig. 1

Either $y_1 \leq y \leq y_2$ or $y_2 \leq y \leq y_1$. In both cases we obtain

$$\ln y - \ln y_1 = k \cdot (\ln y_2 - \ln y)$$
$$\ln y - \ln y_1 = k \cdot (\ln y_2 - \ln y)$$
$$\ln y + k \ln y = k \ln y_2 + \ln y_1$$
$$\ln y = (k \ln y_2 + \ln y_1) \,/\, (k+1)$$
$$y = e^{\left(k \ln y_2 + \ln y_1\right)/(k+1)}$$
$$C = \left(x_1, e^{\left(k \ln y_2 + \ln y_1\right)/(k+1)}\right) = \left(x_2, e^{\left(k \ln y_2 + \ln y_1\right)/(k+1)}\right).$$

Case II: Let C be any point on a half circle $(x_1 \neq x \neq x_2)$ (Fig. 2). If we apply Theorem 1 to triangles BEC and ACD, then

$$\frac{\sin\alpha}{\sinh|EC|} = \frac{\sin 90}{\sinh|CB|} = \frac{\sin(90-\beta)}{\sinh|EB|},$$
$$\frac{\sin\beta}{\sinh|AD|} = \frac{\sin 90}{\sinh|AC|} = \frac{\sin\gamma}{\sinh|CD|}.$$

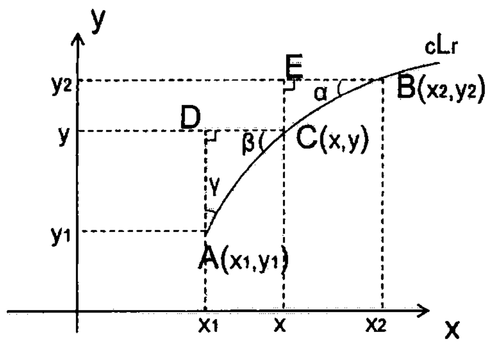

Fig. 2

If these ratios are divided side to side, then

$$\frac{\sin\alpha\sinh|AD|}{\sin\beta\sinh|EC|} = \frac{\sinh|AC|}{\sinh|CB|} = \frac{\sin(90-\beta)\sinh|CD|}{\sin\gamma\sinh|EB|},$$
$$\frac{\sin\alpha\sinh|AD|}{\sin\beta\sinh|EC|} = k,$$
$$\frac{\sin\alpha\sinh\left|\ln\frac{y}{y_1}\right|}{\sin\beta\sinh\left|\ln\frac{y_2}{y}\right|} = k,$$
$$\frac{\sinh|\ln y - \ln y_1|}{\sinh|\ln y_2 - \ln y|} = \frac{k\sin\beta}{\sin\alpha}.$$

Either $y_1 \le y \le y_2$ or $y_2 \le y \le y_1$. In both cases we have

$$\begin{aligned}
&\sinh(\ln y)\cosh(\ln y_1) - \cosh(\ln y)\sinh(\ln y_1) \\
&\qquad = \frac{k\sin\beta}{\sin\alpha}[\sinh(\ln y_2)\cosh(\ln y) - \cosh(\ln y_2)\sinh(\ln y)], \\
&\sinh(\ln y)\cosh(\ln y_1) + \frac{k\sin\beta}{\sin\alpha}\cosh(\ln y_2)\sinh(\ln y) \\
&\qquad - \cosh(\ln y)\sinh(\ln y_1) - \frac{k\sin\beta}{\sin\alpha}\sinh(\ln y_2)\cosh(\ln y) = 0, \\
&\sinh(\ln y)\left[\cosh(\ln y_1) + \frac{k\sin\beta}{\sin\alpha}\cosh(\ln y_2)\right] - \\
&\qquad \cosh(\ln y)\left[\sinh(\ln y_1) + \frac{k\sin\beta}{\sin\alpha}\sinh(\ln y_2)\right] = 0, \\
&K\sinh(\ln y) - L\cosh(\ln y) = 0,
\end{aligned}$$

where

$$\begin{aligned}
K &= \cosh(\ln y_1) + \frac{k\sin\beta}{\sin\alpha}\cosh(\ln y_2), \\
L &= \sinh(\ln y_1) + \frac{k\sin\beta}{\sin\alpha}\sinh(\ln y_2).
\end{aligned}$$

If we divide every term by $L\cosh(\ln y)$, then

$$\begin{aligned}
&\frac{K}{L}\tanh(\ln y) - 1 = 0, \\
&\frac{K}{L}\tanh(\ln y) = 1, \\
&\tanh(\ln y) = \frac{L}{K}, \\
&\ln y = arc\tanh(L/K), \\
&y = e^{arc\tanh(L/K)}.
\end{aligned}$$

Since equation of Poincaré line AB

$$(x-c)^2 + y^2 = r^2,$$

where

$$\begin{aligned}
c &= \left(y_2^2 - y_1^2 + x_2^2 - x_1^2\right)/2(x_2 - x_1), \\
r &= \sqrt{(x_1-c)^2 + y_1^2} = \sqrt{(x_2-c)^2 + y_2^2}.
\end{aligned}$$

then

$$x = c \pm \sqrt{r^2 - y^2}.$$

Thus,

$$x = c \pm \sqrt{r^2 - (e^{arc\tanh(L/K)})^2},$$
$$C = \left(c \pm \sqrt{r^2 - (e^{arc\tanh(L/K)})^2}, e^{arc\tanh(L/K)}\right),$$

where

$$K = \cosh(\ln y_1) + \frac{k \sin\beta}{\sin\alpha} \cosh(\ln y_2),$$
$$L = \sinh(\ln y_1) + \frac{k \sin\beta}{\sin\alpha} \sinh(\ln y_2). \blacksquare$$

2.2. Externally division point

Theorem 3. *Let $A = (x_1,\ y_1)$ and $B = (x_2,\ y_2)$ be any two distinct points in the Poincaŕe upper half plane. If $C = (x, y)$ divides externally the line segment $[AB]$ in the k then,*

$$C(x,y) = \begin{cases} (x_1, e^{(k \ln y_2 - \ln y_1)/(k-1)}), & \text{if } C \text{ is on a half line,} \\ \left(c \pm \sqrt{r^2 - (e^{\arctan h(M/N)})^2}, e^{\arctan h(M/N)}\right), & \\ & \text{if } C \text{ is on a half circle,} \\ (M = \sinh(\ln y_1) - k \sinh(\ln y_2), & \\ N = \cosh(\ln y_1) - k \cosh(\ln y_2)). & \end{cases}$$

Proof. Case I: If C be any point on a half line $(x = x_1 = x_2)$ (Fig. 3) then,

$$\frac{d_H(A,C)}{d_H(B,C)} = k, \qquad \frac{\left|\ln \frac{y}{y_1}\right|}{\left|\ln \frac{y}{y_2}\right|} = k,$$
$$|\ln y - \ln y_1| = k \cdot |\ln y - \ln y_2|,$$

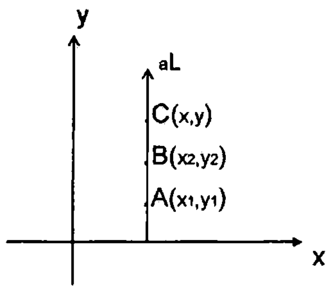

Fig. 3

If $y \geq y_1$, $y \geq y_2$ then,

$$|\ln y - \ln y_1| = k \cdot |\ln y - \ln y_2|,$$
$$\ln y - \ln y_1 = k \cdot (\ln y - \ln y_2),$$
$$k \ln y_2 - \ln y_1 = k \ln y - \ln y,$$
$$\ln y = (k \ln y_2 - \ln y_1) / (k-1),$$
$$y = e^{\left(k \ln y_2 - \ln y_1\right)/(k-1)},$$
$$C = \left(x_1, e^{\left(k \ln y_2 - \ln y_1\right)/(k-1)}\right) = \left(x_2, e^{\left(k \ln y_2 - \ln y_1\right)/(k-1)}\right).$$

If $y < y_1$, $y < y_2$, then it is the same as above.

Case II: Let C be any point on a half circle $(x_1 \neq x \neq x_2)$ (Fig. 4). If we apply Theorem 1 to triangles $BE'C$ and $AD'C$ then,

$$\frac{\sin\beta}{\sinh|BE'|} = \frac{\sin 90}{\sinh|BC|} = \frac{\sin\alpha}{\sinh|CE'|},$$
$$\frac{\sin\beta}{\sinh|AD'|} = \frac{\sin 90}{\sinh|AC|} = \frac{\sin\gamma}{\sinh|CD'|}.$$

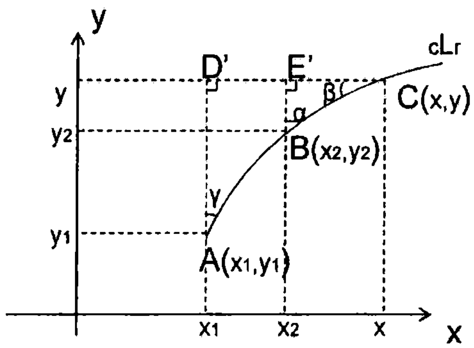

Fig. 4

If these ratios are divided side to side, then

$$\frac{\sinh|AD'|}{\sinh|BE'|} = \frac{\sinh|AC|}{\sinh|BC|} = \frac{\sin\alpha \sinh|CD'|}{\sin\gamma \sinh|CE'|},$$
$$\frac{\sinh|AD'|}{\sinh|BE'|} = k,$$
$$\frac{\sinh\left|\ln\frac{y}{y_1}\right|}{\sinh\left|\ln\frac{y}{y_2}\right|} = k,$$
$$\sinh|\ln y - \ln y_1| = k \sinh|\ln y - \ln y_2|.$$

If $y \geq y_1$, $y \geq y_2$, then

$$\sinh(\ln y)\cosh(\ln y_1) - \cosh(\ln y)\sinh(\ln y_1) = k\sinh(\ln y)\cosh(\ln y_2) - k\cosh(\ln y)\sinh(\ln y_2),$$

$$\sinh(\ln y)\cosh(\ln y_1) - k\sinh(\ln y)\cosh(\ln y_2) - \cosh(\ln y)\sinh(\ln y_1) + k\cosh(\ln y)\sinh(\ln y_2) = 0,$$

$$\frac{\sinh(\ln y)}{\cosh(\ln y)}\left[\cosh(\ln y_1) - k\cosh(\ln y_2)\right] + \frac{\cosh(\ln y)}{\cosh(\ln y)}\left[-\sinh(\ln y_1) + k\sinh(\ln y_2)\right] = \frac{0}{\cosh(\ln y)},$$

$$\tanh(\ln y)\left[\cosh(\ln y_1) - k\cosh(\ln y_2)\right] = \sinh(\ln y_1) - k\sinh(\ln y_2),$$

$$\tanh(\ln y) = \frac{\sinh(\ln y_1) - k\sinh(\ln y_2)}{\cosh(\ln y_1) - k\cosh(\ln y_2)},$$

$$\ln y = \operatorname{arctan} h\left(\frac{\sinh(\ln y_1) - k\sinh(\ln y_2)}{\cosh(\ln y_1) - k\cosh(\ln y_2)}\right),$$

$$y = e^{\operatorname{arctan} h\left(\frac{\sinh(\ln y_1) - k\sinh(\ln y_2)}{\cosh(\ln y_1) - k\cosh(\ln y_2)}\right)}.$$

Since the equation of Poincaré line AB

$$(x-c)^2 + y^2 = r^2,$$

where

$$c = \left(y_2^2 - y_1^2 + x_2^2 - x_1^2\right)/2(x_2 - x_1),$$
$$r = \sqrt{(x_1 - c)^2 + y_1^2} = \sqrt{(x_2 - c)^2 + y_2^2},$$

then

$$x = c \pm \sqrt{r^2 - y^2}.$$

Thus,

$$C = \left(c \pm \sqrt{r^2 - \left(e^{\operatorname{arctan} h(M/N)}\right)^2}, e^{\operatorname{arctan} h(M/N)}\right),$$

where

$$M = \sinh(\ln y_1) - k\sinh(\ln y_2),$$
$$N = \cosh(\ln y_1) - k\cosh(\ln y_2).$$

If $y < y_1$, $y < y_2$, then it is the same as above. ∎

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