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CONVERGENCE OF IMPLICIT ITERATIVE PROCESS FOR A FINITE FAMILY OF I-NONEXPANSIVE MAPPINGS

Abstract. We prove that an implicit iterative process converges weakly and strongly to a common fixed point of a finite family of I-nonexpansive mappings in a Banach space. The results presented in this paper extend and improve the corresponding results of [1, 3, 11, 12]

1. Introduction

Let K be a nonempty subset of uniformly convex Banach space X. Let T be a self-mapping of K. Let $F(T) = \{x \in K : Tx = x\}$ be denoted as the set of fixed points of a mapping T.

We introduce the following definitions and statements which will be used in our main results (see therein references [5], [7], [9]).

A mapping $T : K \longrightarrow K$ is called *nonexpansive* provided

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in K$.

Let $T, I : K \longrightarrow K$. Then T is called I*-nonexpansive* on K if

$$\|Tx - Ty\| \leq \|Ix - Iy\|$$

for all $x, y \in K$.

T is called I*-quasi-nonexpansive* if $F(T) \cap F(I) \neq \emptyset$ and

$$\|Tx - p\| \leq \|Ix - p\|$$

for all $x \in K$ and $p \in F(T) \cap F(I)$.

From the above definitions it follows that if $F(T) \cap F(I)$ is nonempty, a I-nonexpansive mapping must be I-quasi-nonexpansive, and linear I-quasi-nonexpansive mappings are I-nonexpansive mappings. But it is easily seen

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that there exist nonlinear continuous I-quasi-nonexpansive mappings which are not I-nonexpansive.

Recently, concerning the convergence problems of an implicit (or non-implicit) iterative process to a common fixed point for finite family of asymptotically nonexpansive mappings (or nonexpansive mappings) in Hilbert spaces or uniformly convex Banach spaces have been studied by several authors (for example see [1, 3, 11, 12]).

Xu and Ori [11] introduced the following implicit iteration process for a finite family of nonexpansive mappings. Let K be a nonempty closed convex subset of $\mathcal{H}$ Hilbert space. Let $\{T_i\}_{i=1}^{N}$ be N nonexpansive self-maps of K such that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, the set of common fixed points of T_i, $i = 1, \ldots, N$. An implicit iteration process for a finite family of nonexpansive mappings $\{T_i\}_{i=1}^{N}$ is defined as follows, with $\{\alpha_n\} \subset (0,1)$, and an initial point $x_0 \in K$, the sequence $\{x_n\}_{n\geq 1}$ is generated as follows:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + (1-\alpha_1)T_1 x_1\\
x_2 &= \alpha_2 x_1 + (1-\alpha_2)T_2 x_2\\
&\vdots\\
x_N &= \alpha_N x_{N-1} + (1-\alpha_N)T_N x_N\\
x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1})T_{N+1} x_{N+1}\\
&\vdots
\end{aligned}$$

The process is expressed in the following form

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \quad n \geq 1 \tag{1.1}$$

where $T_n = T_{n (\operatorname{mod} N)}$.

Using this iteration process, under some additional condition, Xu and Ori [11] proved the weak convergence of the sequence $\{x_n\}$ defined implicity by (1.1) to a common fixed point of the finite family of nonexpansive mappings defined in Hilbert space.

Zhou and Chang [12] studied the weak and strong convergence of implicit iteration process to a common fixed point for a finite family of nonexpansive mappings in Banach spaces.

Liu [4] and Chidume–Shahzad [1] proved the strong convergence of an implicit iteration process to a common fixed point for a finite family of nonexpansive mappings in Banach spaces.

Gu and Lu [3] studied the weak and strong convergence of implicit iteration process with errors to a common fixed point for a finite family of nonexpansive mappings in Banach spaces.

Let K be a nonempty subset of X Banach space. Let $\{T_i\}_{i=1}^N$ be finite families of I_i nonexpansive self-mappings and $\{I_i\}_{i=1}^N$ be finite families of nonexpansive self-mappings on K. The following implicit iteration process for finite families of I_i- nonexpansive mappings $\{T_i\}_{i=1}^N$ with $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1]$. Then sequence $\{x_n\}_{n\geq 1}$ is generated as follows:

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T_n x_n; \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n I_n y_n \end{cases} \tag{1.2}$$

where $T_n = T_{n(\text{mod}\, N)}, I_n = I_{n(\text{mod}\, N)}$.

Nonexpansive mappings since their introduction have been extensively studied by many authors in different frames work. One is the convergence problem of iterative process of nonexpansive mappings.

The purpose of this paper is to study the weak and strong convergence of implicit iterative sequence $\{x_n\}_{n\geq 1}$ defined by (1.2) to common fixed point for finite family of I_i- nonexpansive mappings in Banach space. We consider also $\{I_i\}_{i=1}^N$ be finite family of nonexpansive self-mappings on K subset of Banach space.

To proceed in this way, we recall some definitions and notations.

2. Preliminaries

Recall that a Banach space X is said to satisfy *Opial's condition* [6] if, for each sequence $\{x_n\}$ in X, the condition $x_n \rightharpoonup x$ implies that

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $y \neq x$. It is well known from [6] that all l_r spaces for $1 < r < \infty$ have this property. However, the L_r space do not have unless $r = 2$.

Lemma 2.1. ([10]) *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real sequences satisfying the following conditions: $\forall n \geq 1$, $a_{n+1} \leq a_n + b_n$, where $\sum_{n=0}^{\infty} b_n < \infty$. Then $\lim_{n\to\infty} a_n$ exists.*

Lemma 2.2. ([8]) *Let K be a nonempty closed bounded convex subset of a uniformly convex Banach space X and $\{\alpha_n\}$ a sequence $[\delta, 1-\delta]$, for some $\delta \in (0,1)$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in K such that*

$$\limsup_{n\to\infty} \|x_n\| \leq c, \qquad \limsup_{n\to\infty} \|y_n\| \leq c, \qquad \limsup_{n\to\infty} \|\alpha_n x_n + (1-\alpha_n) y_n\| = c$$

holds for some $c \geq 0$. Then

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Lemma 2.3. ([2]) *Let X be a uniformly convex Banach space, K a nonempty closed convex subset of X and $T : K \longrightarrow K$ be an nonexpansive mapping. Then $E-T$ is demiclosed at zero, ($T^0 = E$, E denotes the mapping $E : K \to K$ defined by $Ex = x$), i.e. for each sequence $\{x_n\}$ in K, if $\{x_n\}$ converges weakly to $q \in K$ and $(E-T)\{x_n\}$ converges strongly to 0, then $(E-T)q = 0$.*

Definition 2.4.

(1) A mapping $T : K \to K$ is said to satisfy Condition *(A)* if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$, for all $r \in [0,\infty)$ such that $\|x - Tx\| \geq f(d(x,F))$ for all $x \in K$, where $d(x, F(T)) = \inf\{\|x-p\| : p \in F(T)\}$.
For two mappings, Condition (A) can be written as follow.
(2) The mappings $T, I : K \to K$ are said to satisfy Condition *(A′)* if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$, for all $r \in [0,\infty)$ such that $\frac{1}{2}(\|x - Tx\| + \|x - Ix\|) \geq f(d(x,F))$ for all $x \in K$, where $d(x,F) = \inf\{\|x-p\| : p \in F = F(T) \cap F(I)\}$.
(3) A family $\{T_i : i \in \{1,\dots,N\}\}$ of N self-mappings of K with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ is said to satisfy Condition *(B)* on K if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$, for all $r \in [0,\infty)$ such that $\max_{1\leq i\leq N}\{\|x - T_i x\|\} \geq f(d(x,F))$ for all $x \in K$, where $d(x,F) = \inf\{\|x-p\| : p \in F\}$.
(4) Let $\{T_i : i \in \{1,\dots,N\}\}$ be N I_i-nonexpansive self-mappings on K and $\{I_i : i \in \{1,\dots,N\}\}$ be N nonexpansive mappings on K with $F = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i) \neq \emptyset$ are said to satisfy Condition *(B′)* on K if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$, for all $r \in [0,\infty)$ such that $\max_{1\leq i\leq N}\{\frac{1}{2}(\|x - T_i x\| + \|x - I_i x\|)\} \geq f(d(x,F))$ for all $x \in K$, where $d(x,F) = \inf\{\|x-p\| : p \in F\}$.

We need the following useful known lemmas for development of our convergence results.

3. Convergence theorems of implicit iteration process for I-nonexpansive mappings

In this section, we prove strong and weak convergence theorems of the implicit iterative process to common fixed point for a finite family of I-nonexpansive mappings in uniformly convex Banach spaces. We first prove the following lemmas.

Lemma 3.1. *Let X be a uniformly convex Banach space, K be a nonempty closed convex subset of X and let $\{T_i : i \in \{1,\dots,N\}\}$ be N I_i-nonexpansive self-mappings on K and $\{I_i : i \in \{1,\dots,N\}\}$ be N nonexpansive mappings* *on K with $F=\bigcap_{i=1}^{N}F(T_i)\cap F(I_i)\neq\emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}\subset[\delta,1-\delta]$ for some $\delta\in(0,1)$. Suppose that for any given $x\in K$, the sequence $\{x_n\}$ is generated by (1.2). If $F=\bigcap_{i=1}^{N}F(T_i)\cap F(I_i)\neq\emptyset$ then $\lim_{n\to\infty}\|x_n-p\|$ exists.*

Proof. For any $p\in F$

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n I_n y_n+(1-\alpha_n)x_n-p\|\\ &\le (1-\alpha_n)\|x_n-p\|+\alpha_n\|I_n y_n-p\|\\ &\le (1-\alpha_n)\|x_n-p\|+\alpha_n\|y_n-p\|.\end{aligned}\tag{3.1}$$

$$\begin{aligned}\|y_n-p\| &= \|(1-\beta_n)x_n+\beta_n T_n x_n-p\|\\ &\le (1-\beta_n)\|x_n-p\|+\beta_n\|T_n x_n-p\|\\ &\le (1-\beta_n)\|x_n-p\|+\beta_n\|I_n x_n-p\|\\ &\le (1-\beta_n)\|x_n-p\|+\beta_n\|x_n-p\|\\ &\le \|x_n-p\|((1-\beta_n)+\beta_n)\\ &\le \|x_n-p\|.\end{aligned}\tag{3.2}$$

Substituting (3.2) into (3.1), we obtain

$$\begin{aligned}\|x_{n+1}-p\| &\le (1-\alpha_n)\|x_n-p\|+\alpha_n\|x_n-p\|\\ &\le \|x_n-p\|.\end{aligned}\tag{3.3}$$

Thus from (3.3) we obtain

$$\|x_{n+1}-p\|\le\|x_n-p\|.$$

By Lemma 2.1, $\lim_{n\to\infty}\|x_n-p\|$ exists for each $p\in F=\bigcap_{i=1}^{N}F(T_i)\cap F(I_i)$. ∎

Lemma 3.2. *Let X be a uniformly convex Banach space, K be a nonempty closed convex subset of X and let $\{T_i: i\in\{1,\dots,N\}\}$ be N I_i-nonexpansive self-mappings on K and $\{I_i: i\in\{1,\dots,N\}\}$ be N nonexpansive mappings on K with $F=\bigcap_{i=1}^{N}F(T_i)\cap F(I_i)\neq\emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}\subset[\delta,1-\delta]$ for some $\delta\in(0,1)$. Suppose that for any given $x\in K$, define the sequence $\{x_n\}$ by (1.2). Then,*

$$\lim_{n\to\infty}\|T_\ell x_n-x_n\|=\lim_{n\to\infty}\|I_\ell x_n-x_n\|=0,\forall\ell=1,2,\dots,N.$$

Proof. By Lemma 3.1 for any $p\in F=\bigcap_{i=1}^{N}F(T_i)\cap F(I_i)$, $\lim_{n\to\infty}\|x_n-p\|$ exists. Let $\lim_{n\to\infty}\|x_n-p\|=d$. If $d=0$ by continuity of T_ℓ and I_ℓ, then the proof is completed.

Now suppose $d>0$.

Taking lim sup on both sides in (3.2) inequality,

$$\limsup_{n\to\infty}\|y_n-p\|\le d.\tag{3.4}$$

Since $\{I_\ell : \ell \in \{1, \dots, N\}\}$ is N nonexpansive self-mappings on K, we can get that, $\|I_n y_n - p\| \leq \|y_n - p\|$, which on taking $\limsup_{n\to\infty}$ and using (3.4) gives

$$\limsup_{n\to\infty} \|I_n y_n - p\| \leq d.$$

Further,

$$\lim_{n\to\infty} \|x_{n+1} - p\| = d$$

means that

$$\lim_{n\to\infty} \|\alpha_n I_n y_n + (1-\alpha_n)x_n - p\| = d$$

$$\lim_{n\to\infty} \|\alpha_n (I_n y_n - p) + (1-\alpha_n)(x_n - p)\| = d.$$

It follows from Lemma 2.2

$$\lim_{n\to\infty} \|I_n y_n - x_n\| = 0. \tag{3.5}$$

Moreover,

$$\|x_{n+1} - x_n\| = \|\alpha_n [I_n y_n - x_n]\| \leq \alpha_n \|I_n y_n - x_n\|.$$

Thus, from (3.5) we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$$

and

$$\lim_{n\to\infty} \|x_n - x_{n+j}\| = 0, \forall j = 1, \dots, N. \tag{3.6}$$

Now,

$$\begin{aligned} \|x_n - p\| &\leq \|x_n - I_n y_n\| + \|I_n y_n - p\| \\ &\leq \|x_n - I_n y_n\| + \|y_n - p\|, \end{aligned}$$

which on taking $\lim_{n\to\infty}$ implies

$$\begin{aligned} d = \lim_{n\to\infty} \|x_n - p\| &\leq \limsup_{n\to\infty} (\|x_n - I_n y_n\| + \|y_n - p\|) \\ &= \limsup_{n\to\infty} \|y_n - p\| \leq d. \end{aligned}$$

Then we obtain,

$$\limsup_{n\to\infty} \|y_n - p\| = d.$$

Next,

$$\|T_n x_n - p\| \leq \|I_n x_n - p\| \leq \|x_n - p\|.$$

Taking $\lim\limits_{n\to\infty}$ on both sides in the above inequality,

$$\lim_{n\to\infty} \|T_n x_n - p\| \leq \lim_{n\to\infty} \|x_n - p\| = d.$$

Further,

$$\lim_{n\to\infty} \|\beta_n(T_n x_n - p) + (1-\beta_n)(x_n - p)\| = \lim_{n\to\infty} \|y_n - p\| = d.$$

By Lemma 2.2, we have

$$\lim_{n\to\infty} \|T_n x_n - x_n\| = 0. \tag{3.7}$$

We have also,

$$\begin{aligned}
\|I_n x_n - x_n\| &\le \|I_n x_n - I_n y_n\| + \|I_n y_n - x_n\| \\
&\le \|x_n - y_n\| + \|I_n y_n - x_n\| \\
&\le \|x_n - [(1-\beta_n)x_n + \beta_n T_n x_n\| + \|I_n y_n - x_n\| \\
&\le \|\beta_n(T_n x_n - x_n)\| + \|I_n y_n - x_n\| \\
&\le \beta_n\|(T_n x_n - x_n)\| + \|I_n y_n - x_n\|.
\end{aligned}$$

Thus from (3.5) and (3.7), we obtain

$$\lim_{n\to\infty} \|I_n x_n - x_n\| = 0. \tag{3.8}$$

Next,

$$\begin{aligned}
&\|x_n - T_{n+\ell} x_n\| \\
&\quad \le \|x_n - x_{n+\ell}\| + \|x_{n+\ell} - T_{n+\ell} x_{n+\ell}\| + \|T_{n+\ell} x_{n+\ell} - T_{n+\ell} x_n\| \\
&\quad \le \|x_n - x_{n+\ell}\| + \|x_{n+\ell} - T_{n+\ell} x_{n+\ell}\| + \|I_{n+\ell} x_{n+\ell} - I_{n+\ell} x_n\| \\
&\quad \le \|x_n - x_{n+\ell}\| + \|x_{n+\ell} - T_{n+\ell} x_{n+\ell}\| + \|x_{n+\ell} - x_n\|.
\end{aligned}$$

Taking $\lim_{n\to\infty}$ on both sides in the above inequality, then from (3.6) and (3.7) we get

$$\lim_{n\to\infty} \|x_n - T_{n+\ell} x_n\| = 0$$

for all $\ell = \{1, \ldots, N\}$.

Consequently, we have

$$\lim_{n\to\infty} \|x_n - T_\ell x_n\| = 0 \tag{3.9}$$

for all $\ell = \{1, \ldots, N\}$, which lies on the fact that any subsequence of a convergent number sequence converges to the same limit.

Similarly,

$$\begin{aligned}
\|x_n - I_{n+\ell} x_n\| &\le \|x_n - x_{n+\ell}\| + \|x_{n+\ell} - I_{n+\ell} x_{n+\ell}\| + \|I_{n+\ell} x_{n+\ell} - I_{n+\ell} x_n\| \\
&\le \|x_n - x_{n+\ell}\| + \|x_{n+\ell} - I_{n+\ell} x_{n+\ell}\| + \|x_{n+\ell} - x_n\|.
\end{aligned}$$

Taking $\lim\limits_{n\to\infty}$ on both sides in the above inequality, then from (3.7) and (3.8) we get

$$\lim_{n\to\infty} \|x_n - I_{n+\ell} x_n\| = 0$$

for all $\ell = \{1, \dots, N\}$.

Consequently, we have

$$\lim_{n \to \infty} \|x_n - I_\ell x_n\| = 0. \tag{3.10}$$

Then the proof is completed. ■

THEOREM 3.3. *Let X be a uniformly convex Banach space satisfying Opial's condition, K be a nonempty closed convex subset of X and let $\{T_i : i \in \{1, \dots, N\}\}$ be N I_i-nonexpansive self-mappings on K and $\{I_i : i \in \{1, \dots, N\}\}$ be N nonexpansive mappings on K with $F = \bigcap_{i=1}^N F(T_i) \cap F(I_i) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$. Then the implicit iterative sequence $\{x_n\}$ defined by (1.2) converges weakly to an element of F.*

Proof. Let $u \in F$. Then, as in Lemma 3.1, it follows $\lim_{n\to\infty} \|x_n - u\|$ exists and so for $n \geq 1$, $\{x_n\}$ is bounded on K. Since X is uniformly convex, every bounded closed convex subset of X is weakly compact. Let $x \in K$. Then for $u \in \bigcap_{i=1}^N F(T_i)$, we obtain that $D = \{y \in K : \|y - u\| \leq \|x - u\|\}$ is a bounded closed convex subset of K and $x \in D$. Further, for any $y \in D$, we have $T_i y \in K$ and $\|T_i y - u\| \leq \|I_i y - u\| \leq \|y - u\| \leq \|x - u\|$. Then D is invariant under T_i for all i. So, without loss of generality, we may assume that K is bounded. Again, by virtue of boundedness of $\{x_n\}$ in K, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p$ weakly. We assume that $n_k = i(\operatorname{mod} N)$, where i is some positive integer in $\{1, \dots, N\}$. Otherwise, we can take a subsequence $\{x_{n_{k_j}}\} \subset \{x_{n_k}\}$ such that $n_{k_j} = i(\operatorname{mod} N)$. For $\ell \in \{1, \dots, N\}$, there exists an integer $j \in \{1, \dots, N\}$ such that $n_k + j = \ell(\operatorname{mod} N)$. For $\ell \in \{1, \dots, N\}$, from (3.9) and (3.10) we have

$$\lim_{k\to\infty} \|x_{n_k} - T_\ell x_{n_k}\| = 0 \qquad \text{and} \qquad \lim_{k\to\infty} \|x_{n_k} - I_\ell x_{n_k}\| = 0.$$

By Lemma 2.3, for each $\ell \in \{1, \dots, N\}$ we know that $p \in F(T_\ell) \cap F(I_\ell)$. By the arbitrariness of $\ell \in \{1, \dots, N\}$, we have $p \in F = \bigcap_{i=1}^N F(T_i) \cap F(I_i)$. If $F = \bigcap_{i=1}^N F(T_i) \cap F(I_i)$ is a singleton, then the proof is complete. For $p, q \in F = \bigcap_{i=1}^N F(T_i) \cap F(I_i)$, we assume that $F = \bigcap_{i=1}^N F(T_i) \cap F(I_i)$ is not singleton. Suppose $p, q \in w(\{x_n\})$, where $w(\{x_n\})$ denotes the weak limit set of $\{x_n\}$. Let $\{x_{n_k}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ which converge weakly to p and q, respectively. By Lemma 3.2 and Lemma 2.3 guarantees that $p \in \bigcap_{i=1}^N F(T_i)$, $p \in \bigcap_{i=1}^N F(I_i)$ and in the same way $q \in \bigcap_{i=1}^N F(T_i)$ and $q \in \bigcap_{i=1}^N F(I_i)$.

Next, we prove the uniqueness. Assume that $p \neq q$ and $\{x_{n_k}\} \to p$, $\{x_{n_j}\} \to q$. By Opial's condition, we conclude that

$$\lim_{n\to\infty}\|x_n - p\| = \lim_{k\to\infty}\|x_{n_k} - p\| < \lim_{k\to\infty}\|x_{n_k} - q\|$$
$$= \lim_{n\to\infty}\|x_n - q\| = \lim_{j\to\infty}\|x_{n_j} - q\|$$
$$< \lim_{j\to\infty}\|x_{n_j} - p\| = \lim_{n\to\infty}\|x_n - p\|.$$

This is a contradiction. Thus $\{x_n\}$ converges weakly to an element of $F = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i)$.

This completes the proof of Theorem 3.3. ∎

Theorem 3.4. *Let X be a uniformly convex Banach space satisfying Opial's condition, K be a nonempty closed convex subset of X and let $\{T_i : i \in \{1,\dots,N\}\}$ be N I_i- nonexpansive self-mappings on K and $\{I_i : i \in \{1,\dots,N\}\}$ be N nonexpansive mappings on K with $F = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i) \neq \emptyset$. Suppose that $\{T_i : i \in \{1,\dots,N\}\}$ be N I_i-nonexpansive and $\{I_i : i \in \{1,\dots,N\}\}$ be N nonexpansive mappings satisfying Condition (B'). Let $\{\alpha_n\}$ and $\{\beta_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$. Then the implicit iterative sequence $\{x_n\}$ defined by (1.2) converges strongly to an element of F.*

Proof. By Lemma 3.1, for all $p \in F$, $\lim_{n\to\infty}\|x_n - p\|$ exists and

$$\|x_{n+1} - p\| \le \|x_n - p\|$$

for $n \ge 1$. This implies that

$$d(x_{n+1}, F) \le d(x_n, F).$$

So by Lemma 2.1, $\lim_{n\to\infty} d(x_n, F)$ exists and also by Lemma 3.2, for $\ell \in \{1,\dots,N\}$, $\lim_{n\to\infty}\|x_n - T_\ell x_n\| = 0$ and $\lim_{n\to\infty}\|x_n - I_\ell x_n\| = 0$. The condition (B') guarantees that $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is a nondecreasing function and $f(0) = 0$, it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. Next, we show that $\{x_n\}$ is a Cauchy sequence in X. In fact, for any integers m, n, from (3.3), for any $p \in F$, we have

$$\|x_{n+m} - p\| \le \|x_n - p\|.$$

Therefore, since $\lim_{n\to\infty} d(x_n, F) = 0$, for any given $\epsilon > 0$, there exists a positive integer n_0 such that for all $n \ge n_0$, $d(x_n, F) < \frac{\epsilon}{2}$. There exists $p_0 \in F$ such that $\|x_{n_0} - p_0\| < \frac{\epsilon}{2}$.

Hence, for all $n \ge n_0$ and $m \ge 1$, we have

$$\|x_{n+m} - x_n\| \le \|x_{n+m} - p_0\| + \|x_n - p_0\|$$
$$\le \|x_{n_0} - p_0\| + \|x_{n_0} - p_0\|$$
$$\le 2\|x_{n_0} - p_0\| \le 2\frac{\epsilon}{2} = \epsilon,$$

which shows that $\{x_n\}$ is a Cauchy sequence in X.

Thus, the completeness of X implies that $\{x_n\}$ is convergent. Assume that $\{x_n\}$ converges to a point p.

Then $p \in K$, because K is closed subset of X. The set $F = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i)$ is closed. $\lim_{n\to\infty} d(x_n, F) = 0$ gives that $d(p, F) = 0$.

Thus $p \in F = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i)$. This completes the proof. ∎

References

[1] C. E. Chidume, N. Shahzad, *Strong convergence of an implicit iteration process for a finite family of nonexpansive mappings*, Nonlinear Anal. 62 6 (2005), 1149–1156.

[2] J. Gornicki, *Weak convergence theorems for asymptotically nonexpansive mappings in uniformly Banach spaces*, Comment. Math. Univ. Carolin. 30 (1989) 2, 249–252.

[3] F. Gu, J. Lu, *A new composite implicit iteration process for a finite family of non-expansive mappings in Banach spaces*, Fixed Point Theory and Appl. (2006), Article ID 82738, 1–11.

[4] J. Ai Liu, *Some convergence theorems of implicit iterative process for nonexpansive mappings in Banach spaces*, Mathematical Communications 7 (2002), 113–118.

[5] W. V. Petryshyn, T. E. Williamson, *Strong and weak convergence of the sequence of successive approximations for quasi-nonexpansive mappings*, J. Math. Anal. Appl. 43 (1973), 459–497.

[6] Z. Opial, *Weak convergence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. 73 (1967), 591–597.

[7] B. E. Rhoades, S. Temir, *Convergence theorems for I-nonexpansive mapping*, IJMMS, (2006), Article ID 63435, 1–4.

[8] J. Schu, *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bull. Austral. Math. Soc. 43 (1991), no. 1, 153–159.

[9] S. Temir, O. Gul, *Convergence theorem for I-asymptotically quasi-nonexpansive mapping in Hilbert space*, J. Math. Anal. Appl. 329 (2007), 759–765.

[10] K. K. Tan, H. K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iterative process*, J. Math. Anal. Appl. 178 (1993), 301–308.

[11] H. K. Xu, R. G. Ori, *An implicit iteration process for nonexpansive mappings*, Numer. Funct. Anal. Optim. 22 (2001), no. 5–6, 767–773.

[12] Y. Zhou, S. S. Chang, *Convergence of implicit iteration process for a finite family of asymptotically nonexpansive mappings in Banach spaces*, Numer. Funct. Anal. Optim. 23 (2002), 911–921.

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