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POROSITY OF CERTAIN CLASSES OF OPERATORS IN GENERALIZED METRIC SPACES

Abstract. We study the porosity behavior of non-contractive mappings in a generalized metric space, a concept recently introduced in [1]. We also investigate partially the porosity position of a certain class of operators whose condition arises from [8].

1. Introduction

In recent years several authors ([6], [11], [12]) have investigated the category and more importantly porosity position of contraction or contractive mappings in relation to the class of non-expansive mappings. As it is well known that non-expansive mappings in general may not have fixed points, whereas contractive mappings [10] have unique fixed points, the study in [12] actually showed that almost all non-expansive mappings (in the sense of porosity) are contractive [10] and so have unique fixed points. A similar type of investigation was also carried out in [9] for a different class of operators.

In 2000 Branciari [1] introduced a very interesting generalization of a metric space called 'generalized metric space' by replacing the triangle inequality by a more general inequality. As such every metric space is a generalized metric space but the converse is not true (see [1], [5]). However some very important fixed point theorems namely, Banach's fixed point theorem, Ciric's fixed point theorem and very recently Boyd and Wong's fixed point theorem have been proved in such spaces in [1], [8] and [5] respectively.

Encouraged by observations of [8] and [5], in this paper we try to investigate in the line of [12] and show that almost all non-expansive mappings in this more general structure are also contractive [5] under certain general conditions. As in [9] we also investigate the porosity problem for a class of operators whose condition arises from the idea of quasi-contraction [8]. As

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in [1], [8] or [5], due to the absence of triangle inequality, the methods of proofs do not appear to be analogous.

2. Preliminaries

Let R^+ denote the set of all non-negative real numbers and N the set of positive integers.

Definition 1. (cf. [1]) Let X be a set and $d : X^2 \to R^+$ be a mapping such that for all $x, y \in X$ and for all distinct points $z_1, z_2, \dots, z_k \in X (k \geq 2)$ each of them different from x and y, one has

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \leq d(x, z_1) + d(z_1, z_2) + \cdots + d(z_k, y)$.

Then we will say that (X, d) is a generalized metric space (or shortly g.m.s). Throughout this section a g.m.s will be denoted by (X, d) (or sometime by X only).

Any metric space is a g.m.s but the converse is not true ([1]).

In [1] it was claimed that as in a metric space, a topology can be generated in a g.m.s X with the help of the neighborhood basis given by $B = \{B(x, r) : x \in X, r \in R^+ \setminus \{0\}\}$ where $B(x, r) = \{y \in X : d(x, y) < r\}$ is the open ball with centre x and radius r.

However in [5] it was shown through two examples that the topological structure of a g.m.s is somewhat different from a metric space. The following examples were given in [5] which we reproduce here for easy reference.

Example 1. Let us define $X = \{\frac{1}{n} : n = 1, 2, \dots\} \cup \{0\}$,

$$d : X \times X \to R^+, \qquad d(x, y) = \begin{cases} 0 & \text{for } x = y \\ \frac{1}{n} & \text{for } \{x, y\} = \{0, \frac{1}{n}\} \\ 1 & \text{for } x \neq y, x, y \in X \setminus \{0\}. \end{cases}$$

Note that (X, d) satisfies axioms of a generalized metric space, i.e. for all $x, y \in X$

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \leq d(x, z_1) + d(z_1, z_2) + d(z_2, y)$, for distinct points x, y, z_1, z_2.

Observe that $B(\frac{1}{3}, \frac{1}{2}) \cap B(\frac{1}{4}, \frac{1}{2}) = \{0\}$ and hence there is no $r > 0$ with $B(0, r) \subset B(\frac{1}{3}, \frac{1}{2}) \cap B(\frac{1}{4}, \frac{1}{2})$. Therefore the family $\{B(x, r) : x \in X, r > 0\}$ is not a neighborhood basis for any topology on X.

In view of the above example, it seems more reasonable to construct the topology in a g.m.s X by taking the collection B as a sub basis. Further it can be observed from Example 1 that $\lim_{n\to\infty} d(\frac{1}{2},\frac{1}{n}) = 1$ whereas $d(\frac{1}{2},0) = \frac{1}{2} \neq 1$ which shows that d is not continuous in a sense presented in [1].

Consider now the following example

EXAMPLE 2. Let us define $Y = \{\frac{1}{n} : n = 1, 2, ...\} \cup \{0, 2\}$,

$$d_1 : Y \times Y \to R^+, \quad d_1(x,y) = \begin{cases} 0 & \text{for } x = y \\ \frac{1}{n} & \text{for } x \in \{0,2\},\ y = \frac{1}{n} \\ \frac{1}{n} & \text{for } x = \frac{1}{n},\ y \in \{0,2\} \\ 1 & \text{otherwise.} \end{cases}$$

Note that (Y, d_1) is a g.m.s in which the points 0 and 2 do not have any disjoint open balls.

All this points out to the fact that a g.m.s (which is not a metric space) may sometimes be perceived as a much weaker structure than a metric space due to weakening of the triangle inequality. The results of [1], [5] and [8], in a sense, prove the existence of fixed points of contraction mappings, contractive mappings or quasi-contraction mappings in more general spaces.

We also reproduce the following Definition and Theorem from [5] for easy reference.

DEFINITION 2. (cf. [5]) A mapping $T : X \to X$ is said to be contractive if for any two distinct points $x, y \in X$, $d(Tx, Ty) < d(x, y)$.

THEOREM 1. *Let X be a complete g.m.s and let $T : X \to X$ satisfy*

$$d(Tx, Ty) \leq \psi(d(x, y))$$

where $\psi : \overline{P} \to [0, \infty)$ is upper semi-continuous from right on $\overline{P}$ (the closure of the range of d) and satisfes $\psi(t) < t$ for all $t \in \overline{P} \setminus \{0\}$. Then T has a unique fixed point x_0 and $T^n x \to x_0$ for each $x \in X$.

REMARK 1. As in [2] we note that if we take $\psi(t) = \alpha(t)t$ where α is a decreasing function and $\alpha(t) < 1$ for $t > 0$ then we can obtain the Rakotch's fixed point theorem [10] for contractive mappings $T : X \to X$ satisfying the condition

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y) \qquad \forall\, x, y \in X$$

where α is a mapping as defined above.

3. Porosity of non-contractive mappings

In [11] and [12] it was established that the contractive mappings play a very prominent role in the theory of fixed points in normed linear spaces where it was shown that the collection of non-contractive mappings is a σ-porous set (and so a set of first category) in the collection of non-expansive mappings. This result means that almost all non-expansive maps have unique fixed points which is a remarkable observation. In order to investigate the same in our context we need to introduce some more concepts.

So analogous to the idea of a generalized metric space we first introduce the following notion of a generalized normed linear space.

Definition 3. A generalized normed linear space is a vector space with a generalized norm defined on it. We define a generalized norm defined on a real or complex vector space X as a real valued function on X whose value at an $x \in X$ is denoted by $\|x\|$ and which has the following properties

(i) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = \theta$,
(ii) $\|\lambda x\| = |\lambda|\|x\|$, λ is a scalar,
(iii) $\|x + z_1 + z_2 + \cdots + z_k + y\| \leq \|x\| + \|z_1\| + \cdots + \|z_k\| + \|y\|$ when $x, y, z_1, \ldots, z_k \neq \theta$.

Now if we define a function $d : X^2 \to R^+$ as

$$(A) \qquad d(x,y) = \|x - y\|$$

then it can be easily verified that this d becomes a generalized metric on X.

Further by a generalized Banach space we mean a generalized normed space which is complete with respect to the induced generalized metric defined by (A).

Assume that $(X, \|.\|)$ is a generalized Banach space with the additional condition

$(\star)$ there exists a positive integer $k_0 > 1$ such that

$$\|x - z\| \leq k_0\|x - y\| + \|y - z\|$$

and

$$\|x - z\| \leq \|x - y\| + k_0\|y - z\|, \quad \forall\ x, y, z \in X.$$

Remark 2. Clearly every Banach space is a generalized Banach space satisfying the additional condition $(\star)$ with $k_0 = 1$. Also examples of generalized metric spaces satisfying the condition $(\star)$ can be easily constructed. In fact the generalized metric space given in [1] is such a space. However we are unable to construct an example of a generalized Banach space which is not a Banach space satisfying the above condition and we leave it as an open problem.

Let K be a bounded closed convex subset of X. Denote by A_1 the set of all non-expansive mappings T on K i.e. $T : K \to K$ is such that

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in K.$$

Set $d(K) = Sup\{\|x - y\| : x, y \in K\}$.

We define a function h defined on $A_1 \times A_1$ as

$$h(S,T) = Sup\{\|Tx - Sx\| : x \in K\}.$$

Then h obviously satisfies the following properties

(i) $h(S,T) \geq 0$ and $h(S,T) = 0$ iff $S = T$,
(ii) $h(S,T) = h(T,S)$,
(iii) $h(S,T) \leq h(S,S_1) + h(S_1,T_1) + h(T_1,T)$,

if S_1, T_1 are different and also different from S and T (at all $x \in X$). However in the view of the condition ($\star$) it can be shown that

($\star\star$) $h(S,T) \leq h(S,S_1) + k_0 h(S_1,T)$ and $h(S,T) \leq k_0 h(S,S_1) + h(S_1,T)$

where $S, S_1, T \in A_1$.

Now as in a metric space or a g.m.s open balls can be defined in (A_1, h). Observe here that because of condition ($\star\star$), it can be easily proved that given any two open balls $B(S,p)$ and $B(T,s)$, $T_1 \in B(S,p) \cap B(T,s)$ implies $B(T_1,t) \subset B(S,p) \cap B(T,s)$ where $t = \frac{\min\{p-h(S,T_1),\ s-h(T_1,T)\}}{2K_0}$. Hence as in a normed space (or a metric space) the open balls form a base of a topology on (A_1, h). We can then introduce the following concepts of porosity (cf. [13], [15], [16], [17]) on (A_1, h) as follows.

Denote by $B(S,p)$ the open ball with centre $S \in A_1$ and radius $p > 0$. Let $M \subseteq A_1$. Let

$$\gamma(S,p,M) = Sup\{t > 0 : \text{there is a } T \in B(S,p) \text{ such that } B(T,t) \subseteq B(S,p) \text{ and } M \cap B(T,t) = \phi\}.$$

Note that if M is dense in A_1 then $\gamma(S,p,M) = -\infty$. Let

$$\bar{\rho}(S,M) = \lim_{p \to 0^+} \sup \frac{\gamma(S,p,M)}{p}$$

$$\underline{\rho}(S,M) = \lim_{p \to 0^+} \inf \frac{\gamma(S,p,M)}{p}$$

and if $\bar{\rho}(S,M) = \underline{\rho}(S,M)$ then we set

$$\rho(S,M) = \bar{\rho}(S,M) = \underline{\rho}(S,M) = \lim_{p \to 0^+} \sup \frac{\gamma(S,p,M)}{p}.$$

A subset M of A_1 is said to be porous at $S \in A_1$ if $\bar{\rho}(S,M) > 0$ and σ-porous at $S \in A_1$ if it is a countable union of porous subsets in (A_1, h). M is called porous or σ-porous in $\acute{A} \subseteq A_1$ if it is so at each $S \in \acute{A}$.

We also introduce the following definitions which will be needed in the next section. The set M is said to be very porous at $S \in \acute{A}$ if $\underline{\rho}(S,M) > 0$ and very strongly porous at $S \in \acute{A}$ if $\underline{\rho}(S,M) = 1$. Also M is said to be uniformly very porous in $\acute{A} \subseteq A_1$ if there is a $c > 0$ such that for each $S \in \acute{A}$

we have $\underline{\rho}(S, M) \geq c$. M is said to be uniformly σ-very strongly porous in $\acute{A} \subseteq A_1$ if $M = \bigcup_{n=1}^{\infty} M_n$ and each M_n is very strongly porous at each $S \in \acute{A}$.

We say that a mapping $T \in A_1$ is contractive (following Rakotch [10]) if there exists a decreasing function $\phi^T : [0, d(K)] \to [0, 1]$ such that $\phi^T(t) < 1$ for all $t \in (0, d(K)]$ and $\|Tx - Ty\| \leq \phi^T(\|x - y\|)\|x - y\|$ for all $x, y \in K$.

Now we prove our main theorem in this section which shows that almost all non-expansive mappings (in the sense of porosity) have fixed points.

Theorem 2. *There exists a set $F \subseteq A_1$ such that $A_1 \setminus F$ is σ-porous in (A_1, h) and each $T \in F$ is contractive.*

Proof. For each natural number n denote by M_n the set of all $T \in A_1$ which have the following property

$(P1)$ there exists $c \in (0, 1)$ such that $\|Tx - Ty\| \leq c\|x - y\|$ for all $x, y \in K$ satisfying $\|x - y\| \geq \frac{d(K)}{2n}$.

Let $n \geq 1$ be an integer. We shall show that $A_1 \setminus M_n$ is porous in (A_1, h). Set

$$\alpha = \frac{\min\{d(K), 1\}}{8{k_0}^2(2n)[d(K) + 1]}. \tag{1}$$

Fix $\theta \in K$ and let $T \in A_1$ and $r \in (0, 1]$. Set

$$\gamma = \frac{r}{2k_0[d(K) + 1]} \quad \text{and} \quad \delta = \frac{r}{4k_0[d(K) + 1]} \tag{2}$$

and define

$$T_\gamma x = (1 - \gamma)Tx + \gamma\theta \quad \text{and} \quad T_\delta x = (1 - \delta)Tx + \delta\theta, \qquad \text{for all } x \in K.$$

Clearly T_γ and $T_\delta \in A_1$ and $h(T_\gamma, T) \leq \gamma d(K)$ and

$$\begin{aligned} h(T_\delta, T_\gamma) &= Sup\{\|\delta Tx - \gamma Tx + \theta(\gamma - \delta)\| : x \in K\} \\ &= (\gamma - \delta) Sup\{\|Tx - \theta\| : x \in K\} \\ &\leq (\gamma - \delta)d(K) \\ &\leq \frac{r}{4k_0}. \end{aligned} \tag{3}$$

We also note that for all $x, y \in K$

$$\|T_\delta x - T_\delta y\| \leq (1 - \delta)\|Tx - Ty\| \leq (1 - \delta)\|x - y\|. \tag{4}$$

Assume that $S \in A_1$ and

$$h(S, T_\delta) \leq \alpha r. \tag{5}$$

We will show that $S \in M_n$. Let $x, y \in K$ and

$$\|x - y\| \geq \frac{d(K)}{2n}. \tag{6}$$

Now from (4) and (6)

$$\|x-y\| - \|T_\delta x - T_\delta y\| \geq \delta\|x-y\| \geq \frac{\delta d(K)}{2n}. \tag{7}$$

Next we consider the following two cases.

Case I. If Sx, $T_\delta x$, $T_\delta y$ and Sy are all distinct then

$$\begin{aligned}\|Sx - Sy\| &\leq \|Sx - T_\delta x\| + \|T_\delta x - T_\delta y\| + \|T_\delta y - Sy\| \\ &\leq 2\alpha r + \|T_\delta x - T_\delta y\| \qquad \text{(using (5))}.\end{aligned}$$

Case II. If $Sx = T_\delta x$ or $T_\delta y = Sy$ then by the condition $(\star)$ and using (5), we have,

$$\begin{aligned}\|Sx - Sy\| &\leq \|T_\delta x - T_\delta y\| + k_0\|T_\delta y - Sy\| \\ &\leq k_0\alpha r + \|T_\delta x - T_\delta y\|\end{aligned}$$

or

$$\begin{aligned}\|Sx - Sy\| &\leq \|T_\delta x - T_\delta y\| + k_0\|Sx - T_\delta x\| \\ &\leq k_0\alpha r + \|T_\delta x - T_\delta y\|.\end{aligned}$$

Since $k_0 > 1$, combining both the cases, we have,

$$\|Sx - Sy\| \leq k_0\alpha r + \|T_\delta x - T_\delta y\|.$$

Now from (7), (2) and (1) it follows that

$$\begin{aligned}\|x-y\| - \|Sx - Sy\| &\geq \|x-y\| - \|T_\delta x - T_\delta y\| - k_0\alpha r \\ &\geq \frac{\delta d(K)}{2n} - k_0\alpha r \\ &= \frac{rd(K)}{4k_0[d(K)+1]2n} - k_0\alpha r \\ &= \frac{r}{4k_0}\left[\frac{d(K)}{[d(K)+1]2n} - 4k_0{}^2\alpha\right] \\ &= \frac{r}{4k_0}\left[\frac{d(K)}{[d(K)+1]2n} - \frac{\min\{d(K),1\}}{4n[d(K)+1]}\right] \\ &\geq \frac{r}{4k_0}\frac{d(K)}{[d(K)+1]4n}.\end{aligned}$$

Thus

$$\|Sx - Sy\| \leq \|x-y\| - \frac{rd(K)}{16k_0 n[d(K)+1]} \leq \|x-y\|\left(1 - \frac{r}{16k_0 n[d(K)+1]}\right).$$

Since this holds for all $x, y \in K$ satisfying (6), we conclude that $S \in M_n$. Thus we have shown that

$$\{S \in A_1 : h(S, T_\delta) \leq \alpha r\} \subseteq M_n. \tag{8}$$

Next if $T_\delta \neq S$, $\forall x$ and if $S \in A_1$ satisfying (5), then by (3), (1) and (2), we have,

$$\begin{aligned} h(S,T) &\leq h(T,T_\gamma) + h(T_\gamma, T_\delta) + h(T_\delta, S) \\ &\leq \gamma d(K) + \frac{r}{4k_0} + \alpha r \\ &< \frac{r}{2k_0} + \frac{r}{4} + \frac{r}{16} < r. \end{aligned}$$

Also, if $T_\delta = S$, for some x then

$$h(S,T) \leq h(S,T_\gamma) + k_0 h(T_\gamma, T) \leq \frac{r}{4k_0} + k_0\gamma d(K) < r.$$

Thus $\{S \in A_1 : h(S,T_\delta) \leq \alpha r\} \subseteq \{S \in A_1 : h(S,T) < r\}$. When combined with (8), this inclusion implies that $A_1 \setminus M_n$ is porous in (A_1, h). Set $F = \bigcap_{n=1}^{\infty} M_n$. Clearly $A_1 \setminus F$ is σ-porous in (A_1, h). By property $(P1)$ each $T \in F$ is contractive and hence the proof follows.

4. Porosity of a certain class of operators

In this section we investigate a similar type of problem for a different class of mappings whose condition arises from quasi-contraction maps (see [8], [3]). Such an investigation has already been done in a metric space by one of the authors [9]. We do the same here in the more general structure of a g.m.s where due to the absence of triangle inequality the methods of proofs do not appear to be analogous.

We consider the following classes of operators. Denote by A_2 the set of all mappings $T : K \to K$ such that

$$\|Tx - Ty\| \leq \max\{\|x - Tx\|, \|y - Ty\|\} = m_T(x,y) \quad \text{for all } x, y \in X,$$

where as before K is a closed bounded convex subset of a generalized Banach space $(X, \|.\|)$. Now we equip A_2 with the same function h as in A_1.

Let B be the collection of all those $T \in A_2$ such that $\|Tx - Ty\| \leq c(T) m_T(x,y)$ for all $x, y \in K$ where $0 < c(T) < 1$ and $c(T)$ is a constant depending on T only. In view of [6] every member of B has a unique fixed point. As in the previous section we now intend to study the porosity behavior of B or $A_2 \setminus B$ in A_2.

We also recall that if (Y,d) is a metric space and $A \subseteq Y$ be an F_σ-set in Y then A is uniformly σ-very strongly porous in $Y \setminus A$ (see [13]). Now it can be easily proved that a similar result also holds in (A_2, h).

Lemma 1. *B is an F_σ-set in (A_2, h).*

Proof. Clearly $B = \bigcup_{r \in \Delta} B_r$, where Δ is an enumeration of the set of all rationals in $(0,1)$ and

$$B_r = \{T \in A_2 : \|Tx - Ty\| \leq r.m_T(x,y) \quad \text{for all} \quad x, y \in K\}.$$

To prove that B is a F_σ-set, we have to show that for a fixed $r \in \Delta$, B_r is closed. Let $T_n \to T$ as $n \to \infty$, where $T_n \in B_r$ for all n. Now if Tx, T_nx, T_ny and Ty are all distinct then

$$\begin{aligned}\|Tx - Ty\| &\le \|Tx - T_nx\| + \|T_nx - T_ny\| + \|T_ny - Ty\| \\ &\le \|Tx - T_nx\| + \|T_ny - Ty\| + r.m_{T_n}(x,y) \\ &\le \|Tx - T_nx\| + \|T_ny - Ty\| \\ &+ r.\max\{\|x - Tx\| + k_0\|Tx - T_nx\|, \|y - Ty\| + k_0\|Ty - T_ny\|\} \\ &\le r.m_T(x,y) + (1 + k_0 r)(\|Tx - T_nx\| + \|Ty - T_ny\|).\end{aligned}$$

Since $\{T_n\}$ converges to T, then it follows that

$$\|Tx - Ty\| \le r.m_T(x,y) \quad \text{i.e. } T \in B_r.$$

Also, if $Tx = T_nx$ or $Ty = T_ny$ we have by the condition $(\star)$,

$$\|Tx - Ty\| \le \|T_nx - T_ny\| + k_0\|T_ny - Ty\|$$

or

$$\|Tx - Ty\| \le \|T_nx - T_ny\| + k_0\|T_nx - Tx\|.$$

Since $\{T_n\}$ converges to T, we have again in this case also $\|Tx - Ty\| \le r.m_T(x,y)$ *i.e.* $T \in B_r$. This shows that in any case B_r is closed and this completes the proof.

Hence we have the following result.

THEOREM 3. *The set B is uniformly σ-very strongly porous in $A_2 \setminus B$.*

Now, for $\delta > 0$ denote by B_δ the collection of all $T \in A_2$ for which there exists a constant $c(T)$, $0 < c(T) < 1$ such that

$$(B) \qquad \|Tx - Ty\| \le c(T).m_T(x,y) + \delta \quad \text{for all } x, y \in K.$$

Then $B \subseteq B_\delta \subseteq A_2$.

As a corollary to the next theorem, we observe that most of the mapping of A_2 are of the form (B).

THEOREM 4. *For any $\delta > 0$, $A_2 \setminus B_\delta$ is uniformly very porous in (A_2, h).*

Proof. Let $T \in A_2$ and $r \in (0,1]$. Fix $\theta \in K$ and set

$$\alpha = \frac{r}{k_0(d(K)+1)2^m} \quad \text{and} \quad \beta = \frac{r}{2k_0(d(K)+1)2^m}$$

where m is a natural number such that $2^{-m} < \frac{\delta}{2}$.

Now define T_α and T_β by

$$T_\alpha x = (1-\alpha)Tx + \alpha\theta \quad \text{and} \quad T_\beta x = (1-\beta)Tx + \beta\theta \quad \text{for all } x \in K.$$

Then $T_\alpha, T_\beta : K \to K$ and for all $x, y \in K$,

$$\|T_\alpha x - T_\alpha y\| = (1-\alpha)\|Tx - Ty\| \le (1-\alpha).m_T(x,y)$$
$$\|T_\beta x - T_\beta y\| = (1-\beta)\|Tx - Ty\| \le (1-\beta).m_T(x,y).$$

But

$$\begin{aligned}\|x - Tx\| &\le \|x - T_\alpha x\| + \|T_\alpha x - T_\beta x\| + \|T_\beta x - T_x\| \\ &\le \|x - T_\alpha x\| + (\alpha-\beta)\|Tx - \theta\| + \beta\|Tx - \theta\| \\ &\le \|x - T_\alpha x\| + \alpha d(K).\end{aligned}$$

Similarly, $\|y - Ty\| \le \|y - T_\alpha y\| + \alpha d(K)$. Therefore, $m_T(x,y) \le m_{T_\alpha}(x,y) + \alpha d(K)$. Thus

$$\begin{aligned}\|T_\alpha x - T_\alpha y\| &\le (1-\alpha)m_{T_\alpha}(x,y) + \alpha(1-\alpha)d(K) \\ &\le (1-\alpha)m_{T_\alpha}(x,y) + \alpha d(K) \\ &\le (1-\alpha)m_{T_\alpha}(x,y) + \frac{\delta}{2k_0} \\ &< (1-\alpha)m_{T_\alpha}(x,y) + \frac{\delta}{2} \quad \text{(since } k_0 > 1).\end{aligned}$$

Since $0 < 1-\alpha < 1$, $T_\alpha \in B_\delta$. Similarly

$$\begin{aligned}\|T_\beta x - T_\beta y\| &\le (1-\beta)m_{T_\beta}(x,y) + \beta(1-\beta)d(K) \\ &\le (1-\beta)m_{T_\beta}(x,y) + \beta d(K) \\ &\le (1-\beta)m_{T_\beta}(x,y) + \frac{\delta}{4k_0} \\ &< (1-\beta)m_{T_\beta}(x,y) + \frac{\delta}{2} \quad \text{(since } k_0 > 1).\end{aligned}$$

Hence from the relation $0 < 1-\beta < 1$, we conclude that $T_\beta \in B_\delta$.

Also, $h(T, T_\alpha) < \alpha d(K) < \frac{r}{2k_0} < \frac{r}{2}$.

Further we choose the positive integer m in such a way that $2^{-m} < \frac{\delta}{8(k_o+1)}$.

Let $S \in A_2$ be such that $h(T_\alpha, S) < \gamma r$ where $\gamma = (k_0+1)2^{-m} < \frac{\delta}{8}$. Since for any $x \in K$

$$\|x - T_\alpha x\| \le \|x - Sx\| + k_0\|Sx - T_\alpha x\| \le \|x - Sx\| + k_0\gamma r$$

and

$$\|y - T_\alpha y\| \le \|y - Sy\| + k_0\|Sy - T_\alpha y\| \le \|y - Sy\| + k_0\gamma r$$

we have, $m_{T_\alpha}(x,y) \le m_S(x,y) + k_0\gamma r$. Hence

$$\begin{aligned}\|Sx - Sy\| &\le \|Sx - T_\alpha x\| + \|T_\alpha x - T_\alpha y\| + \|T_\alpha y - Sy\| \\ &\le (1-\alpha)m_{T_\alpha}(x,y) + \frac{\delta}{2} + 2\gamma r\end{aligned}$$

$$\begin{aligned}&\leq (1-\alpha)m_S(x,y) + 2\gamma r + (1-\alpha)k_0\gamma r + \frac{\delta}{2}\\&\leq (1-\alpha)m_S(x,y) + \frac{\delta}{4} + \frac{\delta}{8} + \frac{\delta}{2}\\&\leq (1-\alpha)m_S(x,y) + \delta\end{aligned}$$

if Sx, $T_\alpha x$, $T_\alpha y$, and Sy are not distinct then some easy calculations will again prove the same result.

This shows that $\{S \in A_2 : h(T_\alpha, S) \leq \gamma r\} \subseteq B_\delta$. Also we have, if S, T_α, T, and T_β are all distinct at all $x \in K$, then

$$\begin{aligned}h(S,T) &\leq h(S,T_\alpha) + h(T_\alpha,T_\beta) + h(T_\beta,T)\\&\leq \gamma r + (\alpha-\beta)d(K) + \beta d(K)\\&\leq \gamma r + \alpha d(K) < r.\end{aligned}$$

Also if $Sx = T_\alpha x$ for some $x \in K$ then, we have,

$$\begin{aligned}h(S,T) &\leq h(S,T_\beta) + k_0 h(T_\beta,T)\\&\leq (\alpha-\beta)d(K) + k_0\beta d(K)\\&< k_0(\alpha-\beta)d(K) + k_0\beta d(K)\\&= k_0\alpha d(K) < r.\end{aligned}$$

Therefore $\{S \in A_2 : h(S,T_\alpha) \leq \gamma r\} \subseteq \{S \in A_2 : h(S,T) \leq r\}$. This proves that $A_2 \setminus B_\delta$ is uniformly very porous in (A_2, h). This completes the proof of the theorem.

Concluding remark. Since porous sets should be nowhere dense (in a metric space), we could only use the well known definitions of porosity in a more general structure of a generalized metric space when it satisfies an additional condition $(\star)$. We are not sure whether the condition $(\star)$ is essential for introducing the notion of porosity, as it exists in the literature. However it appears that, as the topology on a generalized metric space (without any additional property) is often generated by finite intersection of open balls only (see Example 1), this may cause that the porosity (as defined here in accordance with the literature) in a g.m.s may not imply nowhere dense in the induced topology. Under the circumstances the following open questions seems natural.

Problem 1. Investigate the category position of non-contractive mappings in a generalized metric space or a normed space without any additional condition.

Problem 2. Define the notion of porosity as an extension of nowhere dense sets in a generalized metric space.

PROBLEM 3. Investigate the porosity position of non-contractive mappings in a generalized metric space without any condition or under some condition which is weaker than $(\star)$.

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