DEMONSTRATIO MATHEMATICA
Vol. XLII No 1 2009

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SOME RESULTS ON COMMON FIXED POINTS FOR WEAKLY COMPATIBLE MAPPINGS SATISFYING ALTMAN INTEGRAL TYPE CONTRACTION

Abstract. In this paper, a common fixed point theorem for two pairs of weakly compatible mappings satisfying Altman integral type contraction in a metric space is proved. Our result extends and improves several known results.

1. Introduction

Let A and S be two self-maps of a metric space (X, d). Sessa [14] defined A and S to be *weakly commuting* if $d(ASx, SAx) \leq d(Ax, Sx)$, for all $x \in X$. Jungck [5] defined A and S to be *compatible* if $\lim_n d(ASx_n, SAx_n) = 0$, whenever there exists a sequence $\{x_n\}$ in X such that $\lim_n Sx_n = \lim_n Ax_n = t$, for some $t \in X$. Pathak et al. [9] defined A and S to be *compatible of type (P)* if $\lim_n d(AAx_n, SSx_n) = 0$, whenever there exists a sequence $\{x_n\}$ in X such that $\lim_n Sx_n = \lim_n Ax_n = t$, for some $t \in X$. On the other hand, Jungck et al. [7] defined A and S to be *compatible of type (A)* if $\lim_n d(ASx_n, SSx_n) = \lim_n d(SAx_n, AAx_n) = 0$, whenever there exists a sequence $\{x_n\}$ in X such that $\lim_n Sx_n = \lim_n Ax_n = t$, for some $t \in X$. Clearly, commuting mappings are weakly commuting and weakly commuting mappings are compatible but neither implication is reversible. In 1998, Jungck [6] defined A and S to be *weakly compatible* if $SAx = ASx$ whenever $Ax = Sx$ for some x in X.

There exist examples showing that weakly compatible maps need not be compatible (compatible of type (P) or compatible of type (A)). However, $Ax = Sx$, for some $x \in X$ with compatibility (compatible of type (P) or compatible of type (A)) implies that $ASx = SSx = SAx = AAx$.

2000 *Mathematics Subject Classification*: 54H25, 47H10.
Key words and phrases: Altman integral type contraction, compatible mappings, compatible mappings of type (P), compatible mappings of type (A), weakly compatible mappings.

2. Altman condition

In 1975, Altman [1] introduced a generalized contraction. Let (X, d) be a metric space and $f : X \to X$. Then f is called a *generalized contraction* if, for all $x, y \in X$,

$$d(fx, fy) \leq G(d(x, y)) \tag{2.1}$$

where $G : [0, +\infty) \longrightarrow \mathbb{R}$ is a nondecreasing function satisfying the following conditions:

(a) $0 < G(t) < t$, for all $t > 0$, $G(0) = 0$,
(b) $g(t) = \frac{t}{t - G(t)}$ is non-increasing on $(0, \infty)$,
(c) $\int_0^\tau g(t)dt < +\infty$ for each $\tau > 0$.

Henceforth, we shall denote by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ the set of real numbers, the set of nonnegative real numbers and the set of natural numbers, respectively.

After Altman's theorem on metric spaces, Carbone and Singh [2], Rhoades and Watson [12], Watson, Meade and Norris [16] etc. proved fixed point theorems for generalized contractions. We shall use more general contraction condition than the Altman type in our main result.

3. Preliminaries

The following theorem was proved by Sahu and Dewangan [13]. In its statement, $\mathcal{G}_0$ denotes the family of real-valued functions G on the set $D = \mathrm{cl}(\mathrm{ran}\, d)$ which are nondecreasing on D and satisfy (a), and conditions (b), (c) on $D \setminus \{0\}$, that is,

$$\mathcal{G}_0 = \{G : D \to \mathbb{R} : \; G \text{ nondecreasing and satisfying (a) and (b), (c) on } D \setminus \{0\}\}.$$

Theorem A. *Let S and T be self-mappings of a complete metric space (X, d). Let $\{A_i\}_{i \in \mathbb{N}}$ and $\{B_i\}_{i \in \mathbb{N}}$ be sequences of self-maps on X satisfying the following conditions:*

(i) $A_i(X) \subset T(X)$, $B_i(X) \subset S(X)$,
(ii) $d(A_i x, B_i y) \leq G(m(x, y))$, *for all $x, y \in X$, where $G \in \mathcal{G}_0$, the family of real-valued functions G, and* $m(x, y) = \max\{d(Sx, Ty), d(A_i x, Sx), d(B_i y, Ty), \frac{1}{2}[d(B_i y, Sx) + d(A_i x, Ty)]\}$,
(iii) *one of A_i, B_i, S or T is continuous and*
(iv) *A_i and S and B_i and T are compatible of type (A).*

Then each A_i, B_i, S and T have a unique common fixed point in X.

Let $\mathcal{F}$ be the set of all functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that

(*) f is isotone, i.e., if $t_1 \leq t_2$ then $f(t_1) \leq f(t_2)$, for all $t_1, t_2 \in \mathbb{R}_+$,

(**) f is upper semi-continuous,
(***) $f(t) < t$, for each $t > 0$.

In the light of the above notation, the following theorem was proved by Popa and Pathak [11].

Theorem B. *Let A, B, S and T be self-mappings of a complete metric space (X,d) satisfying the conditions:*

(i) $A(X) \subset T(X)$, $B(X) \subset S(X)$,
(ii) *the inequality*

$$[1 + p\, d(Sx,Ty)]d(Ax,By) \\ \leq p \max\{d(Ax,Sx)d(By,Ty), d(Ax,Ty)d(By,Sx)\} \\ + f\left(\max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), \frac{1}{2}[d(By,Sx) + d(Ax,Ty)]\}\right),$$

holds for all $x, y \in X$, where $p \geq 0$ and $f \in \mathcal{F}$,
(iii) *one of A, B, S or T is continuous, and*
(iv) *pairs (A,S) and (B,T) are compatible of type (A).*

Then A, B, S and T have a common fixed point z. Further, z is the unique common fixed point of A and S and of B and T.

Our aim in this paper is to prove a common fixed point theorem for two pairs of weakly compatible mappings satisfying Altman type contraction condition and to derive a few known results as corollaries. In our main result, we have: dropped the completeness of the whole space X in Theorem B, by choosing the range space of one of the four mappings complete; relaxed the duality of conditions on mappings in compatibility of type (A) by taking weakly compatible mappings and dropped the requirement of the continuity of one of the four mappings.

4. Main results

We now state and prove our main theorem.

Theorem 4.1. *Let A, B, S and T be four self-mappings of a metric space (X,d) satisfying the following conditions:*

$$A(X) \subset T(X), \qquad B(X) \subset S(X), \tag{4.1}$$

$$\int_0^{d(Ax,By)} \psi(t)\, dt + p \int_0^{d(Sx,Ty)d(Ax,By)} \psi(t)\, dt \tag{4.2}$$

$$\leq p \int_0^{\max\{d(Ax,Sx)d(By,Ty),\ d(Ax,Ty)d(By,Sx)\}} \psi(t)\,dt$$

$$+ G\Big(\int_0^{\max\{d(Sx,Ty),d(Ax,Sx),d(By,Ty),\frac{d(Ax,Ty)+d(By,Sx)}{2}\}} \psi(t)\,dt\Big),$$

for all $x, y \in X$, where $p \geq 0$, $G : [0, +\infty) \to \mathbb{R}$ is nondecreasing and satisfies the Altman type conditions (a)–(c) *and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonnegative, Lebesgue measurable mapping which is summable on each compact interval, and such that*

$$\int_0^{\epsilon} \psi(t)\,dt > 0 \text{ for each } \epsilon > 0. \tag{4.3}$$

Assume also the following hypothesis:

(H1) *ψ is a nonincreasing function.*

If one of $A(X)$, $B(X)$, $S(X)$ or $T(X)$ is a complete subspace of X, then

(i) *(A, S) have a coincidence point.*
(ii) *(B, T) have a coincidence point.*

Moreover, if both the pairs (A, S) and (B, T) are weakly compatible then A, B, S and T have a unique common fixed point.

Proof. Pick $x_0 \in X$, then by (4.1) we can choose a sequence $\{x_n\}$ in X such that

$$x_0 = y_0, Ax_{2n} = Tx_{2n+1} = y_{2n+1} \text{ and } Bx_{2n+1} = Sx_{2n+2} = y_{2n+2},$$

for all $n = 0, 1, 2, \ldots$.

We now show that the sequence $\{y_n\}$ defined above is a Cauchy sequence in X. Let us denote $d(y_n, y_{n+1})$ by d_n, for each $n = 0, 1, 2, \ldots$. First, we show that $\int_0^{d_{n+1}} \psi(t)\,dt \leq G(\int_0^{d_n} \psi(t)\,dt)$. Now we claim that

$$\lim_{n\to\infty} d_n = 0$$

and then show that $\{y_n\}$ is a Cauchy sequence in X. For this, putting x_{2n} for x and x_{2n+1} for y in (4.2), we obtain

$$\int_0^{d_{2n+1}} \psi(t)\,dt + p \int_0^{d_{2n}d_{2n+1}} \psi(t)\,dt \leq p \int_0^{\max\{d_{2n}d_{2n+1},\ 0\}} \psi(t)\,dt$$

$$+ G\Big(\int_0^{max\{d_{2n},d_{2n},d_{2n+1},\frac{1}{2}d(y_{2n},y_{2n+2})\}} \psi(t)\,dt\Big)$$

i.e.,

$$\int_0^{d_{2n+1}} \psi(t)\,dt \le G\Big(\int_0^{max\{d_{2n},d_{2n},d_{2n+1},\frac{1}{2}d(y_{2n},y_{2n+2})\}} \psi(t)\,dt\Big).$$

But, from the triangle inequality for metric d, we have

$$\frac{1}{2}\,d(y_{2n},y_{2n+2}) \le \frac{1}{2}[d(y_{2n},y_{2n+1}) + d(y_{2n+1},y_{2n+2})]$$
$$= \frac{1}{2}[d_{2n} + d_{2n+1}] \le max\{d_{2n},d_{2n+1}\}.$$

Using this in above, we obtain

$$\int_0^{d_{2n+1}} \psi(t)\,dt \le G\Big(\int_0^{\max\{d_{2n},\ d_{2n+1}\}} \psi(t)\,dt\Big)$$
$$= G\Big(\max\Big\{\int_0^{d_{2n}} \psi(t)\,dt,\ \int_0^{d_{2n+1}} \psi(t)\,dt\Big\}\Big).$$

If we choose $\int_0^{d_{2n+1}} \psi(t)\,dt$ as "max" in above, then $d_{2n+1} > 0$ and we have

$$\int_0^{d_{2n+1}} \psi(t)\,dt \le G\Big(\int_0^{d_{2n+1}} \psi(t)\,dt\Big) < \int_0^{d_{2n+1}} \psi(t)\,dt,$$

a contradiction. Hence,

$$\int_0^{d_{2n+1}} \psi(t)\,dt \le G\Big(\int_0^{d_{2n}} \psi(t)\,dt\Big). \tag{4.4}$$

Similarly, by setting x_{2n+2} for x and x_{2n+1} for y in (4.2), we obtain

$$\int_0^{d_{2n+2}} \psi(t)\,dt + p\int_0^{d_{2n+1}d_{2n+2}} \psi(t)\,dt \le p\int_0^{max\{d_{2n+2}d_{2n+1},\ 0\}} \psi(t)\,dt$$
$$+ G\Big(\int_0^{\max\{d_{2n+1},d_{2n+2},d_{2n+1},\frac{1}{2}d(y_{2n+1},y_{2n+3})\}} \psi(t)\,dt\Big),$$

i.e.,

$$\int_0^{d_{2n+2}} \psi(t)\,dt \le G\Big(\int_0^{\max\{d_{2n+1},\ d_{2n+2},\ d_{2n+1},\frac{1}{2}d(y_{2n+1},y_{2n+3})\}} \psi(t)\,dt\Big)$$

i.e.,

$$\int_0^{d_{2n+2}} \psi(t)\,dt \le G\Big(\int_0^{\max\{d_{2n+1},d_{2n+2}\}} \psi(t)\,dt\Big),$$

whence

$$\int_0^{d_{2n+2}} \psi(t)\,dt \le G\Big(\int_0^{d_{2n+1}} \psi(t)\,dt\Big). \tag{4.5}$$

Unifying (4.4) and (4.5), we obtain

$$\int_0^{d_{n+1}} \psi(t)\,dt \le G\Big(\int_0^{d_n} \psi(t)\,dt\Big),$$

for all $n = 0, 1, 2, \ldots$.

Next, define a sequence $\{t_n\}$ by $t_{n+1} = G(t_n)$, with

$$t_1 = \int_0^{d_0} \psi(t)\,dt = \int_0^{d(y_0,y_1)} \psi(t)\,dt.$$

It then follows by assumption (a) that, $0 < G(t_n) = t_{n+1} < t_n < t_1$, $\forall n \ge 1$, if $t_1 > 0$. If $t_1 = 0$, then $t_n = 0$, for every n.

Furthermore, by induction, we show that $\int_0^{d_n} \psi(t)\,dt \le t_{n+1}$, for every $n \in \mathbb{N}$. If $n = 1$, then by putting x_0 for x and x_1 for y in (4.2), we have

$$\int_0^{d(y_1,y_2)} \psi(t)\,dt + p \int_0^{d(y_0,y_1)d(y_1,y_2)} \psi(t)\,dt \le p \int_0^{max\{d(y_0,y_1)d(y_1,y_2),\ 0\}} \psi(t)\,dt$$
$$+ G\Big(\int_0^{max\{d(y_0,y_1),d(y_1,y_2),\frac{1}{2}d(y_2,y_0)\}} \psi(t)\,dt\Big),$$

whence

$$\int_0^{d_1} \psi(t)\,dt = \int_0^{d(y_1,y_2)} \psi(t)\,dt$$
$$\le G\Big(\int_0^{\max\{d(y_0,y_1),d(y_1,y_2),\frac{1}{2}d(y_2,y_0)\}} \psi(t)\,dt\Big)$$
$$= G\Big(\int_0^{max\{d(y_0,y_1),d(y_1,y_2)\}} \psi(t)\,dt\Big)$$
$$= G\Big(\int_0^{d(y_0,y_1)} \psi(t)\,dt\Big) = G\Big(\int_0^{d_0} \psi(t)\,dt\Big) = G(t_1) = t_2;$$

because if we choose $d(y_1, y_2)$ as "max", then $d(y_1, y_2) > 0$ and it yields $\int_0^{d_1} \psi(t)\,dt \le G(\int_0^{d_1} \psi(t)\,dt) < \int_0^{d_1} \psi(t)\,dt$, which is a contradiction.

Thus, for $n = 1$, we observe that $\int_0^{d_1} \psi(t)\,dt \le t_2$.

Assume, for some fixed n, that $\int_0^{d_n} \psi(t)\,dt \le t_{n+1}$ is true. Then, by induction; we have, since G is nondecreasing,

$$\int_0^{d_{n+1}} \psi(t)\,dt \le G\Big(\int_0^{d_n} \psi(t)\,dt\Big) \le G(t_{n+1}) = t_{n+2}.$$

Thus, it follows that $\int_0^{d_n} \psi(t)\,dt \le t_{n+1}$, for all $n \in \mathbb{N}$.

Note that, if $t_1 = 0$, then $d_n = 0$ for every n, so that we consider the case where $t_n > 0$, for every n.

Now, by conditions (a)–(c) and $t_{n+1} = G(t_n)$, $n \in \mathbb{N}$, which shows that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} d_n = 0$, it follows that $\{y_n\}$ is a Cauchy sequence. Indeed, if $m, n \in \mathbb{N}$ with $m \ge n$, then using that hypothesis (H1) implies

$$\begin{aligned}
&\int_0^{\sum_{k=n}^{m-1} d_k} \psi(t)\,dt \\
&= \int_0^{d_n} \psi(t)\,dt + \int_{d_n}^{d_n+d_{n+1}} \psi(t)\,dt + \int_{d_n+d_{n+1}}^{d_n+d_{n+1}+d_{n+2}} \psi(t)\,dt + \cdots + \int_{\sum_{k=n}^{m-2} d_k}^{\sum_{k=n}^{m-1} d_k} \psi(t)\,dt \\
&\le \int_0^{d_n} \psi(t)\,dt + \int_0^{d_{n+1}} \psi(t)\,dt + \int_0^{d_{n+2}} \psi(t)\,dt + \cdots + \int_0^{d_{m-1}} \psi(t)\,dt \\
&= \sum_{k=n}^{m-1} \int_0^{d_k} \psi(t)\,dt,
\end{aligned}$$

we obtain

$$\begin{aligned}
\int_0^{d(y_m,y_n)} \psi(t)\,dt &\le \int_0^{\sum_{k=n}^{m-1} d_k} \psi(t)\,dt \le \sum_{k=n}^{m-1} \int_0^{d_k} \psi(t)\,dt \le \sum_{k=n}^{m-1} t_{k+1} \\
&= \sum_{k=n+1}^{m} t_k = \sum_{k=n+1}^{m} \frac{t_k(t_k - t_{k+1})}{t_k - G(t_k)} \le \sum_{k=n+1}^{m} \int_{t_{k+1}}^{t_k} g(t)dt \\
&\le \int_{t_{m+1}}^{t_{n+1}} g(t)dt.
\end{aligned}$$

Since the sequence $\{t_n\}$ is convergent and $\int_0^{\tau} g(t)dt < +\infty$ for each $\tau \in (0, \int_0^K \psi(t)\,dt]$, where $ran\, d \subseteq [0, K]$, then the last term tends to zero as $n \to \infty$ and, hence, $\{y_n\}$ is a Cauchy sequence in X.

Now, we suppose that the range of one of the four mappings is complete.

Case I. Suppose that $T(X)$ is a complete subspace of X, then the subsequence $\{y_{2n+1}\} = \{Tx_{2n+1}\}$ is a Cauchy sequence in $T(X)$ and hence converges to a limit, say z in X. Since $\{y_n\}$ is Cauchy and its subsequence $\{y_{2n+1}\}$ is convergent to z, so $\{y_n\}$ also converges to z. Hence its subsequence $\{y_{2n+2}\}$ is also convergent to z. Thus we have

$$\lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Sx_{2n} = z.$$

Let $v \in T^{-1}z$, then $Tv = z$. We claim that $Bv = z$. For this, setting $x = x_{2n}$ and $y = v$ in the implicit relation (4.2) we have

$$\int_0^{d(Ax_{2n},Bv)} \psi(t)\,dt + p \int_0^{d(Sx_{2n},Tv)d(Ax_{2n},Bv)} \psi(t)\,dt$$
$$\leq p \int_0^{\max\{d(Ax_{2n},Sx_{2n})d(Bv,Tv),d(Ax_{2n},Tv)d(Bv,Sx_{2n})\}} \psi(t)\,dt$$
$$+ G\Big(\int_0^{\max\{d(Sx_{2n},Tv),\ d(Ax_{2n},Sx_{2n}),\ d(Bv,Tv),\frac{1}{2}[d(Ax_{2n},Tv)+d(Bv,Sx_{2n})]\}} \psi(t)\,dt\Big).$$

If we suppose that $d(z, Bv) > 0$, then we have, for n large enough,

$$\int_0^{d(Ax_{2n},Bv)} \psi(t)\,dt + p \int_0^{d(Sx_{2n},Tv)d(Ax_{2n},Bv)} \psi(t)\,dt$$
$$\leq p \int_0^{max\{d(Ax_{2n},Sx_{2n})d(Bv,Tv),d(Ax_{2n},Tv)d(Bv,Sx_{2n})\}} \psi(t)\,dt + G\Big(\int_0^{d(Bv,z)} \psi(t)\,dt\Big).$$

Letting $n \to \infty$, it yields

$$\int_0^{d(z,Bv)} \psi(t)\,dt \leq G\Big(\int_0^{d(z,Bv)} \psi(t)\,dt\Big) < \int_0^{d(z,Bv)} \psi(t)\,dt,$$

which is a contradiction. Thus $d(Bv, z) = 0$, so that $Bv = z$. Hence $z = Bv = Tv$, showing that v is a coincidence point of B and T.

Further, since $B(X) \subset S(X)$, $Bv = z$ implies that $z \in S(X)$. Let $u \in S^{-1}z$, then $Su = z$. Now, we claim that $Au = z$. For this, putting $x = u$ and $y = v$ in (4.2), we have

$$\int_0^{d(Au,z)} \psi(t)\,dt + p \int_0^{0\cdot d(Au,z)} \psi(t)\,dt \leq p \int_0^{max\{0,\ 0\}} \psi(t)\,dt$$
$$+ G\Big(\int_0^{max\{0,\ d(Au,z),\ 0,\frac{1}{2}d(Au,z)\}} \psi(t)\,dt\Big)$$

i.e.,

$$\int_0^{d(Au,z)} \psi(t)\,dt \le G\Big(\int_0^{d(Au,z)} \psi(t)\,dt\Big) < \int_0^{d(Au,z)} \psi(t)\,dt,$$

if $d(Au,z) > 0$, getting a contradiction. Thus $Au = z$. Hence $z = Au = Su$, showing that u is a coincidence point of (A,S).

Case II. If we assume $S(X)$ to be a complete subspace of X, then analogous arguments establish the earlier conclusion. Indeed, in this case, the subsequence $\{y_{2n+2}\} = \{Sx_{2n+2}\}$ is a Cauchy sequence in $S(X)$ and hence converges to a limit, say z in $S(X)$. Similarly to Case I,

$$\lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Sx_{2n} = z.$$

Let $v \in X$ be such that $Sv = z$. To prove that $Av = z$, we take $x = v$ and $y = x_{2n+1}$ in the implicit relation (4.2), hence, assuming that $d(Av,z) > 0$, we get, for n large enough,

$$\int_0^{d(Av,Bx_{2n+1})} \psi(t)\,dt + p \int_0^{d(Sv,Tx_{2n+1})d(Av,Bx_{2n+1})} \psi(t)\,dt$$
$$\le p \int_0^{\max\{d(Av,z)d(Bx_{2n+1},Tx_{2n+1}),d(Av,Tx_{2n+1})d(Bx_{2n+1},Sv)\}} \psi(t)\,dt$$
$$+ G\Big(\int_0^{d(Av,z)} \psi(t)\,dt\Big),$$

hence, taking the limit as $n \to \infty$, we obtain

$$\int_0^{d(Av,z)} \psi(t)\,dt \le G\Big(\int_0^{d(Av,z)} \psi(t)\,dt\Big) < \int_0^{d(Av,z)} \psi(t)\,dt,$$

which is a contradiction. Hence $Av = Sv = z$.

On the other hand, since $A(X) \subset T(X)$, then $z = Tu$, for some $u \in X$. To check that $Bu = z$, we take $x = v$ and $y = u$ in (4.2), achieving

$$\int_0^{d(z,Bu)} \psi(t)\,dt \le G\Big(\int_0^{d(Bu,z)} \psi(t)\,dt\Big) < \int_0^{d(Bu,z)} \psi(t)\,dt,$$

if $d(Bu,z) > 0$, getting a contradiction. This proves that $Bu = Tu = z$.

The remaining two cases are essentially the same as the previous cases. Indeed, if $A(X)$ is complete, then by (4.1), $z \in A(X) \subset T(X)$. Similarly, if $B(X)$ is complete, then $z \in B(X) \subset S(X)$.

Thus pairs (A,S) and (B,T) have coincidence points. Hence in all we have $z = Au = Su = Bv = Tv$. This proves our assertions in (i) and (ii).

Now, the weak compatibility of (A, S) gives $Az = ASu = SAu = Sz$; i.e., $Az = Sz$. Similarly, the weak compatibility of (B, T) gives $Bz = BTv = TBv = Tz$; i.e., $Bz = Tz$.

To show that z is a coincidence point of A, B, S and T, we have to check that $Az = Bz$. For this, putting $x = z$ and $y = z$ in (4.2), we have

$$\int_0^{d(Az,Bz)} \psi(t)\,dt + p \int_0^{d(Sz,Tz)d(Az,Bz)} \psi(t)\,dt \le p \int_0^{max\{d(Az,Sz)d(Bz,Tz),d(Az,Tz)d(Bz,Sz)\}} \psi(t)\,dt + G\Big(\int_0^{max\{d(Sz,Tz),\ d(Az,Sz),\ d(Bz,Tz),\ \frac{1}{2}[d(Az,Tz)+d(Bz,Sz)]\}} \psi(t)\,dt\Big)$$

i.e.,

$$\int_0^{d(Az,Bz)} \psi(t)\,dt \le G\Big(\int_0^{d(Az,Bz)} \psi(t)\,dt\Big) < \int_0^{d(Az,Bz)} \psi(t)\,dt,$$

if $d(Az, Bz) > 0$, which is a contradiction. Thus $Az = Bz$. Hence $Az = Sz = Bz = Tz$.

To show that z is a common fixed point, putting $x = z$ and $y = v$ in (4.2), we have

$$\int_0^{d(Az,Bv)} \psi(t)\,dt + p \int_0^{d(Sz,Tv)d(Az,Bv)} \psi(t)\,dt \le p \int_0^{\max\{d(Az,Sz)d(Bv,Tv),\ d(Az,Tv)d(Bv,Sz)\}} \psi(t)\,dt + G\Big(\int_0^{\max\{d(Sz,Tv),d(Az,Sz),d(Bv,Tv),\frac{1}{2}[d(Az,Tv)+d(Bv,Sz)]\}} \psi(t)\,dt\Big)$$

i.e.,

$$\int_0^{d(Az,z)} \psi(t)\,dt \le G\Big(\int_0^{d(Az,z)} \psi(t)\,dt\Big) < \int_0^{d(Az,z)} \psi(t)\,dt,$$

if $d(Az, z) > 0$, getting a contradiction. Thus, we obtain $z = Az = Bz = Sz = Tz$. Uniqueness of common fixed point z follows easily by (4.2). This completes the proof.

We remark that G in Theorem 4.1 must be defined, at least, in $[0, \int_0^K \psi(s)\,ds]$, where $\mathrm{cl}(\mathrm{ran}\, d) \subset [0, K]$.

If we take $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (H1), then ψ is measurable, summable on each compact interval, and condition (4.3) holds if $\int_0^\epsilon \psi(t)\,dt$ is positive and finite for an $\epsilon > 0$.

Note that condition (H1) is valid for constant functions ψ, but it is not true for functions of the type $\psi(t) = Rt$, $t > 0$, where $R > 0$.

THEOREM 4.2. *In Theorem* 4.1, *hypothesis* (H1) *can be replaced by the following one:*

(H2) $\psi(t) > 0$, $\forall t > 0$, *and* $G(\int_0^x \psi(t)\,dt) \le \int_0^{G(x)} \psi(t)\,dt$, $\forall x > 0$.

Proof. We have to justify that the sequence $\{y_n\}$ defined in the proof of Theorem 4.1 is a Cauchy sequence. Using that

$$\int_0^{d_{n+1}} \psi(t)\,dt \le G\Big(\int_0^{d_n} \psi(t)\,dt\Big),$$

for all $n = 0, 1, 2, \ldots$, and (H2), we get

$$\int_0^{d_{n+1}} \psi(t)\,dt \le \int_0^{G(d_n)} \psi(t)\,dt,$$

for all $n = 0, 1, 2, \ldots$, and $d_{n+1} \le G(d_n)$, for all $n = 0, 1, 2, \ldots$.

We define a sequence $\{t_n\}$ by $t_1 = d_0$, $t_{n+1} = G(t_n)$, $\forall n \in \mathbb{N}$. If $t_1 = d_0 = 0$, then $d_n = 0$ for every n. Consider $t_1 > 0$, hence $t_{n+1} = G(t_n) < t_n$, $\forall n \in \mathbb{N}$ and $t_n \to 0$. Besides, it can be easily obtained that $d_n \le t_{n+1}$, for all $n = 0, 1, 2, \ldots$.

Now, for $m, n \in \mathbb{N}$ with $m \ge n$, we get

$$d(y_m, y_n) \le \sum_{k=n}^{m-1} d_k \le \sum_{k=n}^{m-1} t_{k+1} = \sum_{k=n+1}^{m} t_k \le \int_{t_{m+1}}^{t_{n+1}} g(t)dt,$$

and the sequence $\{y_n\}$ is a Cauchy sequence, since $\int_0^\tau g(t)dt < +\infty$ for each $\tau > 0$.

Note that condition $G(\int_0^x \psi(t)\,dt) \le \int_0^{G(x)} \psi(t)\,dt$, $\forall x > 0$, is trivially satisfied if $\psi \equiv 1$ and reduces to $G(Rx) \le RG(x)$, $\forall x > 0$, if $\psi \equiv R$. In fact such condition can be dropped, as established in the following result.

THEOREM 4.3. *In Theorem* 4.1, *hypothesis* (H1) *can be replaced by the following one:*

(H3) $\psi(t) > 0$, *for every* $t > 0$.

Proof. In the proof of Theorem 4.1, the following inequality was obtained:

$$\int_0^{d_{n+1}} \psi(t)\,dt \le G\Big(\int_0^{d_n} \psi(t)\,dt\Big), \qquad \text{for all } n = 0, 1, 2, \ldots.$$

We define a sequence $\{t_n\}$ by $t_1 = d_0 = d(y_0, y_1)$, and t_{n+1} such that $G(\int_0^{t_n} \psi(t)\,dt) = \int_0^{t_{n+1}} \psi(t)\,dt$, for every $n \in \mathbb{N}$. Note that, for $t_1 > 0$, $\int_0^{t_n} \psi(t)\,dt > G(\int_0^{t_n} \psi(t)\,dt) = \int_0^{t_{n+1}} \psi(t)\,dt$, for every n and, hence, $t_{n+1} < t_n$, for every n. Then, by induction, it can be proved that

$$\int_0^{d_n} \psi(t)\,dt \le \int_0^{t_{n+1}} \psi(t)\,dt, \qquad \text{for every } n \in \mathbb{N}.$$

Indeed, if $n = 1$,

$$\int_0^{d_1} \psi(t)\,dt \le G\Big(\int_0^{d_0} \psi(t)\,dt\Big) = G\Big(\int_0^{t_1} \psi(t)\,dt\Big) = \int_0^{t_2} \psi(t)\,dt.$$

If, for some fixed n, $\int_0^{d_n} \psi(t)\,dt \le \int_0^{t_{n+1}} \psi(t)\,dt$ is true, then

$$\int_0^{d_{n+1}} \psi(t)\,dt \le G\Big(\int_0^{d_n} \psi(t)\,dt\Big) \le G\Big(\int_0^{t_{n+1}} \psi(t)\,dt\Big) = \int_0^{t_{n+2}} \psi(t)\,dt.$$

Using (H3), we have $d_n \le t_{n+1}$, for every $n \in \mathbb{N}$. By the properties of $\{t_n\}$, we have that $\{t_n\} \to L$. We claim that $L = 0$. Indeed, suppose that $L > 0$, then $\int_0^{t_n} \psi(t)\,dt \to \int_0^{L} \psi(t)\,dt = P > 0$ and $\int_0^{t_n} \psi(t)\,dt > P$, for every $n \in \mathbb{N}$. From the properties (a), (b), we deduce that G is subadditive and, hence,

$$\begin{aligned}
G\Big(\int_0^{t_n} \psi(t)\,dt\Big) &= G\Big(\int_0^{L} \psi(t)\,dt + \int_L^{t_n} \psi(t)\,dt\Big)\\
&\le G\Big(\int_0^{L} \psi(t)\,dt\Big) + G\Big(\int_L^{t_n} \psi(t)\,dt\Big)\\
&= G\Big(\int_0^{L} \psi(t)\,dt\Big) + G\Big(\int_0^{t_n} \psi(t)\,dt - \int_0^{L} \psi(t)\,dt\Big)\\
&\le G\Big(\int_0^{L} \psi(t)\,dt\Big) + \int_0^{t_n} \psi(t)\,dt - \int_0^{L} \psi(t)\,dt.
\end{aligned}$$

This inequality, joint to the property

$$\lim_{n\to+\infty} G\Big(\int_0^{t_n} \psi(t)\,dt\Big) = \lim_{n\to+\infty} \int_0^{t_{n+1}} \psi(t)\,dt = \int_0^{L} \psi(t)\,dt,$$

produces that

$$\int_0^{L} \psi(t)\,dt \le G\Big(\int_0^{L} \psi(t)\,dt\Big) < \int_0^{L} \psi(t)\,dt,$$

which is a contradiction. Thus $L = 0$ and $\{t_n\} \to 0$. This also implies that $(y_n) \to 0$ and $(d_n) \to 0$.

Finally, if $m, n \in \mathbb{N}$ with $m \geq n$, we get

$$d(y_m, y_n) \leq \sum_{k=n}^{m-1} d_k \leq \sum_{k=n}^{m-1} t_{k+1} = \sum_{k=n+1}^{m} t_k \leq \int_{t_{m+1}}^{t_{n+1}} g(t)dt,$$

therefore $\{y_n\}$ is a Cauchy sequence.

If ψ is nondecreasing, then (H1) is satisfied only for ψ a constant function. However, condition (H3) could be fulfilled. Note that if $\psi = 1$ in Theorems 4.1-4.3, then (4.2) is reduced to inequality (ii) in Theorem B (see [11]). On the other hand, if $p = 0$, then Theorems 4.1-4.3 reduce to the following Corollary.

Corollary 4.4. *Let A, B, S and T be four self-mappings of a metric space (X, d) satisfying* (4.1) *and*

$$(4.6) \qquad \int_0^{d(Ax,By)} \psi(t)\,dt \leq G\Big(\int_0^{\max\left\{d(Sx,Ty),d(Ax,Sx),d(By,Ty),\frac{d(Ax,Ty)+d(By,Sx)}{2}\right\}} \psi(t)\,dt\Big),$$

for all $x, y \in X$, where $G : [0, +\infty) \to \mathbb{R}$ is nondecreasing and satisfies the Altman type conditions (a)-(c) and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonnegative, Lebesgue measurable mapping which is summable on each compact interval, and satisfies (4.3). *Assume that one of the hypotheses* (H1), (H2) *or* (H3) *holds. If one of $A(X)$, $B(X)$, $S(X)$ or $T(X)$ is a complete subspace of X, then*

(i) *(A, S) have a coincidence point.*
(ii) *(B, T) have a coincidence point.*

Moreover, if both the pairs (A, S) and (B, T) are weakly compatible then A, B, S and T have a unique common fixed point.

Remark 4.5. If $\{A_i\}_{i \in \mathbb{N}}$, S and T are self-mappings of a metric space (X, d) then we have the following Corollary as a generalization of the results of Popa and Pathak [11].

Corollary 4.6. *Let $\{A_i\}_{i \in \mathbb{N}}$, S and T be self-mappings of a metric space (X, d) such that*

$$(4.7) \qquad A_i(X) \subset T(X), \qquad A_{i+1}(X) \subset S(X),$$

$$(4.8)\quad \int_0^{d(A_ix,A_{i+1}y)} \psi(t)\,dt + p\int_0^{d(Sx,Ty)d(A_ix,A_{i+1}y)} \psi(t)\,dt$$
$$\le p\int_0^{\max\{d(A_ix,Sx)d(A_{i+1}y,Ty),\ d(A_ix,Ty)d(A_{i+1}y,Sx)\}} \psi(t)\,dt$$
$$+ G\Big(\int_0^{\max\{d(Sx,Ty),d(A_ix,Sx),d(A_{i+1}y,Ty),\frac{d(A_ix,Ty)+d(A_{i+1}y,Sx)}{2}\}} \psi(t)\,dt\Big),$$

for all $x, y \in X$, where $p \geq 0$, $G : [0, +\infty) \to \mathbb{R}$ is nondecreasing and satisfies the Altman's conditions (a)–(c) *and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonnegative, Lebesgue measurable mapping which is summable on each compact interval, and such that* (4.3) *holds. Assume that one of the hypotheses* (H1), (H2) *or* (H3) *holds. If one of $A_i(X)$, $S(X)$ or $T(X)$ is a complete subspace of X, and if the pairs (A_i, S) and (A_{i+1}, T) are weakly compatible, then $\{A_i\}_{i\in\mathbb{N}}$, S and T have a unique common fixed point.*

Remark 4.7. If we take sequences $\{A_i\}_{i\in\mathbb{N}}$ and $\{B_i\}_{i\in\mathbb{N}}$ instead of A and B in Theorem 4.1, then we get the following Corollary as a generalization of Theorem A [13], in which the completeness of X and compatibility of type (A) are relaxed by completeness of one subspace and weak compatibility.

Corollary 4.8. *Let S and T be self-maps of a metric space (X, d). Let $\{A_i\}_{i\in\mathbb{N}}$ and $\{B_i\}_{i\in\mathbb{N}}$ be two sequences of self-mappings of the metric space (X, d) satisfying the conditions:*

$$(4.9)\quad A_i(X) \subset T(X), \qquad B_i(X) \subset S(X),$$

$$(4.10)\quad \int_0^{d(A_ix,B_iy)} \psi(t)\,dt + p\int_0^{d(Sx,Ty)d(A_ix,B_iy)} \psi(t)\,dt$$
$$\le p\int_0^{\max\{d(A_ix,Sx)d(B_iy,Ty),d(A_ix,Ty)d(B_iy,Sx)\}} \psi(t)\,dt + G\Big(\int_0^{m(x,y)} \psi(t)\,dt\Big),$$

for all $x, y \in X$, where $p \geq 0$, $G : [0, +\infty) \to \mathbb{R}$ is nondecreasing and satisfies the Altman type conditions (a)–(c), *$\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonnegative, Lebesgue measurable mapping which is summable on each compact interval, and such that* (4.3) *holds, and*

$$m(x, y) = \max\{d(Sx, Ty), d(A_ix, Sx), d(B_iy, Ty), \tfrac{1}{2}[d(A_ix, Ty) + d(B_iy, Sx)]\}.$$

Assume that one of the hypotheses (H1), (H2) *or* (H3) *holds. If one of $A_i(X)$, $B_i(X)$, $S(X)$ or $T(X)$ is a complete subspace of X, then*

(i) (A_i, S) *have a coincidence point.*
(ii) (B_i, T) *have a coincidence point.*

Moreover, if both the pairs (A_i, S) *and* (B_i, T) *are weakly compatible then* A_i, B_i, S *and* T *have a unique common fixed point.*

Now we give an example to show the validity of the main results Theorems 4.1–4.3.

Example 4.9. Let A, B, S and T be four self-mappings of the metric space $X = [0,1]$, endowed with the usual metric d. Define the mappings $A, B, S, T : X \to X$ by:

$$Ax = 1, \quad Sx = 2x \bmod 1, \quad Bx = 1, \quad \text{and} \quad Tx = \frac{1}{2}(1+x), \quad \forall x \in X.$$

Let $G : [0,\infty) \to \mathbb{R}$ be the nondecreasing function defined by $G(t) = \frac{t}{2}$, and $\psi(t) = 2t$, for all $t \in \mathbb{R}_+$. Then we observe that $\int_0^\tau g(t)\,dt = 2\tau < +\infty$, for every $\tau > 0$, and:

(i) $A(X) = \{1\} \subset T(X) = \left[\frac{1}{2}, 1\right]$ and $B(X) = \{1\} \subset S(X) = [0,1]$;

Condition (4.2) is trivially satisfied since $d(Ax, By) = 0$, for every x, $y \in [0,1]$.

(ii) When $0 \le x \le \frac{1}{2}$ and $y \in [0,1]$, we have $d(Ax, By) = 0$, $d(Sx, Ty) = |2x - \frac{1}{2}(y+1)|$, $d(Ax, Sx) = |1-2x|$, $d(By, Ty) = \frac{1}{2}|1-y| = d(Ax, Ty)$ and $d(By, Sx) = |1-2x|$. Then condition (4.2) yields

$$\int_0^0 2t\,dt + p\int_0^0 2t\,dt \le p \int_0^{\max\{|1-2x|\,\frac{1}{2}|1-y|,\ \frac{1}{2}|1-y|\,|1-2x|\}} 2t\,dt$$
$$+ G\Big(\int_0^{\max\{|2x-\frac{1}{2}(y+1)|,|1-2x|,\frac{1}{2}|1-y|,\frac{\frac{1}{2}|1-y|+|1-2x|}{2}\}} 2t\,dt\Big),$$

or,

$$0 \le \frac{p}{4}|1-2x|^2 \cdot |1-y|^2 + G((m(x,y))^2),$$

where

$$m(x,y) = \max\{\tfrac{1}{2}|4x-y-1|, |1-2x|, \tfrac{1}{2}|1-y|, \tfrac{1}{2}(\tfrac{1}{2}|1-y| + |1-2x|)\} \ge 0.$$

Thus condition (4.2) is true for all $x \in [0, \frac{1}{2}]$, $y \in [0,1]$, and $p \ge 0$.

(iii) When $\frac{1}{2} < x \le 1$ and $y \in [0,1]$, we can similarly show, as in (ii), that condition (4.2) is true for all $p \ge 0$.

Further, when $0 \le x \le \frac{1}{2}$ and $y \in [0,1]$, we see that $m(x,y) = 0$ if and only if

$$\tfrac{1}{2}|4x-y-1| = 0 = |1-2x| = \tfrac{1}{2}|1-y| = \tfrac{1}{2}(\tfrac{1}{2}|1-y| + |1-2x|)$$

i.e.,

$$x = \frac{1}{2}, \qquad y = 1.$$

Thus, $m(\frac{1}{2}, 1) = 0$ and, therefore, $G(m(\frac{1}{2}, 1)^2) = G(0) = 0$.

We observe that $T(X)$ (and also $S(X)$) are complete subspaces of X. Further, we have $g(t) = 2$, so that $\int_0^\tau g(t)dt = 2\tau < +\infty$, for every $\tau \in (0, 1]$.

We notice that A and S have as coincidence points $x \in [\frac{1}{2}, 1]$, where $ASx = SAx$, and B, T have the coincidence point $x = 1$, where they commute. So that (A, S) and (B, T) are weakly compatible. Thus all the conditions of Theorem 4.1 are satisfied, with the exception of (H1). Note that (H3) holds. Moreover, the only common fixed point of A, B, S and T is $x = 1$. This validates Theorem 4.3.

The following example also shows the validity of our main Theorem 4.3.

Example 4.10. Let A, B, S and T be four self-mappings on $X = [-2\pi, 2\pi]$ with $|\cdot|$ the usual metric. Suppose that $G(t) = \frac{1}{4}t$, for all $t > 0$, and $G(0) = 0$. Then $0 < G(t) < t$, for every $t > 0$. Besides, $g(t) = \frac{t}{t - G(t)} = \frac{4}{3}$, for all $t > 0$, and so $\int_0^\tau g(t)dt = \frac{4}{3}\tau < +\infty$, where $\tau \in (0, 4\pi]$. Suppose also that $p = 0$ and $\psi(t) = \frac{2}{3}t$, and (H3) holds.

Define the four mappings $A, B, S, T : [-2\pi, 2\pi] \to [-2\pi, 2\pi]$ by

$$Ax = \frac{1}{8}\sin x, \quad Sx = \frac{1}{4}x, \quad Bx = \frac{1}{8}\sin(2x), \quad Tx = \frac{1}{2}x, \quad \forall x \in [-2\pi, 2\pi].$$

Then, we observe that

(i) $A(X) = [-\frac{1}{8}, \frac{1}{8}] \subset T(X) = [-\pi, \pi]$ and $B(X) = [-\frac{1}{8}, \frac{1}{8}] \subset S(X) = [-\frac{\pi}{2}, \frac{\pi}{2}]$.

(ii) Now, $d(Sx, Ty) = \frac{1}{4}|x - 2y|$, $d(Ax, By) = \frac{1}{8}|\sin x - \sin(2y)|$, $d(Ax, Sx) = |\frac{1}{8}\sin x - \frac{1}{4}x|$, $d(By, Sx) = |\frac{1}{8}\sin(2y) - \frac{1}{4}x|$, $d(By, Ty) = |\frac{1}{8}\sin(2y) - \frac{1}{2}y|$, and $d(Ax, Ty) = |\frac{1}{8}\sin x - \frac{1}{2}y|$, for all $x, y \in X$.

Note that the function $\Psi(\epsilon) = \int_0^\epsilon \psi(t)\,dt$ is a nondecreasing function in $\epsilon > 0$ and $G(t)$ is also a nondecreasing function in $t > 0$.

Now, we see that

$$\begin{aligned}\int_0^{d(Ax,By)} \psi(t)\,dt &= \int_0^{\frac{1}{8}|\sin x - \sin(2y)|} \psi(t)\,dt \le \int_0^{\frac{1}{8}|x-2y|} \psi(t)\,dt = \frac{1}{192}|x - 2y|^2 \\ &= G\Big(\int_0^{\frac{1}{4}|x-2y|} \psi(t)\,dt\Big) = G\Big(\int_0^{d(Sx,Ty)} \psi(t)\,dt\Big) \\ &\le G\Big(\int_0^{\max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), \frac{1}{2}[d(Ax,Ty)+d(By,Sx)]\}} \psi(t)\,dt\Big).\end{aligned}$$

Thus, condition (4.2) is satisfied for all $x, y \in X$. Further,

$$\begin{aligned} m(x,y) = 0 \Leftrightarrow\ & \max\{\tfrac{1}{4}|x-2y|, |\tfrac{1}{8}\sin x - \tfrac{1}{4}x|, |\tfrac{1}{8}\sin(2y) - \tfrac{1}{2}y|, \\ & \tfrac{1}{2}[|\tfrac{1}{8}\sin x - \tfrac{1}{2}y| + |\tfrac{1}{8}\sin(2y) - \tfrac{1}{4}x|]\} = 0 \\ \Leftrightarrow\ & \text{each of the values } \tfrac{1}{4}|x-2y|, |\tfrac{1}{8}\sin x - \tfrac{1}{4}x|, |\tfrac{1}{8}\sin(2y) - \tfrac{1}{2}y| \\ & \text{and } \tfrac{1}{2}[|\tfrac{1}{8}\sin x - \tfrac{1}{2}y| + |\tfrac{1}{8}\sin(2y) - \tfrac{1}{4}x|] \text{ must be zero} \\ & \text{separately} \\ \Leftrightarrow\ & x = 0,\ y = 0, \end{aligned}$$

and $G(0) = G(m(0,0)) = 0$. We also observe that $T(X)$ and $S(X)$ are complete subspaces of X.

Thus, all the conditions of Theorem 4.3 are satisfied. The coincidence point of the pairs (A, S) and that of (B, T) is $x = 0$. Clearly, $x = 0$ is the only common fixed point of A, B, S and T in $[-2\pi, 2\pi]$. This validates Theorems 4.1–4.3.

Example 4.11. In order to give examples in the context of nonlinear contractions, we analyze the meaning of property (b). If we seek a differentiable function G, using (a), we deduce that function g is also differentiable and $g'(t) = \frac{tG'(t)-G(t)}{(t-G(t))^2}$, for every t, hence if we choose G satisfying that

$$0 \le G'(t) \le \frac{G(t)}{t}, \qquad \forall t > 0,$$

then g is a nonincreasing function. On the other hand, if g is a nonincreasing function, then

$$\frac{t_2}{t_2 - G(t_2)} \le \frac{t_1}{t_1 - G(t_1)}, \qquad \forall t_2 \ge t_1 > 0,$$

and, using property (a), we obtain that

$$G(t_2) \le \frac{G(t_1)}{t_1} t_2, \qquad \forall t_2 \ge t_1 > 0,$$

which means that the point $(t_2, G(t_2))$ must be in the region which is below all the lines which join $(0,0)$ and each point of the graph of G before $(t_2, G(t_2))$.

Our interest is to find an example of nonlinear contraction with $G'(0) = 1$. Consider the nondecreasing function $G(t) = \ln(t+1)$, $t \ge 0$, which satisfies $G(0) = 0$ and $0 < G(t) < t$, for every $t > 0$. Note that $G'(0) = 1$ but $G'(t) < 1$ for $t > 0$. Besides, g defined by $g(t) = \frac{t}{t-\ln(t+1)}$ is nonincreasing on $(0, +\infty)$, since the sign of its derivative coincides with the sign of $\beta(t) = t - (t+1)\ln(t+1)$, $t > 0$, which is negative on $(0, +\infty)$.

On the other hand,

$$\int_0^\tau g(t)\,dt = \int_0^\tau \frac{t}{t-\ln(t+1)}\,dt < +\infty,$$

for every $\tau > 0$. Indeed, we check that there exists $\delta > 0$ such that $\frac{t}{t-\ln(t+1)} < \frac{1}{t^\alpha}$, for every $t \in (0,\delta)$, for a certain $\alpha > 1$. Hence $\int_0^\delta \frac{t}{t-\ln(t+1)}\,dt < +\infty$ and, by continuity, $\int_\delta^\tau \frac{t}{t-\ln(t+1)}\,dt < +\infty$. To prove this inequality, we check that $\rho(t) = \ln(t+1) - t + t^{\alpha+1} < 0$, for every $t \in (0,\delta)$ and a certain $\alpha > 1$. We have $\rho(0) = 0$ and $\rho'(t) = \frac{\sigma(t)}{t+1}$, $\forall t$, where $\sigma(t) = -t + (\alpha+1)t^\alpha(t+1) < 0$ for $t > 0$ small enough. This follows from $\sigma(0) = 0$ and $\sigma'(t) = -1 + (\alpha+1)\mu(t)$, where $\mu(t) = \alpha t^{\alpha-1}(t+1) + t^\alpha$ is a continuous function on $[0,+\infty)$ with $\mu(0) = 0$, and σ' is continuous on $[0,+\infty)$ with $\sigma'(0) = -1 < 0$, therefore, for a fixed $\alpha > 1$ and $t > 0$ small enough, we obtain $\sigma'(t) < 0$, hence (c) is valid.

It is easy to check that conditions in Theorem 4.3 are satisfied for functions defined in Example 4.9 for the following choice of function G

$$G(t) = \ln(t+1), \qquad t \ge 0,$$

and every $p \ge 0$.

Example 4.12. Consider the functions in Example 4.10, where function G is taken as $G(t) = \ln(t+1)$, $t \ge 0$, and $p = 0$. To check condition (4.2), we prove that, for every $x, y \in X = [-2\pi, 2\pi]$,

$$\begin{aligned}\int_0^{d(Ax,By)} \psi(t)\,dt &\le \int_0^{\frac{1}{8}|x-2y|} \psi(t)\,dt = \frac{1}{192}|x-2y|^2\\ &\le \ln\left(1 + \frac{|x-2y|^2}{48}\right) = G\Big(\int_0^{\frac{1}{4}|x-2y|} \psi(t)\,dt\Big) = G\Big(\int_0^{d(Sx,Ty)} \psi(t)\,dt\Big)\\ &\le G\Big(\int_0^{\max\{d(Sx,Ty),d(Ax,Sx),d(By,Ty),\frac{1}{2}[d(Ax,Ty)+d(By,Sx)]\}} \psi(t)\,dt\Big).\end{aligned}$$

Note that this property is deduced from the inequality

$$\frac{1}{192}z \le \ln\left(1 + \frac{1}{48}z\right), \qquad \forall z \in [0,(6\pi)^2],$$

since $|x-2y|^2 \le (6\pi)^2$, for every $x, y \in [-2\pi, 2\pi]$. Figure 1 shows the graph of function $\phi(z) = \ln\left(1 + \frac{1}{48}z\right) - \frac{1}{192}z$, which is nonnegative on $[0,(6\pi)^2]$.

Remark 4.13. Our Theorems 4.1–4.3 remain true if the pairs (A,S) and (B,T) are R-weakly commuting [8] instead of weakly compatible.

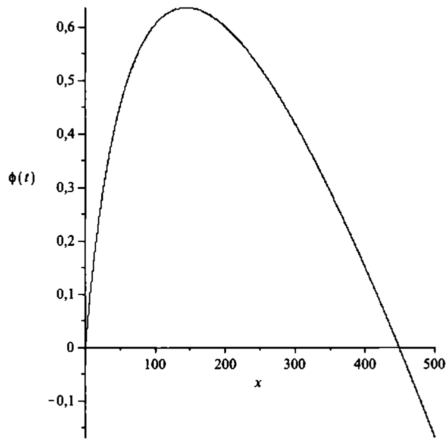

Fig. 1. Graph of function ϕ

REMARK 4.14. The main results in this paper can be established equivalently by using the function $\Psi(t) := \int_0^t \psi(s)\,ds$, for $t \geq 0$, in such a way that condition (4.2) would be written as

$$\begin{aligned}&\Psi(d(Ax,By)) + p\,\Psi(d(Sx,Ty)d(Ax,By))\\&\leq p\,\Psi(\max\{d(Ax,Sx)d(By,Ty), d(Ax,Ty)d(By,Sx)\})\\&+G\Big(\Psi\Big(\max\Big\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), \frac{d(Ax,Ty)+d(By,Sx)}{2}\Big\}\Big)\Big),\end{aligned}$$

for all $x, y \in X$, where $p \geq 0$, $G : [0, +\infty) \to \mathbb{R}$ is nondecreasing and satisfies the Altman type conditions (a)–(c). The proof is made for functions Ψ of integral type $\Psi(t) := \int_0^t \psi(s)\,ds$, for $t \geq 0$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonnegative, Lebesgue measurable mapping which is summable on each compact interval and satisfies (4.3), assuming one of the hypotheses (H1)–(H3). Analogous interpretations can be made for conditions (4.6), (4.8) and (4.10). Note that for the function Ψ of integral type described, we have that Ψ is continuous. Condition (4.3) on function ψ produces the property of Ψ

$$\Psi(t) > 0, \quad \forall t > 0.$$

On the other hand, hypothesis (H2) of function ψ provides that Ψ is increasing on $(0, +\infty)$ and $G(\Psi(x)) \leq \Psi(G(x))$, $\forall x > 0$. Obviously, (H3) implies the increasing character of Ψ on $(0, +\infty)$.

Using this formulation, Theorem B is obtained by choosing Ψ the identity mapping.

Acknowledgments. We thank the referee for the interesting comments and suggestions towards the improvement of the paper.

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Received May 21, 2007; revised version August 30, 2008.