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SOME RANDOM FIXED POINT THEOREMS FOR RANDOM ASYMPTOTICALLY REGULAR OPERATORS

Abstract. Let (Ω, Σ) be a measurable space, X a Banach space, C a weakly compact convex subset of X and $T : \Omega \times C \to C$ a random operator. Let $WCS(X)$ be the weakly convergent sequence coefficient of X and $\kappa_\omega(X)$ its Lifschitz characteristic. If T is asymptotically regular and assume that there exists $\omega \in \Omega$ and constant c such that

$$\sigma(T(\omega, \cdot)) \leq c < \frac{1 + \sqrt{1 + 4WCS(X)(\kappa_\omega(X) - 1)}}{2},$$

we prove that T has a random fixed point. Our results also give stochastic version generalization of some results of Domínguez [*Fixed point theorems for uniformly Lipschitziane mappings and asymptotically regular mappings*, Nonlinear Anal. 32 No. 1 (1998), 15–27].

1. Introduction

The study of random fixed point theorems was initiated by the Prague school of probability in the 1950s. Random operator theory is needed for the study of various classes of random equations (see [9] and references therein). Random fixed point theory has received much attention for the last two decades because of its importance in probabilistic functional analysis; the reader is referred to Beg and Shahzad [2], Shahzad and Latif [11] and Tan and Yaun [12]. Generalizations of the random fixed point theorems for continuous selfmaps to the case of non-selfmaps have been considered by many authors (see e.g. Beg et al. [2], and Shahzad and Latif [11]). On the other hand, the first fixed point theorem for uniformly Lipschitzian mapping in Banach spaces was given by Goebel and Kirk [8] who state a relationship between the existence of fixed point for uniformly Lipschitzian mappings and clarkson modulus of convexity.

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In [3], Casini and Maluta prove the existence of fixed points of uniformly k-Lipschitzian mapping T with $k < \sqrt{N(X)}$ in a space X with uniform normal structure (where, N(X) is the normal structure coefficient of X). In the framework of random operators Xu [14] proved this result as above. In 1995, Domínguez and Xu [7] link the coefficients $\kappa_w(X)$ and $WCS(X)$ to fixed points of uniformly Lipschitzian mappings. Letter, Domínguez [4] was improved a result in [7] and given a class of spaces X whose $\kappa_w(X) < WCS(X)$.

In 2002, Ramírez prove the random analogues of several results due to Domínguez [4].

THEOREM 1.1. (Ramírez [10, Theorem 4.1]) *Let (Ω, Σ) be a measurable spaces with Σ a sigma-algebra of subsets of Ω. Let X be a Banach space, C a closed bounded convex and separable subset of X and $T : \Omega \times C \to C$ a random uniformly Lipschitzian operator. If there exists a constant $c \in \mathbb{R}$ such that*

$$k(\omega) \leq c < \frac{1 + \sqrt{1 + 4N(X)(\kappa_0(X) - 1)}}{2}$$

for all $\omega \in \Omega$, then T has a random fixed point.

In 1998, Domínguez [4] proved that if C is a bounded closed convex subset of a reflexive Banach spaces X and $T : C \to C$ is a asymptotically regular mapping with $\sigma(T(\omega)) \leq c < (1 + \sqrt{1 + 4WCS(X)(\kappa_\omega(X) - 1)})/2$, then T has a fixed point.

The aim of this paper is to establish some random fixed point theorems for asymptotically regular operator. We start the random version of a fixed point result based on the Lifshitz's constant of a Banach space due to Domínguez [4].

2. Preliminaries and notations

Through this paper we will consider a measurable space (Ω, Σ) (where Σ is a $\sigma-$algebra of subset of Ω) and (X, d) will be a metric space. We denote by $CL(X)$ (resp. $CB(X)$, $KC(X)$) the family of all nonempty closed (resp. closed bounded, compact) subset of X.

A set-valued operator $T : \Omega \to 2^X$ is call (Σ)-measurable if, for any open subset B of X,

$$T^{-1}(B) := \{\omega \in \Omega : T(\omega) \cap B \neq \emptyset\}$$

belongs to Σ. A mapping $x : \Omega \to X$ is said to be a *measurable selector* of a measurable set-valued operator $T : \Omega \to 2^X$ if $x(\cdot)$ is measurable and $x(\omega) \in T(\omega)$ for all $\omega \in \Omega$. Let M be a nonempty closed subset of X. An operator $T : \Omega \times M \to 2^X$ is call a random operator if, for each fixed $x \in M$, the operator $T(\cdot, x) : \Omega \to 2^X$ is measurable. We will denote by $F(\omega)$ the fixed point set of $T(\omega, \cdot)$, i.e.,

$$F(\omega) := \{x \in M : x \in T(\omega, x)\}.$$

Note that if we do not assume the existence of fixed point for the deterministic mapping $T(\omega, \cdot) : M \to 2^X, F(\omega)$ may be empty. A measurable operator $x : \Omega \to M$ is said to be a *random fixed point of a operator* $T : \Omega \times M \to 2^X$ if $x(\omega) \in T(\omega, x(\omega))$ for all $\omega \in \Omega$. Recall that $T : \Omega \times M \to 2^X$ is continuous if, for each fixed $\omega \in \Omega$, the operator $T : (\Omega, \cdot) \to 2^X$ is continuous.

Let C be a closed bounded convex subset of a Banach space X. A random operator $T : \Omega \times C \to C$ is said to be *nonexpansive* if, for fixed $\omega \in \Omega$ the map $T : (\omega, \cdot) \to C$ is nonexpansive. We will say that T is *uniformly Lipschitzian* if there exists a function $k : \Omega \to [1, +\infty)$ such that

$$\|T^n(\omega, x) - T^n(\omega, y)\| \leq k(\omega)\|x - y\|$$

for all $x, y \in C$ and for each integer $n \geq 1$. Here $T^n(\omega, x)$ is the valued at x of the nth iterate of the map $T(\omega, \cdot)$. We will say that T is *asymptotically nonexpansive* if there exists a sequence of function $k_n : \Omega \to [1, +\infty)$ such that for each fixed $\omega \in \Omega, \lim_{n\to\infty} k_n(\omega) = 1$ and

$$\|T^n(\omega, x) - T^n(\omega, y)\| \leq k_n(\omega)\|x - y\|$$

for all $x, y \in C$ and integer $n \geq 1$. The nonexpansive random map T is called *asymptotically regular* if for each $x \in K$,

$$\lim_{n\to\infty} \|T^{n+1}(\omega, x) - T^n(\omega, x)\| = 0$$

for each $\omega \in \Omega$.

Now recall the weakly convergent sequence coefficient $WCS(X)$ [7] of X is defined by

$$WCS(X) = \inf\left\{ \frac{A(\{x_n\})}{\inf_{y\in\overline{co}\{x_n\}} \limsup_{n\to\infty} \|x_n - y\|} : \{x_n\} \text{ is a weakly convergent sequence which is not norm-convergent} \right\},$$

where $A(\{x_n\}) = \limsup_{n\to\infty}\{\|x_i - x_j\| : i, j \geq n\}$ is the asymptotic diameter of $\{x_n\}$. We will use next relationship between the asymptotically center of a sequence and the characteristic of convexity of the space. Let C be a nonempty bounded closed subset of Banach spaces X and $\{x_n\}$ bounded sequence in X, we use $r(C, \{x_n\})$ and $A(C, \{x_n\})$ to denote the asymptotic radius and the asymptotic center of $\{x_n\}$ in C, respectively, i.e.

$$r(C, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}, \text{ where } \{r(x, \{x_n\})\} = \limsup_n \|x_n - x\|,$$

$$A(C, \{x_n\}) = \{x \in C : \{r(x, \{x_n\})\} = r(C, \{x_n\})\}.$$

If D is a bounded subset of X, the *Chebyshev radius* of D relative to C is defined by

$$r_C(D) := \inf\{\sup\{\|x-y\| : y \in D\} : x \in C\}.$$

DEFINITION 2.1. Let $\{x_n\}$ and C be a nonempty bounded closed subset of Banach spaces X. Then $\{x_n\}$ is called regular with respect to C if $r(C, \{x_n\}) = r(C, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.

We are going to list several result related to the concept of measurability which will be used repeatedly in next section.

THEOREM 2.2. (cf. Wagner [13]) *Let (X,d) be a complete separable metric spaces and $F : \Omega \to CL(X)$ a measurable map. Then F has a measurable selector.*

THEOREM 2.3. (cf. Tan and Yuan [12]) *Let X be a separable metric spaces and Y a metric spaces. If $f : \Omega \times X \to Y$ is a measurable in $\omega \in \Omega$ and continuous in $x \in X$, and if $x : \Omega \to X$ is measurable, then $f(\cdot, x(\cdot)) : \Omega \to Y$ is measurable.*

Follows form the separability of C and form Theorem 1.2 of Bharucha-Reid's book [1], we can easily prove the following proposition.

PROPOSITION 2.4. *Let C be a closed convex separable subset of a Banach space X and (Ω, Σ) be a measurable space. Suppose $f : \Omega \to C$ is a function that is w-measurable, i.e., for each $x^* \in X^*$, the dual space of X, the numerically-valued function $x^*f : \Omega \to (-\infty, \infty)$ is measurable, then f is measurable.*

THEOREM 2.5. (Benavidel, Lopez and Xu, cf. [5]) *Suppose C is a weakly closed nonempty separable subset of a Banach space $X, F : \Omega \to 2^X$ a measurable with weakly compact values, $f : \Omega \times C \to \mathbb{R}$ is a measurable, continuous and weakly lower semicontinuous function. Then the marginal function $r : \Omega \to \mathbb{R}$ defined by*

$$r(\omega) := \inf_{x \in F(x)} f(\omega, x)$$

and the marginal map $R : \Omega \to X$ defined by

$$R(\omega) := \{x \in F(x) : f(\omega, x) = r(\omega)\}$$

are measurable.

PROPOSITION 2.6. (Xu cf.[14]) *Let M be a separable metric space and $f : \Omega \times C \to \mathbb{R}$ be a Carathéodory map, i.e., for every $x \in M$, then the map $f(\cdot, x) : \Omega \to \mathbb{R}$ is measurable and for every $\omega \in \Omega$, the map $f(\omega, \cdot) : M \to \mathbb{R}$ is continuous. Then for any $s \in \mathbb{R}$, the map $F_s : \Omega \to M$ defined by*

$$F_s(\omega) = \{x \in M : f(\omega, x) < s, \omega \in \Omega\}$$

is measurable.

Let M be a bounded convex sunset of a Banach space X. We recall that the Lifschitz characteristic for asymptotically regular mappings, is defined;

(i) A number $b \geq 0$ is said to have property (P_ω) with respect to M if there exists some $a > 1$ such that for all $x, y \in M$ and $r > 0$ with $\|x - y\| > r$ and each weakly convergent sequence $\{\xi_n\} \subset M$ for which $\limsup \|\xi_n - x\| \leq ar$ and $\limsup \|\xi_n - y\| \leq br$, there exists some $z \in M$ such that $\liminf \|\xi_n - z\| \leq r$;

(ii) $\kappa_\omega(M) = \sup\{b > 0 : b \text{ has property } (P_\omega) \text{ w.r.t. } M\}$;

(iii) $\kappa_\omega(X) = \inf\{\kappa_\omega(m) : M \text{ as above}\}$.

If S is a mapping from a set C into itself, then we use the symbol $|S|$ to denote the Lipscitz constant of S, i.e.

$$|S| = \sup\left\{\frac{\|Sx - Sy\|}{\|x - y\|} : x, y \in C, x \neq y\right\}.$$

For a mapping T on C, we set

$$\sigma(T) = \liminf_{n \to \infty} |T^n|.$$

We recall that a random operator $T(\omega, \cdot)$ on C has *property* (P) if there exists subsequence $\{T^{n_j}(\omega, \cdot)\}$ of $\{T^n(\omega, \cdot)\}$ converges uniformly to $\liminf_{n \to \infty} |T^n(\omega, \cdot)|$.

3. Main results

The following is the stochastic version of Theorem 3 of Domínguez Benavides in [4] for asymptotically regular random operators.

THEOREM 3.1. *Let X be a reflexive Banach space, C be a nonempty bounded convex separable subset of X and $T : \Omega \times C \to C$ be a random asymptotically regular operators and $T(\omega, \cdot)$ has property (P). If there exist a constant $c \in \mathbb{R}$ such that*

$$\sigma(T(\omega, \cdot)) \leq c < \frac{1 + \sqrt{1 + 4WCS(X)(\kappa_\omega(X) - 1)}}{2}$$

for all $\omega \in \Omega$, then T has a random fixed point.

Proof. Denote $W = WCS(X)$ and $\kappa_\omega = \kappa_\omega(X)$. According to the stochastic version of Banach's contraction principle [1], we only need to prove the result if $(1 + \sqrt{1 + 4WCS(X)(\kappa_w(X) - 1)})/2 > 1$. Then we can assume that $c > 1$. Furthermore, since $\kappa_\omega \leq W$ (see Lemma 2 [4]) we have

$$\frac{1 + \sqrt{1 + 4W(\kappa_\omega - 1)}}{2} \leq W.$$

Hence $c < W$. On the other hand, the condition $c < 1+\sqrt{1+4W(\kappa_\omega - 1)}/2$ is equivalent to $c(c-1) < W(\kappa_\omega - 1)$ choose $b < \kappa_\omega$ such that $c(c-1) < W(b-1)$. We shall consider a fixed element $x_0 \in C$, and for every $\omega \in \Omega$, define

$$R(\omega, x_0) = \inf\{\liminf_n \|T^n(\omega, y) - x_0\| : y \in C\}.$$

We start by proving that $R(\cdot, x_0)$ is a measurable function. Set for each $\omega \in \Omega$,

$$f(\omega, y) = \liminf_n \|T^n(\omega, y) - x_0\|.$$

We can apply Theorem 2.3 to deduce that $f(\cdot, y)$ is measurable for each $y \in C$. Since C is a separable subset of X, it follows that there exists a countable dens subset $\{y_n\}$ of C. Therefore for each $\omega \in \Omega$ we have

$$R(\omega, x_0) = \inf_{n\geq 1} f(\omega, y_n),$$

which implies that $R(\cdot, y)$ is measurable. Take $\varepsilon > 0$ such that $(1+\varepsilon)/a := \alpha < 1$. Set

$$G(\omega) = \{y \in C : f(\omega, y) \leq R(\omega, x_0)(1+\varepsilon)\}.$$

It is clear that $G(\omega)$ is a nonempty subset C and since $f(\omega, \cdot)$ is continuous in C, it follows from Proposition 2.6 that $G(\cdot)$ is measurable. Hence, by Theorem 2.2, we can find a measurable selector $y(\omega)$ of $G(\omega)$, which verifies

$$\liminf_n \|T^n(\omega, y(\omega)) - x_0\| < R(\omega, x_0)(1+\varepsilon).$$

Since $T(\omega, \cdot)$ has *property* (P), then we can parsing through a subsequence, assume that $\sigma(T(\omega, \cdot)) = \lim_{n\to\infty} |T^n(\omega, \cdot)|$ and set $L_n(\omega) = |T^n(\omega, \cdot)|$ for all $\omega \in \Omega$. Choose an arbitrary $x_0 \in C$ and set $x_0(\omega) = x_0$, this x_0 is obvious measurable. Consider in Ω the partition given by the set:

$$\Omega_1 := \left\{\omega \in \Omega : \sup_n \|x_0 - T^n(\omega, x_0)\| \leq \frac{WR(\omega, x_0)(1+\varepsilon)}{ca}\right\}$$

and

$$\Omega_2 := \left\{\omega \in \Omega : \sup_n \|x_0 - T^n(\omega, x_0)\| > \frac{WR(\omega, x_0)(1+\varepsilon)}{ca}\right\}.$$

It is easy to prove that both set are measurable.

(*Case 1*) Assume that $\Omega_1 \neq \emptyset$, and fix $\omega \in \Omega_1$.

Now define a map $G_0 : \Omega \to C$ by

$$G(\omega) := w - cl\{T^n((\omega), x_0)\}, \quad \omega \in \Omega.$$

(Here $w - cl$ denote the closure under the weak topology of X). Then $G : \Omega \to C$ is w- measurable. By Lemma 2.2, G_0 has a w- measurable selector $u : \Omega \to C$. Since C is separable $\{u\}$ is actually measurable by

Proposition 2.4. Since for each $\omega \in \Omega$, $u(\omega)$ is a weak cluster point of $\{T^n((\omega), x_0)\}$. In fact, by definition of G_0, for a fix $\omega \in \Omega$, we can choose a subsequence $\{n_k\}$ of positive integer $\{n\}$ such that $\{T^{n_k}((\omega), x_0)\}$ converging weakly to $u(\omega)$ and

$$\lim_{k\to\infty} \|x_0 - T^{n_k}(\omega, x_0)\| = \limsup_{n\to\infty} \|x_0 - T^n(\omega, x_o)\|.$$

Since T is asymptotically regular, it follows that

$$\lim_{k\to\infty} \|T^{n_k+m}(\omega, x_0) - T^{n_k}(\omega, x_o)\| = 0$$

for any fixed m. Thus, we have

$$\begin{aligned}
&\limsup_{p\to\infty} \limsup_{q\to\infty} \|T^{n_p}(\omega, x_0) - T^{n_q}(\omega, x_0)\| \\
&\qquad \le \limsup_{p\to\infty} \limsup_{q\to\infty} L_p(\omega) \|T^{n_q - n_p}(\omega, x_0) - x_0\| \\
&\qquad \le \limsup_{p\to\infty} L_p(\omega) \limsup_{q\to\infty} \|T^{n_q - n_p}(\omega, x_0) - T^{n_q}(\omega, x_0)\| \\
&\qquad\quad + \limsup_{p\to\infty} L_p(\omega) \limsup_{q\to\infty} \|T^{n_q}(\omega, x_0) - x_0\| \\
&\qquad = \limsup_{p\to\infty} L_p(\omega) \limsup_{n\to\infty} \|T^n(\omega, x_0) - x_0\| \\
&\qquad \le \limsup_{p\to\infty} L_p(\omega) \frac{WR(\omega, x_0)(1+\varepsilon)}{ca} \\
&\qquad \le \frac{WR(\omega, x_0)(1+\varepsilon)}{a}.
\end{aligned}$$

Since C is weakly compact set. Define a function $E : \Omega_1 \to 2^C$ by

$$E(\omega) := \{u \in C : \limsup_k \|T^{n_k}(\omega, x_0) - u(\omega)\| \le D(\{T^{n_k}\})/W\}$$

is measurable in Ω_1, where

$$D(\{T^n\}) = \limsup_n \limsup_m \|T^n(\omega, x_0) - T^m(\omega, x_0)\|.$$

It follows from Proposition 2.6 that $E(\omega)$ is measurable in Ω_1. It follows from Theorem 2.2 that there exists a measurable selector. Take $z_1 : \Omega_1 \to C$ a measurable selector of $E(\cdot)$. By Theorem 1 in [6] we know that

$$W = \inf \left\{ \frac{\limsup_n \limsup_m \|x_n - x_m\|}{\limsup_n \|x_n - x_\infty\|} \right\},$$

where the infimum is taken over all weakly (not strong) converging sequences $\{x_n\}$ and $x_\infty = w - \lim x_n$. Then, the above form of W gives

$$\limsup_k \|z_1(\omega) - T^{n_k}(\omega, x_0)\| \le \frac{R(\omega, x_0)(1+\varepsilon)}{a}$$

which implies

$$\liminf_n \|z_1(\omega) - T^n(\omega, x_0)\| \le \frac{R(\omega, x_0)(1+\varepsilon)}{a}$$

and so

$$R(\omega, z_1(\omega)) \le \frac{(1+\varepsilon)R(\omega, x_0)}{a} = \alpha R(\omega, x_0).$$

Furthermore, from the weakly-lower semicontinuous of the norm we have $\|z_1(\omega) - x_0\| \le \|z_1(\omega) - u(\omega)\| + \|u(\omega) - x_0\|$. This implies that

$$\begin{aligned}\|z_1(\omega) - x_0\| &\le \liminf_k \|z_1(\omega) - T^{n_k}(\omega, x_0(\omega))\| \\ &\quad + \liminf_k \|T^{n_k}(\omega, x_0(\omega)) - x_0\| \\ &\le R(\omega, z_1(\omega)) + \frac{WR(\omega, x_0)(1+\varepsilon)}{ca} \\ &\le \alpha R(\omega, x_0) + \alpha \frac{WR(\omega, x_0)}{c} = \alpha\Big(1 + \frac{W}{c}\Big)R(\omega, x_0).\end{aligned}$$

(*Case 2*) Assume that $\Omega_2 \neq \emptyset$. In this case, for $\omega \in \Omega$, there exists $m \in \mathbb{N}$ such that $\|x_0 - T^m(\omega, x_0)\| > WR(\omega, x_0)(1+\varepsilon)/ca$ and $c(L_m(\omega) - 1) < W(\frac{b}{a} - 1)$. If $\sigma(T(\omega, \cdot)) > 1$, then we can assume that $L_m(\omega) > 1$. Choose $y \in C$ such that $\liminf_n \|x_0 - T^n(\omega, y)\| < R(\omega, x_0)(1+\varepsilon)$. Similarly with case $\Omega_1 \neq \emptyset$, we define a map $G_1 : \Omega \to C$ by

$$G_1(\omega) := w - cl\{T^n((\omega), y)\}, \quad \omega \in \Omega.$$

(Here $w - cl$ denote the closure under the weak topology of X). Then $G_1 : \Omega \to C$ is w- measurable. By Lemma 2.2, G_1 has a w- measurable selector $v : \Omega \to C$. Since C is separable $\{v\}$ is actually measurable by Proposition 2.4. Since for each $\omega \in \Omega$, $v(\omega)$ is a weak cluster point of $\{T^n((\omega), y)\}$. In fact, by definition of G_1, for a fix $\omega \in \Omega$, we can choose a subsequence $\{T^{n_k}(\omega, y)\}$ of $\{T^n(\omega, y)\}$ such that $\liminf_n \|x_0 - T^n(\omega, y)\| = \lim_k \|x_0 - T^{n_k}(\omega, y)\|$ and $\{T^{n_k}((\omega), x_0)\}$ converging weakly to $v(\omega)$. Using again the asymptotic regularity of T we obtain

$$\begin{aligned}\limsup_k \|T^m(\omega, x_0) - T^{n_k}(\omega, y)\| &\le L_m(\omega) \limsup_k \|T^{n_k}(\omega, y) - x_0\| \\ &= L_m(\omega) \liminf_n \|T^n(\omega, y) - x_0\| \\ &\le L_m(\omega) R(\omega, x_0)(1+\varepsilon).\end{aligned}$$

Choose $\lambda \in [0,1]$ such that $\frac{c}{W} < \lambda < (\frac{b}{a} - 1)/(L_m(\omega) - 1)$ if $L_m(\omega) > 1$ or $\lambda = 1$ otherwise. Then

$$\begin{aligned}\limsup_k \|T^{n_k}(\omega,y) - \lambda T^m(\omega,x_0) - (1-\lambda)x_0\| &\le \limsup_k \lambda\|T^{n_k}(\omega,y) - T^m(\omega,x_0)\| \\ &\quad + (1-\lambda)\limsup_k \|T^{n_k}(\omega,y) - x_0\| \\ &\le \lambda L_m(\omega)R(\omega,x_0)(1+\varepsilon) + (1-\lambda)R(\omega,x_0)(1+\varepsilon) \\ &\le \frac{bR(\omega,x_0)(1+\varepsilon)}{a}.\end{aligned}$$

Furthermore,

$$\begin{aligned}\|x_0 - \lambda T^m(\omega,x_0) - (1-\lambda)x_0\| &= \lambda\|T^m(\omega,x_0) - x_0\| \\ &\ge \frac{\lambda W R(\omega,x_0)(1+\varepsilon)}{ac} > \frac{R(\omega,x_0)(1+\varepsilon)}{a}.\end{aligned}$$

By the condition which b satisfies there exist $z(\omega) \in C$ such that

$$\liminf_k \|T^{n_k}(\omega,y) - z(\omega)\| \le \frac{R(\omega,x_0)(1+\varepsilon)}{a} = \alpha R(\omega,x_0).$$

Since

$$\liminf_n \|T^n(\omega,y) - z(\omega)\| \le \liminf_k \|T^{n_k}(\omega,y) - z(\omega)\|$$

we obtain $R(\omega,z) \le \alpha R(\omega,x_0)$. Therefore, from Proposition 2.6 the mapping $F : \Omega_2 \to 2^C$ given by

$$F(\omega) = \{z \in C : \limsup_n \|T^n(\omega,y(\omega)) - z(\omega)\| \le \alpha R(\omega,x_0)\}$$

is measurable. Thus it admit a measurable selector $z_2 : \Omega_2 \to C$ which satisfies $R(\omega,z_2(\omega)) < \alpha R(\omega,x_0)$. Hence by the weakly lower semicontinuous of the function norm, we have $\|x_0 - z_2(\omega)\| \le \|x_0 - v(\omega)\| + \|v(\omega) - z_2(\omega)\|$. This implies that

$$\begin{aligned}\|x_0 - z_2(\omega)\| &\le \liminf_k \|x_0 - T^{n_k}(\omega,y(\omega))\| + \liminf_k \|T^{n_k}(\omega,y(\omega)) - z_2(\omega)\| \\ &\le \liminf_n \|x_0 - T^n(\omega,y(\omega))\| + \limsup_n \|T^n(\omega,y(\omega)) - z_2(\omega)\| \\ &\le (1+\varepsilon)R(\omega,x_0) + \alpha R(\omega,x_0) = (1+\varepsilon+\alpha)R(\omega,x_0).\end{aligned}$$

Consider $z : \Omega \to C$ given by $z(\omega) = z_1(\omega)$ if $\omega \in \Omega_1$ and $z(\omega) = z_2(\omega)$ if $\omega \in \Omega_2$. Clearly $z(\cdot)$ is a measurable function. Choose an arbitrary $x_0 \in C$ and set $x_0(\omega) = x_0$. We defined $x_1(\omega) = z(\omega)$ as above and then we can inductively construct a sequence $\{x_m(\omega)\}$ of measurable functions

$x_m : \Omega \to C$ such that for each $\omega \in \Omega$ we have

$$R(\omega, x_m(\omega)) \leq \alpha R(\omega, x_{m-1}(\omega)) \leq \ldots \leq \alpha^m R(\omega, x_0(\omega)).$$

We shall prove that $\{x_m(\omega)\}$ is a Cauchy sequence. Indeed, if $M = \max\{1 + \varepsilon + \alpha, \alpha(1 + \frac{W}{c})\}$ we have

$$\|x_m(\omega - x_{m+1}(\omega))\| \leq MR(\omega, x_m(\omega)) \leq M\alpha^m R(\omega, x_0(\omega)).$$

Thus $\{x_m(\omega)\}$ converges to some $x(\omega) \in C$. It is readily see that $R(\omega, x(\omega)) = 0$ for every $\omega \in \Omega$ which implies that $x(\omega)$ is a random fixed point of T. Indeed, if $R(\omega, x(\omega)) = 0$ for any $\varepsilon > 0$, $\omega \in \Omega$ there exist $y(\omega) \in C$ such that

$$\liminf_n \|x(\omega) - T^n(\omega, y(\omega))\| \leq \varepsilon.$$

By definition of G and G_1, for a fix $\omega \in \Omega$, we can take a subsequence $\{T^{n_j}(\omega, y(\omega))\}$ of $\{T^n(\omega, y(\omega))\}$ such that

$$\liminf_n \|x(\omega) - T^n(\omega, y(\omega))\| = \lim_j \|x(\omega) - T^{n_j}(\omega, y(\omega))\|.$$

By assumption there exists a positive integer m such that $|T^m(\omega, \cdot)| < \infty$. Then

$$\limsup_j \|x(\omega) - T^{n_j+m}(\omega, y(\omega))\| = \limsup_j \|x(\omega) - T^{n_j}(\omega, y(\omega))\| \leq \varepsilon.$$

Thus

$$\begin{aligned}
&\|x(\omega) - T^m(\omega, x(\omega))\| \\
&\quad \leq \limsup_j (\|x(\omega) - T^{n_j+m}(\omega, y(\omega))\| + \|T^{n_j+m}(\omega, y(\omega)) - T^m(\omega, x(\omega))\|) \\
&\quad \leq \limsup_j \|x(\omega) - T^{n_j+m}(\omega, y(\omega))\| + |T^m(\omega, \cdot)| \lim_j \|T^{n_j}(\omega, y(\omega)) - x(\omega)\| \\
&\quad = (1 + L_m(\omega)) \liminf_n \|x(\omega) - T^n(\omega, y(\omega))\| \\
&\quad \leq (1 + L_m(\omega))\varepsilon \to 0, \quad \text{as } \varepsilon \to 0,
\end{aligned}$$

yielding for each $\omega \in \Omega$, $T^m(\omega, x(\omega)) \to x(\omega)$ and thus $x(\omega) = T(\omega, x(\omega))$ by the continuity and asymptotic regularity of $T(\omega, \cdot)$. This completes the proof. ∎

Remark 3.2. From the Theorem 3.1 we can come to conclusion that the random fixed set M may be set in general at least the continuous power and the random point appears as the probability set and as the subset of the Banach space.

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