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ON THE MAZUR–ULAM THEOREM IN METRIC GROUPS

Abstract. Let X, Y be abelian uniquely 2-divisible groups with metrics d_X, d_Y respectively, invariant with respect to the translations and let there exist a constant $c > 1$ such that $d_Y(2y, 0) \geq c d_Y(y, 0)$ for $y \in Y$. We prove that each surjective isometry $U : X \to Y$ has a form $U(x) = a(x) + U(0)$ for $x \in X$, where $a : X \to Y$ is a homomorphism.

1. Introduction

S. Mazur and S. M. Ulam proved the following well-known theorem concerning isometries between normed spaces [3], which is a useful tool in proving many results.

Theorem (S. Mazur, S. M. Ulam). *Let X, Y be real normed spaces and let $U : X \to Y$ be a surjective isometry with $U(0) = 0$. Then U is linear.*

Let X be a linear-metric space. A set $A \subset X$ is said to be bounded, if for each neighbourhood U of zero there is a number a such that $A \subset aU$. The space X is said to be locally bounded, if it contains a bounded neighbourhood of zero. The above theorem has several generalizations. For example, the Theorem IX.3.1 in [4] states, that the assertion of the Mazur-Ulam Theorem is true, if X, Y are locally bounded spaces with F-norms $\|\cdot\|_X$, $\|\cdot\|_Y$ such that for all $x \in X$, $y \in Y$ the functions $t \mapsto \|tx\|_X$, $t \mapsto \|ty\|_Y$ are concave for positive t.

It is natural to investigate the form of isometries between metric groups. We obtain two results, analogous to that of the Theorem, one when the groups admit total families of semimetrics (Def. 2) and another when the group F-norms are regular (Def. 3).

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2. Invariant metrics and F-norms

We recall that by a semimetic on a fixed set X we mean a function $d : X \to [0, \infty)$ satisfying for $x, y, z \in X$ the following conditions: $d(x, x) = 0$, $d(x, y) = d(y, x)$, $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 1. Let X be an abelian group. A function $\|\cdot\| : X \to [0, \infty)$ satisfying for $x, y \in X$:

(i) $\|0\| = 0$,
(ii) $\|-x\| = \|x\|$,
(iii) $\|x + y\| \leq \|x\| + \|y\|$

we call the F-seminorm. An F-seminorm satisfying for $x \in X$ the condition

(i′) $\|x\| = 0 \Leftrightarrow x = 0$

we call the F-norm.

If $(X, +, d)$ is an abelian group with a semimetric [a metric] d invariant with respect to the translations, i.e.

$$d(x + z, y + z) = d(x, y) \qquad \text{for } x, y, z \in X,$$

then the function

$$\|x\| := d(x, 0) \qquad \text{for } x \in X$$

is an F-seminorm [F-norm].

Conversely, if $(X, +)$ is an abelian group and a function $\|\cdot\| : X \to [0, \infty)$ is an F-seminorm [F-norm], then $d(x, y) := \|x - y\|$ for $x, y \in X$ is an invariant semimetric [metric] and $d(x, 0) = \|x\|$ for $x \in X$.

Because of the natural one-to-one corespondence between invariant semimetrics [metrics] and F-seminorms [F-norms], we will use these notions exchangeable and formulate our results in the language either of metrics or F-norms. We will consider isometries between abelian groups with invariant metrics $(X, +, d_X)$, $(Y, +', d_Y)$ in the case where the both groups are uniquely 2-divisible and the both F-norms $\|\cdot\|_X := d_X(\cdot, 0)$, $\|\cdot\|_Y := d_Y(\cdot, 0)$ satisfy the condition:

(1) there exists a constant $c > 1$ such that for all x: $\|2x\| \geq c\|x\|$.

Let us note, that if an F-norm has property (1) and $X \neq \{0\}$, then $c \leq 2$.

Indeed, from condition (iii) of definition of F-norm and from (1) we have $c\|x\| \leq \|2x\| \leq 2\|x\|$ for $x \in X$. Thus $c \leq 2$.

3. Examples

Example 1. Additive groups $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, with the F-norm $d(x, y) = |x - y|$ are the abelian uniquely 2-divisible groups with the F-norm $\|\cdot\| = d(x, 0)$ satisfying (1) with $c = 2$.

Example 2. Let X be a real linear space with F-norm satisfying for some $p \in (0,1]$ the condition:

(v) $\|tx\| = |t|^p \|x\|$ for $t \in \mathbb{R}$, $x \in X$.

Then $\|\cdot\|$ satisfies (1) with $c = 2^p$.

Example 3. Let X be real linear spaces with F-norms $\|\cdot\|_1$, $\|\cdot\|_2$ satisfying (v) with p_1 and p_2, respectively, let:

$$\|\cdot\|_{max} = \max\{\|\cdot\|_1, \|\cdot\|_2\},$$

and let

$$\|\cdot\|_s = (\|\cdot\|_1^s + \|\cdot\|_2^s)^{\frac{1}{s}},$$

for $s \in [1,\infty)$. Then $\|\cdot\|_s$ and $\|\cdot\|_{max}$ are F-norms satisfying (1) with $c = 2^{\min\{p_1,p_2\}}$.

Example 4. Let $a : \mathbb{R} \to \mathbb{R}$ be an additive injective function, $p \in (0,1]$ and let $\|x\|_a = |a(x)|^p$ for $x \in X$. Then $\|\cdot\|_a$ is an F-norm, which satisfies (1) with $c = 2^p$.

Example 5. Let $(X, \|\cdot\|$ be a normed space and let a function $g : [0,\infty) \to [0,\infty)$ satisfies, for $x, y \in X$, the following conditions:

(a) $g(x) = 0 \Leftrightarrow x = 0$,
(b) $g(2x) = 2^p g(x)$,
(c) $g(x+y) \leq g(x) + g(y)$.

Then $|||\cdot|||$ defined by $|||x||| = g(\|x\|)$ for $x \in X$ is an F-norm satisfying (1) with $c = 2^p$.

For arbitrary $p \in (0,1]$ the function $g : [0,\infty) \to [0,\infty)$,

$$g(x) = 2^{-n}(2^{(n+1)p} - 2^{np})x + 2^{np+1} - 2^{(n+1)p} \quad \text{for } x \in [2^n, 2^{n+1}]$$

and $g(0) = 0$ is an example of a function satisfying (a), (b), (c).

Our main result states, that if X, Y are the abelian uniquely 2-divisible groups with invariant metrics satisfing (1) and $U : X \to Y$ is a surjective isometry with $U(0) = 0$, then U is a homomorphism.

4. Total family of semimetrics

Let X be an abelian unique 2-divisible group. We say that $\frac{x+y}{2}$ is the algebraic center of points $x, y \in X$. The Lemma 1 shows that if $U : X \to Y$ is a surjective isometry between abelian unique 2-divisible groups with invariant semimetrics satisfying (1), then U preserves the algebraic center of each two points, modulo the subgroup $\{y \in Y : d_Y(y,0) = 0\}$.

Lemma 1. *Let X, Y be abelian unique 2-divisible groups with invariant semimetrics d_X, d_Y, let $\|\cdot\|_X := d_X(\cdot,0)$, $\|\cdot\|_Y := d_Y(\cdot,0)$ and let*

F-seminorm $\|\cdot\|_Y$ *satisfy* (1). *If a function* $U : X \to Y$ *is a surjective isometry, then*

$$\left\| U\left(\frac{x+y}{2}\right) - \frac{U(x)+U(y)}{2} \right\|_Y = 0 \tag{2}$$

for $x, y \in X$.

Proof. We follow the idea of the proof of Lemma 15.3 from [1].

Let $x, y \in X$ and let

$$p := \frac{x+y}{2}, \quad q := \frac{U(x)+U(y)}{2}, \quad d := \left\| \frac{U(x)-U(y)}{2} \right\|.$$

We define a sequence of mappings $(g_n)_{n\in\mathbf{N}_0}$, $g_n : Y \to Y$ and a sequence $(q_n)_{n\in N}$ of points as follows

$$\begin{gathered} g_0(u) := U(2p - U^{-1}(u)) \quad \text{for } u \in Y, \\ g_1(u) := 2q - u \quad \text{for } u \in Y, \\ g_{n+1} := g_{n-1} \circ g_n \circ g_{n-1}^{-1} \quad \text{for } n \in N, \\ q_1 = q, \\ q_{n+1} := g_{n-1}(q_n) \quad \textit{for } n \in N. \end{gathered}$$

We will estimate the „distance", with respect to the semimetric d_Y, between q and $U(p)$. To do it we will estimate the norms of differences $q_n - q_{n-1}$.

It is obvious that for all $n \in \mathbb{N}_0$, g_n is an isometry.

One can prove by induction that

$$g_n(U(x)) = U(y), \quad g_n(U(y)) = U(x). \tag{3}$$

for $n \in \mathbb{N}_0$. In turn, we prove inductively that

$$\|q_n - U(x)\| = d, \quad \|q_n - U(y)\| = d \tag{4}$$

for $n \in \mathbb{N}$.

We have

$$\|q_1 - U(x)\| = \left\| \frac{U(x)+U(y)}{2} - U(x) \right\| = \left\| \frac{U(y)-U(x)}{2} \right\| = d$$

and similarly

$$\|q_1 - U(y)\| = d.$$

Hence (4) is valid for $n = 1$. Assume that (4) is valid for some $n \in \mathbb{N}$. Then making use of (3) and the induction hypothesis, we obtain

$$\|q_{n+1} - U(x)\| = \|g_{n-1}(q_n) - g_{n-1}(U(y))\| = \|q_n - U(y)\| = d.$$

In a similiar way we check that $\|q_{n+1} - U(y)\| = d$.

From (4) we have

$$\|q_n - q_{n-1}\| \le (\|q_n - U(x)\| + \|U(x) - q_{n-1}\|) = 2d.$$

Consequently

$$\|q_n - q_{n-1}\| \le 2d \qquad \text{for } n \ge 2 \quad n \in \mathbb{N}. \tag{5}$$

We prove by induction, that

$$\|g_n(u) - u\| \ge c\|q_n - u\| \tag{6}$$

for $u \in Y, n \in \mathbb{N}$.

We have

$$\|g_1(u) - u\| = \|2q - u - u\| \ge c\|q - u\| \ge c\|q - u\|.$$

By the induction hypothesis we receive that for all $u \in Y$

$$\begin{aligned}\|g_{n+1}(u) - u\| &= \|g_{n-1}g_n g_{n-1}^{-1}(u) - g_{n-1}g_{n-1}^{-1}(u)\| \\ &= \|g_n g_{n-1}^{-1}(u) - g_{n-1}^{-1}(u)\| \ge c\|q_n - g_{n-1}^{-1}(u)\| \\ &= c\|g_{n-1}^{-1}g_{n-1}(q_n) - g_{n-1}^{-1}(u)\| = c\|g_{n-1}(q_n) - u\| \\ &= c\|q_{n+1} - u\|.\end{aligned}$$

Substituting $u = q_{n+1}$ in (6), we obtain for $n \in \mathbb{N}$

$$\|q_{n+2} - q_{n+1}\| \ge c\|q_{n+1} - q_n\|. \tag{7}$$

Hence, for $n \ge 3$, $n \in \mathbb{N}$

$$\|q_n - q_{n-1}\| \ge c^{n-2}\|q_2 - q_1\|. \tag{8}$$

From inequality (8) we have

$$\|q_2 - q_1\| \le \frac{1}{c^{n-2}}\|q_n - q_{n-1}\| \qquad \text{for } n \ge 3.$$

By this inequality and (5) we get, for $n \ge 3$, $n \in \mathbb{N}$

$$\|q_2 - q_1\| \le \frac{2}{c^{n-2}}d. \tag{9}$$

On the other hand we have

$$\begin{aligned}\|q_2 - q_1\| &= \|U(2p - U^{-1}(q)) - UU^{-1}(q)\| = \|2p - 2U^{-1}(q)\| \\ &\ge c(\|U(p) - q\| = c\|U(p) - q\| \ge c\|U(p) - q\|.\end{aligned}$$

From this inequality and from (9) we get

$$c\|U(p) - q\| \le \|q_2 - q_1\| \le \frac{2}{c^{n-2}}d$$

for $n \ge 3$, $n \in \mathbb{N}$.

Consequently

$$\left\|U\left(\frac{x+y}{2}\right) - \frac{U(x) + U(y)}{2}\right\| \le \frac{2}{c^{n-1}}\left\|\frac{x-y}{2}\right\| \le \frac{2}{c^{n-1}}\frac{1}{c}\|x-y\| = \frac{2}{c^n}\|x-y\|$$

for $n \in \mathbb{N}$. Passing to the limit when $n \to \infty$, we get the assertion. ∎

Definition 2. Let X be a nonempty set and let $\{d_s\}_{s\in S}$ be a family of semimetrics defined on X. We say that this family is total, if for each x, $y \in X$ such that $x \neq y$ there exists $s \in S$ such that $d_s(x,y) > 0$.

Theorem 1. *Let X, Y be the abelian unique 2-divisible groups with families $\{d_s\}_{s\in S}$, $\{\rho_s\}_{s\in S}$ of invariant semimetrics, let each ρ_s satisfy (1) and let the family $\{\rho_s\}_{s\in S}$ be total. If a function $U : X \to Y$ is a surjection with $U(0) = 0$ satisfying*

$$\rho_s(U(x), U(y)) = d_s(x,y) \qquad \text{for } x, y \in X, s \in S,$$

then U is a homomorphism.

Proof. By Lemma 1 we obtain that for each $x, y \in X$ and each $s \in S$

$$\rho_s\left(U\left(\frac{x+y}{2}\right) - \frac{U(x)+U(y)}{2}, 0\right) = 0.$$

By the assumption $\{\rho_s\}_{s\in S}$ is total, thus U satisfies the Jensen equation:

$$U\left(\frac{x+y}{2}\right) = \frac{U(x)+U(y)}{2} \qquad \text{for } x, y \in X.$$

Since $U(0) = 0$, we have $U(\frac{z}{2}) = \frac{U(z)}{2}$ for $z \in X$. For arbitrary $x, y \in X$ we have

$$U(x+y) = 2U\left(\frac{x+y}{2}\right) = 2\frac{U(x)+U(y)}{2} = U(x) + U(y).$$

Thus U is a homomorphism. ∎

Obviously, if d is a metric, then the family consisting of the singleton d is total. Thus we have the following

Corollary 1. *Let X, Y be abelian unique 2-divisible groups with invariant metrics and let F-norm in Y satisfies (1). If a function $U : X \to Y$ is a surjective isometry with $U(0) = 0$, then U is a homomorphism.*

Remark 1. The $c > 1$ is essential in Corollary 1. Indeed, let $c_1, c_2 > 0$, $c_1 \leq 2c_2$, and let $\|x\| = c_1$ for $x \in \mathbb{Q} \setminus \{0\}$, $\|x\| = c_2$ for $x \in \mathbb{R} \setminus \mathbb{Q}$, $\|0\| = 0$. Then $\|\cdot\|$ is an F-norm, satisfying condition (1) and $(\mathbb{R}, +,)$ is an abelian uniquely 2-divisible group. For an arbitrary bijection $g : \mathbb{Q} \setminus \{0\} \to \mathbb{Q} \setminus \{0\}$ a function $U : \mathbb{R} \to \mathbb{R}$, $U(x) = g(x)$ for $x \in \mathbb{Q} \setminus \{0\}$, and $U(x) = x$ for $x \in \mathbb{R} \setminus \mathbb{Q}$, $U(0) = 0$, is a surjective isometry from $(\mathbb{R}, \|\cdot\|)$ onto itself, but U is not additive if $g(x) \neq x$ for an $x \in \mathbb{Q} \setminus \{0\}$.

5. Regular F-norms

In this section we consider isometries between real linear spaces with invariant metrics satisfying (1).

DEFINITION 3. Let X be a real linear space with an invariant metric d. We say that the F-norm $\|\cdot\| := d(\cdot, 0)$ is regular, if for every $x \in X$ a function $\mathbb{R} \ni t \mapsto \|tx\| \in \mathbb{R}$ is bounded on a set $A \subset \mathbb{R}$ of the positive inner Lebesgue measure or is of the second category with the Baire property.

Let us note that if for every $x \in X$ the function $\mathbb{R} \ni t \mapsto \|tx\| \in \mathbb{R}$ is Lebesgue- or Baire measurable, then the F-norm $\|\cdot\|$ is regular.

LEMMA 2. *Let X be a real linear space with an invariant metric d satisfying* (1)*. Then the following conditions are equivalent:*

(a) *the F-norm $\|\cdot\|$ is regular ,*
(b) *for every $x \in X$ function $\mathbb{R} \ni t \mapsto tx \in X$ is continuous on $\mathbb{R}$.*

Proof. Because of the continuity of F-norm (b) implies (a). We will show that (a) implies (b). Let $(t_n) \subset \mathbb{R}$, $t_n \to 0$ and $x \in X$. From Theorem XVI.2.6 [2], we obtain that a function $\mathbb{R} \ni t \mapsto \|tx\| \in \mathbb{R}$ is bounded on every bounded set. In particular, there exists a constant $K(x) > 0$ such that $\|tx\| \leq K(x)$ for all $t \in (-1, 1)$. One can prove that there is a sequence (l_n), $l_n \in \mathbb{R}$ such that $l_n \to \infty$ and $2^{l_n} t_n \to 0$. Thus $|2^{l_n} t_n| < 1$ for sufficient large n. Using (1) we get, for such n, that

$$c^{l_n}\|t_n x\| \leq \|t_n 2^{l_n} x\| \leq K(x).$$

Hence $\|t_n x\| \leq \frac{K(x)}{c^{l_n}} \to 0$ as $n \to \infty$. Consequently we obtain that the function $\mathbb{R} \ni t \mapsto tx \in X$ is continuous at 0. Since it is additive, it is continuous on $\mathbb{R}$. ∎

THEOREM 2. *Let X, Y be real spaces with invariant metrics d_X, d_Y, let $\|\cdot\|_X := d_X(\cdot, 0)$, $\|\cdot\|_Y := d_Y(\cdot, 0)$ the latter F-norm satisfying* (1)*. If a function $U : X \to Y$ is a surjective isometry with $U(0) = 0$, then U is additive.*

Moreover, if F-norms $\|\cdot\|_X$, $\|\cdot\|_Y$ are regular, then U is linear.

Proof. The first part of the assertion immediately follows from Corollary 1.

Suppose that $\|\cdot\|_X$, $\|\cdot\|_Y$ are regular. For an arbitrary $x \in X$ we have

$$\|2x\|_X = \|U(2x)\|_Y = \|2U(x)\|_Y \geq c\|U(x)\|_Y = c\|x\|_X.$$

Hence the F-norm in X also satisfies (1).

By the additivity of U we have $U(tx) = tU(x)$ for $t \in \mathbb{Q}$, $x \in X$. From Lemma 2 we have, that for all $x \in X$, $y \in Y$ the functions $\mathbb{R} \ni t \mapsto tx \in X$ and $\mathbb{R} \ni t \mapsto ty \in Y$ are continuous. Let $t \in \mathbb{R}$, $x \in X$ and let (t_n), $t_n \in \mathbb{Q}$, $\lim_{n\to\infty} t_n = t$. Then because of the continuity of both the two function just introduced and the isometry U we get

$$U(tx) = U(\lim_{n\to\infty}(t_n x)) = \lim_{n\to\infty} U(t_n x) = \lim_{n\to\infty}(t_n U(x)) = tU(x).$$

Consequently U is also homogeneous, therefore linear. ∎

Remark 2. The assumptions that F-norms are regular are essential for the linearity of U in Theorem 2. Indeed, let $U : \mathbb{R} \to \mathbb{R}$ be an additive, bijective and discontinuous function, $p \in (0,1]$, and let $\|x\|_1 = |U(x)|^p$, $\|x\|_2 = |x|^p$. Then $\|\cdot\|_1$, $\|\cdot\|_2$ are F-norms in $\mathbb{R}$ satisfying (1), with $c = 2^p$. Obviously the F-norm $\|\cdot\|_1$ is not regular. Since $\|U(x) - U(y)\|_2 = \|U(x-y)\|_2 = \|x-y\|_1$ for $x, y \in \mathbb{R}$, U is a surjective isometry from $(\mathbb{R}, \|\cdot\|_1)$ onto $(\mathbb{R}, \|\cdot\|_2)$, but it is not homogeneous.

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