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ON q-BASKAKOV TYPE OPERATORS

Abstract. In the present paper we introduce two q-analogous of the well known Baskakov operators. For the first operator we obtain convergence property on bounded interval. Then we give the montonity on the sequence of q-Baskakov operators for n when the function f is convex. For second operator, we obtain direct approximation property on unbounded interval and estimate the rate of convergence.

One can say that, depending on the selection of q, these operators are more flexible then the classical Baskakov operators while retaining their approximation properties.

1. Introduction

Phillips [13] introduced the generalization of Bernstein polynomials based on *q-integers*. Very recently Aral [4] introduced the q-Szász-Mirakyan operators. Aral and Gupta [5] extended the study and established some approximation properties for q-Szász Mirakyan operators. We now try to define some other q-analogue of exponential type operators. Before introducing the operators, we mention some properties of q-calculus (see [9] and [12]).

For any fixed real number $q > 0$ and non-negative integer r, the *q-integers* of the number r is defined by

$$(1.1) \qquad [r]_q = \begin{cases} (1-q^r)/(1-q), & q \neq 1 \\ r, & q = 1 \end{cases}.$$

Also we have $[0]_q = 0$.

The *q-factorial* is defined in the following:

$$[r]_q! = \begin{cases} [r]_q[r-1]_q \dots [1]_q, & r = 1, 2, \dots \\ 1, & r = 0 \end{cases}$$

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and *q-binomial* coefficient is defined as

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[r]_q![n-r]_q!}$$

for integers $n \geq r \geq 0$. Also, let us recall the following identity

$$\sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q x^k = \frac{1}{(x;q)_n}, \quad |x| < 1, \tag{1.2}$$

where

$$(x;\ q)_n = (1-x)(1-qx)\dots\left(1-q^{n-1}x\right)$$

(see [3, p. 420]).

Let

$$\left(\frac{x}{1+x};\ q\right)_n = \left(1-\frac{x}{1+x}\right)\left(1-q\frac{x}{1+x}\right)\cdots\left(1-q^{n-1}\frac{x}{1+x}\right). \tag{1.3}$$

We can easily see that

$$\begin{aligned} \left(\frac{qx}{1+x};\ q\right)_n &= (1+x)\left(\frac{x}{1+x};\ q\right)_{n+1}, \\ \left(\frac{x}{q(1+x)};\ q\right)_{n+1} &= \frac{(q+x(q-1))}{(1+x(1-q^n))\,q(1+x)}\left(\frac{qx}{1+x};\ q\right)_n, \\ \left(\frac{x}{q(1+x)};\ q\right)_{n+2} &= \frac{(q+x(q-1))}{q(1+x)^2}\left(\frac{qx}{1+x};\ q\right)_n. \end{aligned} \tag{1.4}$$

Motivated by the generalization of the Bernstein polynomial based on q-integers, by Phillips [13] and subsequent work in this direction (see e. g. [4], [5], [12] etc.), we introduce a new Baskakov type operators based on *q-integers* as follows:

For $f \in C[0,\infty)$, $q \in (0,1)$ and each positive integer n, q-Baskakov operators are defined as

$$B_{n,q}(f;x) = \left(\frac{qx}{1+x};q\right)_n \sum_{k=0}^{\infty} f\left(\frac{[k]_q}{q^k [n]_q}\right) \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \left(\frac{qx}{1+x}\right)^k, \tag{1.5}$$

where $x \in \mathbb{R}^+ := [0,\infty)$.

It is observed that in a special case if $q = 1$, the operators (1.5) reduce to the well known Baskakov operators [6], defined by:

$$B_n(f,x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right).$$

The well known Baskakov operators $B_n(f,x)$ and their different generalization were studied by many researchers. Pethe [11] and Altomare and Mongino [2] studied some approximation properties of certain generalized Baskakov operators. Very recently Cao et al. [7] studied multivariate Baskakov operators and gave some shape preserving properties such as monotony, semi-additivity and convexity.

In the present paper, we study the approximation properties of q-Baskakov operators defined by (1.5), we first give uniform convergence of $B_{n,q}(f,x)$ on a compact subset of $\mathbb{R}^+$ using Bohman and Korovkin Theorem. For convex function f, we also establish the monotonicity property for the sequence of these operators. In the last section we propose another q-generalization of Baskakov operators, we obtain direct approximation result on unbounded interval and give rate of convergence of these operators.

2. Convergence of q-Baskakov operators

In the sequel we shall require the following lemma:

Lemma 1. *For $q \in (0,1)$, we have*

$$B_{n,q}(1;x) = 1, \quad B_{n,q}(t;x) = x, \quad \textit{for } x \in \mathbb{R}^+,$$
$$B_{n,q}(t^2;x) = \frac{x^2}{(q+x(q-1))} + \frac{1}{[n]_q}\frac{x^2+x}{(q+x(q-1))} \quad \textit{for } x \in \left[0, \frac{q}{1-q}\right).$$

Proof. We deduce from (1.5) that $B_{n,q}(1;x) = 1$, to calculate $B_{n,\ q}(t;x)$ using (1.4) and (1.2) we proceed as follows:

$$\begin{aligned}
B_{n,q}(t;x) &= \left(\frac{qx}{1+x};\ q\right)_n \sum_{k=1}^{\infty} \frac{[k]_q}{q^k\,[n]_q} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \left(\frac{qx}{1+x}\right)^k \\
&= \left(\frac{qx}{1+x};\ q\right)_n \sum_{k=1}^{\infty} \begin{bmatrix} n+k-1 \\ k-1 \end{bmatrix}_q \left(\frac{x}{1+x}\right)^k \\
&= \frac{x}{1+x}\left(\frac{qx}{1+x};\ q\right)_n \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{1+x}\right)^k \\
&= \frac{x}{1+x}\frac{\left(\frac{qx}{1+x};\ q\right)_n}{\left(\frac{x}{1+x};\ q\right)_{n+1}} \\
&= x.
\end{aligned}$$

Next using the identity $[k]_q^2 = [k]_q\left(q\,[k-1]_q + 1\right)$ then we have

$$
\begin{aligned}
B_{n,q}(t^2;x) &= \left(\frac{qx}{1+x};\, q\right)_n \sum_{k=1}^{\infty} \frac{[k]_q^2}{q^{2k}[n]_q^2} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \left(\frac{qx}{1+x}\right)^k \\
&= \left(\frac{qx}{1+x};\, q\right)_n \sum_{k=2}^{\infty} \frac{q[k]_q[k-1]_q}{q^{2k}[n]_q^2} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \left(\frac{qx}{1+x}\right)^k \\
&\quad + \left(\frac{qx}{1+x};\, q\right)_n \sum_{k=1}^{\infty} \frac{[k]_q}{q^{2k}[n]_q^2} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \left(\frac{qx}{1+x}\right)^k \\
&= \frac{x^2}{q(1+x)^2} \frac{[n+1]_q}{[n]_q} \left(\frac{qx}{1+x};\, q\right)_n \sum_{k=0}^{\infty} \begin{bmatrix} n+k+1 \\ k \end{bmatrix}_q \left(\frac{x}{q(1+x)}\right)^k \\
&\quad + \frac{1}{[n]_q} \frac{x}{q(1+x)} \left(\frac{qx}{1+x};\, q\right)_n \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{q(1+x)}\right)^k \\
&= \frac{x^2}{q(1+x)^2} \frac{[n+1]_q}{[n]_q} \frac{\left(\frac{qx}{1+x};\, q\right)_n}{\left(\frac{x}{q(1+x)};\, q\right)_{n+2}} \\
&\quad + \frac{1}{[n]_q} \frac{x}{q(1+x)} \frac{\left(\frac{qx}{1+x};\, q\right)_n}{\left(\frac{x}{q(1+x)};\, q\right)_{n+1}}.
\end{aligned}
$$

Using the equalities (1.4), we have

$$
\begin{aligned}
B_{n,q}(t^2;x) &= \frac{x^2}{(q+x(q-1))} \frac{[n+1]_q}{[n]_q} + \frac{1}{[n]_q} \frac{x(1+x(1-q^n))}{(q+x(q-1))} \\
&= \frac{x^2}{(q+x(q-1))} \left(q + \frac{1}{[n]_q}\right) + \frac{1}{[n]_q} \frac{x(1+x(1-q^n))}{(q+x(q-1))} \\
&= \frac{qx^2}{(q+x(q-1))} + \frac{1}{[n]_q} \frac{x^2+x}{(q+x(q-1))} + \frac{(1-q)x^2}{(q+x(q-1))} \\
&= \frac{x^2}{(q+x(q-1))} + \frac{1}{[n]_q} \frac{x^2+x}{(q+x(q-1))}.
\end{aligned}
$$

This completes the proof of Lemma 1. ∎

Let $C[0,\ a]$ denote the space of all real-valued continuous functions on $[0,\ a]$, $a > 0$. By $C_M[0,\ a]$ be denote the space of all functions f which are continuous on $[0,\ a]$ and bounded on $\mathbb{R}^+ := [0,\ \infty)$. The spaces of bounded functions, endowed with the norm

$$
\|f\|_{C[0,\ a]} = \sup_{x\in[0,\ a]} |f(x)|,
$$

where $f \in C[0,\ a]$.

The uniform convergence for the q-Baskakov operators can be deduced as a consequence of Bohman & Korovkin Theorem (See [8, pp. 67]).

THEOREM 1. *Let* (q_n) *denote a sequence such that* $q_n \to 1$ *as* $n \to \infty$ *and the inequality* $\frac{a}{a+1} < q_n < 1$ *holds for fixed* $a > 0$ *and* n *large enough. If* $f \in C_M\,[0,\ a]$ *then*

$$\lim_{n\to\infty} \|B_{n,q_n}(f;x) - f\,(x)\|_{C[0,\ a]} = 0.$$

Let $C_x\,[0,\ \infty)$ be the space of continuous functions f on $\mathbb{R}^+$ such that the condition

$$|f\,(x)| \le M\,(1+x)$$

holds. For any positive b we denote the modulus of continuity of function f on closed interval $[0,\ b]$ with

$$\omega_b\,(f,\ \delta) = \sup_{\substack{t,x\in[0,\ b]\\ |t-x|\le\delta}} |f\,(t) - f\,(x)|\,.$$

In the following theorem, we give a estimate of approximation of unbounded functions via modulus of continuity of derivative of function (see [10]).

THEOREM 2. *Let* $a > 0$ *and* $\frac{a}{a+1} < q < 1$. *If the function* $f' \in C_x\,[0,\ \infty)$, *then we have*

$$\|B_{n,q}(f;x) - f\,(x)\|_{C[0,\ a]} \le 2\delta_n\,(q)\,\omega_{a+1}\left(f',\ \delta_n\,(q)\right) + M\,(3+2a)\,\delta_n^2\,(q)\,,$$

where $\delta_n(q) = \sqrt{A\left(\frac{1}{q} + \frac{1}{q[n]_q} - 1\right)}$ *and* A, M *are positive constant.*

Proof. By the mean value theorem there exist $\xi \in (t,x)$ such that

$$f\,(t) - f\,(x) = (t-x)\,f'\,(x) + (t-x)\left(f'\,(\xi) - f'\,(x)\right)$$

holds. Because of the positivity we can apply $B_{n,q}$ to this equality and after using Lemma 1 we have

$$|B_{n,q}(f\,(t) - f\,(x)\,;x)| \le B_{n,q}(|t-x|\left|f'\,(\xi) - f'\,(x)\right|;x). \tag{2.1}$$

Besides, since $f' \in C_x\,[0,\ \infty)$ we have for $x \in [0,\ a]$ and $t > a+1$

$$\begin{aligned}\left|f'\,(\xi) - f'\,(x)\right| &\le M\,(2+\xi+x)\\ &\le M\,(3+2a)\,|t-x|\end{aligned} \tag{2.2}$$

where $|t-x| > 1$. Also we have for $x \in [0,\ a]$ and $t \in [0,\ a+1]$

$$\begin{aligned}\left|f'\,(\xi) - f'\,(x)\right| &\le \omega_{a+1}\left(f',\ |t-x|\right)\\ &\le \omega_{a+1}\left(f',\ \delta_n\,(q)\right)\left(1 + \frac{|t-x|}{\delta_n\,(q)}\right).\end{aligned} \tag{2.3}$$

Now (2.2) and (2.3) imply that for $x \in [0,\ a]$ and $t \in [0,\ \infty)$

$$\left|f'(\xi) - f'(x)\right| \leq \omega_{a+1}\left(f',\ \delta_n(q)\right)\left(1 + \frac{|t-x|}{\delta_n(q)}\right) + M(3+2a)|t-x|.$$

Then the Cauchy–Schwartz inequality for positive functionals and (2.1) lead to

$$B_{n,q}(|t-x|\left|f'(\xi) - f'(x)\right|; x) \leq \omega_{a+1}\left(f',\ \delta_n(q)\right)\left(\frac{1}{\delta_n(q)}B_{n,q}((t-x)^2; x) + \sqrt{B_{n,q}((t-x)^2; x)}\right) + M(3+2a)B_{n,q}((t-x)^2; x).$$

If we choose $\delta_n^2(q) = \max_{x\in[0,\ a]} B_{n,q}((t-x)^2; x)$, then we have desired result. ∎

Remark 1. Since $x \in [0, a]$, from Lemma 1 we get

$$\begin{aligned}
B_{n,q}((t-x)^2; x) &= B_{n,q}(t^2; x) - 2xB_{n,q}(t; x) + x^2 B_{n,q}(1; x)\\
&= x^2\left(\frac{1}{q + x(q-1)} - 1\right) + \frac{1}{[n]_q}\frac{x^2+x}{q+x(q-1)}\\
&= x^2\left(\frac{1-q+x(1-q)}{q+x(q-1)}\right) + \frac{1}{[n]_q}\frac{x^2+x}{q+x(q-1)}\\
&= (1-q)\left(\frac{x^2(1+x)}{q+x(q-1)} + \frac{1}{1-q^n} \times \frac{x^2+x}{q+x(q-1)}\right)\\
&\leq M\left(\frac{1}{q+x(q-1)} + \frac{1}{1-q^n} \times \frac{1}{q+x(q-1)}\right)\\
&\leq M\left(\frac{1}{q+a(q-1)} + \frac{1}{1-q^n} \times \frac{1}{q+a(q-1)}\right) =: D(q),
\end{aligned}$$

where $M = (1-q)\max\left\{a^2(1+a), a^2+a\right\}$. We choose $q = q_n$ such that $q_n \to 1$ and $q_n^n \to c$ (c is a constant) in the assumptions of Theorem 2. From Lemma 1 we get $\delta_n^2(q_n) = B_{n,q_n}((t-x)^2; x) \leq D(q_n)$ which tends to zero as $n \to \infty$, so that $\delta_n^2(q_n) \leq \delta_n(q_n)$ for n large enough. Since $\delta_n(q_n) \leq \omega_{a+1}\left(f',\ \delta_n(q)\right)$ for $f' \neq const$ on the interval $[0,\ a+1]$, in the preceding theorem we can write

$$\left\|B_{n,q_n}(f; x) - f(x)\right\|_{C[0,\ a]} \leq C\omega_{a+1}\left(f',\ \delta_n(q_n)\right),$$

for n large enough, where C is a positive constant which depends on f'.

3. Monotony for the sequence of q-Baskakov operator

Note that, Phillips [12, pp.270] proved that the sequence of the q-Bernstein operators are decreases as n, when f is convex. However, it is shown in

[5] that the q-Szász Mirakyan operator does not satisfy this property. But, it is interesting that the q-Baskakov operator defined by (1.5) satisfies similar property as in the q-Bernstein operators.

THEOREM 3. *If f is a convex function defined on $\mathbb{R}^+$, then the q-Baskakov operator $B_{n,q}(f,\cdot)$ defined by (1.5) is strictly monotonically non-decreasing in n for all $q \in (0,\ 1)$, unless f is the linear function (in which case $B_{n,q}(f,\cdot) = B_{n+1,q}(f,\cdot)$ for all n).*

Proof. From (1.5) we can write

$$\begin{aligned}
&B_{n,q}(f;x) - B_{n+1,q}(f;x)\\
&= \left(\frac{qx}{1+x};\ q\right)_n \sum_{k=0}^{\infty} f\left(\frac{[k]_q}{q^k[n]_q}\right) \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \left(\frac{qx}{1+x}\right)^k\\
&- \left(\frac{qx}{1+x};\ q\right)_{n+1} \sum_{k=0}^{\infty} f\left(\frac{[k]_q}{q^k[n+1]_q}\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{qx}{1+x}\right)^k.
\end{aligned}$$

By (1.3), we have

$$\begin{aligned}
\left(\frac{qx}{1+x};\ q\right)_{n+1} &= \left(1 - q^{n+1}\frac{x}{1+x}\right)\left(\frac{qx}{1+x};\ q\right)_n\\
&= \left(\frac{1+x\left(1-q^{n+1}\right)}{1+x}\right)\left(\frac{qx}{1+x};\ q\right)_n
\end{aligned}$$

and this equality leads to

$$\begin{aligned}
(3.1)\quad &B_{n,q}(f;x) - B_{n+1,q}(f;x)\\
&= \left(\frac{qx}{1+x};\ q\right)_n \sum_{k=0}^{\infty} \left\{ f\left(\frac{[k]_q}{q^k[n]_q}\right) \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \right.\\
&\left. - \left(\frac{1+x\left(1-q^{n+1}\right)}{1+x}\right) f\left(\frac{[k]_q}{q^k[n+1]_q}\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q \right\} \left(\frac{qx}{1+x}\right)^k\\
&= \left(\frac{qx}{1+x};\ q\right)_n \sum_{k=1}^{\infty} \left\{ f\left(\frac{[k]_q}{q^k[n]_q}\right) \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \right.\\
&\left. - f\left(\frac{[k]_q}{q^k[n+1]_q}\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q \right\} \left(\frac{qx}{1+x}\right)^k\\
&+ \left(\frac{qx}{1+x};\ q\right)_n \sum_{k=0}^{\infty} \left(1 - \frac{1+x\left(1-q^{n+1}\right)}{1+x}\right)
\end{aligned}$$

$$\times f\left(\frac{[k]_q}{q^k[n+1]_q}\right)\begin{bmatrix} n+k \\ k \end{bmatrix}_q\left(\frac{qx}{1+x}\right)^k$$

$$= \left(\frac{qx}{1+x};\, q\right)_n \sum_{k=0}^{\infty}\left\{ f\left(\frac{[k+1]_q}{q^{k+1}[n]_q}\right)\begin{bmatrix} n+k \\ k+1 \end{bmatrix}_q - f\left(\frac{[k+1]_q}{q^{k+1}[n+1]_q}\right)\begin{bmatrix} n+k+1 \\ k+1 \end{bmatrix}_q + q^n f\left(\frac{[k]_q}{q^k[n+1]_q}\right)\begin{bmatrix} n+k \\ k+1 \end{bmatrix}_q \right\}\left(\frac{qx}{1+x}\right)^{k+1}.$$

If we take

$$\lambda_1 = \frac{\begin{bmatrix} n+k \\ k+1 \end{bmatrix}_q}{\begin{bmatrix} n+k+1 \\ k+1 \end{bmatrix}_q}, \qquad \lambda_2 = q^n \frac{\begin{bmatrix} n+k \\ k \end{bmatrix}_q}{\begin{bmatrix} n+k+1 \\ k+1 \end{bmatrix}_q}$$

and

$$x_1 = \frac{[k+1]_q}{q^{k+1}[n]_q}, \qquad x_2 = \frac{[k]_q}{q^k[n+1]_q}$$

then we have

$$\lambda_1 x_1 + \lambda_2 x_2 = \frac{[n]_q}{[n+k+1]_q}\frac{[k+1]_q}{q^{k+1}[n]_q} + q^{n+1}\frac{[k+1]_q}{[n+k+1]_q}\frac{[k]_q}{q^{k+1}[n+1]_q}$$

$$= \frac{[k+1]_q}{q^{k+1}[n+k+1]_q}\left(1 + \frac{q^{n+1}[k]_q}{[n+1]_q}\right) = \frac{[k+1]_q}{q^{k+1}[n+1]_q}$$

and

$$\lambda_1 + \lambda_2 = 1.$$

Since f is convex, we obtain

$$f\left(\frac{[k+1]_q}{q^{k+1}[n+1]_q}\right) \le \frac{\begin{bmatrix} n+k \\ k+1 \end{bmatrix}_q}{\begin{bmatrix} n+k+1 \\ k+1 \end{bmatrix}_q} f\left(\frac{[k+1]_q}{q^{k+1}[n]_q}\right) + q^n \frac{\begin{bmatrix} n+k \\ k \end{bmatrix}_q}{\begin{bmatrix} n+k+1 \\ k+1 \end{bmatrix}_q} f\left(\frac{[k]_q}{q^k[n+1]_q}\right).$$

Thus, from (3.1) we have desired result. ∎

4. Another version of q-Baskakov operator

We can easily see that

$$(4.1)\quad \begin{aligned} \left(\frac{q^2x}{1+x};\,q\right)_n &= \frac{(1+x)^2}{(1+x(1-q))}\left(\frac{x}{1+x};\,q\right)_{n+2} \\ \left(\frac{q^2x}{1+x};\,q\right)_n &= \frac{(1+x)\left(1+x\left(1-q^{n+1}\right)\right)}{(1+x(1-q))}\left(\frac{x}{1+x};\,q\right)_{n+1} \\ \left(\frac{q^2x}{1+x};\,q\right)_n &= \frac{(1+x)}{(1+x(1-q))}\left(\frac{qx}{1+x};\,q\right)_{n+1}. \end{aligned}$$

For $f \in C[0,\infty)$, $q \in (0,1)$ and each positive integer n, another version of q-Baskakov operators are defined as

(4.2)

$$B^*_{n,q}(f;x) = \left(\frac{q^2x}{1+x};\,q\right)_n \sum_{k=0}^{\infty} f\left(\frac{[k]_q}{q^{k+1}[n]_q}\right)\begin{bmatrix} n+k+1 \\ k \end{bmatrix}_q \left(\frac{q^2x}{1+x}\right)^k.$$

Lemma 2. *For $q \in (0,\ 1)$ and $x \in \mathbb{R}^+$, we have*

$$B^*_{n,q}(1;x) = 1, \quad B^*_{n,q}(t;x) = \frac{x}{(1+x(1-q))},$$

$$B^*_{n,q}(t^2;x) = \frac{x^2}{q(1+x(1-q))} + \frac{1}{[n]_q}\frac{x^2+x}{q^2(1+x(1-q))}.$$

Proof. By (4.2), it is obvious that $B^*_{n,q}(1;x) = 1$, to estimate $B^*_{n,q}(t^i;x) = x^i$, $\imath = 1,2$ using (4.1) we proceed as follows:

$$\begin{aligned} B^*_{n,q}(t;x) &= \left(\frac{q^2x}{1+x};\,q\right)_n \sum_{k=1}^{\infty}\frac{[k]_q}{q^{k+1}[n]_q}\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \left(\frac{q^2x}{1+x}\right)^k \\ &= \left(\frac{q^2x}{1+x};\,q\right)_n \sum_{k=1}^{\infty}\frac{1}{q}\begin{bmatrix} n+k-1 \\ k-1 \end{bmatrix}_q \left(\frac{qx}{1+x}\right)^k \\ &= \frac{x}{1+x}\left(\frac{q^2x}{1+x};\,q\right)_n \sum_{k=0}^{\infty}\begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{qx}{1+x}\right)^k \\ &= \frac{x}{(1+x)}\frac{\left(\frac{q^2x}{1+x};\,q\right)_n}{\left(\frac{qx}{1+x};\,q\right)_{n+1}} = \frac{x}{1+x(1-q)}. \end{aligned}$$

Next using the identity $[k]_q^2 = [k]_q\left(q\,[k-1]_q + 1\right)$, we have

$$
\begin{aligned}
B_{n,q}^{*}(t^2;x) &= \left(\frac{q^2x}{1+x};\, q\right)_n \sum_{k=1}^{\infty} \frac{[k]_q^2}{q^{2k+2}\,[n]_q^2} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \left(\frac{q^2x}{1+x}\right)^k \\
&= \left(\frac{q^2x}{1+x};\, q\right)_n \sum_{k=2}^{\infty} \frac{1}{q^2} \frac{q\,[k]_q\,[k-1]_q}{q^{2k}\,[n]_q^2} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \left(\frac{x}{1+x}\right)^k \\
&\quad + \left(\frac{q^2x}{1+x};\, q\right)_n \sum_{k=1}^{\infty} \frac{[k]_q}{q^{2k+2}\,[n]_q^2} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \left(\frac{q^2x}{1+x}\right)^k \\
&= \frac{x^2}{q\,(1+x)^2} \frac{[n+1]_q}{[n]_q} \left(\frac{q^2x}{1+x};\, q\right)_n \sum_{k=0}^{\infty} \begin{bmatrix} n+k+1 \\ k \end{bmatrix}_q \left(\frac{x}{1+x}\right)^k \\
&\quad + \frac{x}{q^2\,(1+x)} \frac{1}{[n]_q} \left(\frac{q^2x}{1+x};\, q\right)_n \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{1+x}\right)^k \\
&= \frac{x^2}{q\,(1+x)^2} \frac{[n+1]_q}{[n]_q} \frac{\left(\frac{q^2x}{1+x};\, q\right)_n}{\left(\frac{x}{(1+x)};\, q\right)_{n+2}} \\
&\quad + \frac{1}{[n]_q} \frac{x}{q^2\,(1+x)} \frac{\left(\frac{q^2x}{1+x};\, q\right)_n}{\left(\frac{x}{1+x};\, q\right)_{n+1}}.
\end{aligned}
$$

Using the equalities (4.1), we have

$$
\begin{aligned}
B_{n,q}^{*}(t^2;x) &= \frac{[n+1]_q}{[n]_q} \frac{x^2}{q\,(1+x\,(1-q))} + \frac{1}{[n]_q} x \frac{\left(1+x\left(1-q^{n+1}\right)\right)}{q^2\,(1+x\,(1-q))} \\
&= \frac{x^2}{q\,(1+x\,(1-q))} \left(q+\frac{1}{[n]_q}\right) + \frac{1}{[n]_q} x \frac{\left(1+x\left(1-q^{n+1}\right)\right)}{q^2\,(1+x\,(1-q))} \\
&= \frac{x^2}{q\,(1+x\,(1-q))} \left(q+\frac{1}{[n]_q}\right) + \frac{1}{[n]_q} x \frac{\left(1+(1-q)\,x\,[n+1]_q\right)}{q^2\,(1+x\,(1-q))} \\
&= \frac{x^2}{q\,(1+x\,(1-q))} \left(q+\frac{1}{[n]_q}\right) + \frac{1}{[n]_q} \frac{x}{q^2\,(1+x\,(1-q))} \\
&\quad + \left(q+\frac{1}{[n]_q}\right) \frac{(1-q)\,x^2}{q^2\,(1+x\,(1-q))} \\
&= \frac{x^2}{q^2\,(1+x\,(1-q))} \left(q+\frac{1}{[n]_q}\right) + \frac{1}{[n]_q} \frac{x}{q^2\,(1+x\,(1-q))} \\
&= \frac{x^2}{q\,(1+x\,(1-q))} + \frac{1}{[n]_q} \frac{x^2+x}{q^2\,(1+x\,(1-q))}. \ \blacksquare
\end{aligned}
$$

REMARK 2. We observe that the behavior of q-Baskakov operators defined by (4.2) is different from the usual Baskakov operators. From Lemma 2 we observe that the q-Baskakov operators defined by (4.2) reproduce only the constant functions not the linear ones, while the Baskakov operators i.e. $q = 1$ and q-Baskakov operators defined by (1.5), reproduce constant as well as linear functions.

Let $C(\mathbb{R}^+)$ be a space that all real valued continuous functions on $\mathbb{R}^+$. For $\alpha > 0$ we define following weighted space

$$E_\alpha := \left\{ f \in C\left(\mathbb{R}^+\right) : \lim_{x\to\infty} \frac{f(x)}{1+x^\alpha} = 0 \right\}$$

with the norm

$$\|f\|_\alpha := \sup_{0\le x<\infty} \frac{|f(x)|}{1+x^\alpha}.$$

Let f be a uniform continuos function in $[0, \infty)$. Then the modulus of continuity of f defined as

$$\omega(f;\ \delta) := \sup_{0\le x<\infty,\ |t-x|<\delta} |f(t) - f(x)| \tag{4.3}$$

exist on the entire positive half-axis.

It is known that, for a uniform continuous function f, we have

$$\lim_{\delta\to 0} \omega(f;\ \delta) = 0$$

and, for any $\delta > 0$,

$$|f(t) - f(x)| \le \omega(f;\ \delta)\left(1 + \frac{|t-x|}{\delta}\right). \tag{4.4}$$

THEOREM 4. *Let $q = q_n$ satisfies $0 < q_n < 1$ and $q_n \to 1$ as $n \to \infty$. For every $f \in E_\alpha$ $(\alpha > 2)$ one has*

$$\left\|B^*_{n,q_n}(f;x) - f(x)\right\|_\alpha = 0. \tag{4.5}$$

If f is uniform continuous function in $\mathbb{R}^+$, then we have

$$\left|B^*_{n,q_n}(f;x) - f(x)\right| \le \omega\left(f; \sqrt{\frac{x(1+x)}{[n]_{q_n}}}\right)\sqrt{\left(\frac{1}{q_n[n]_{q_n}} + \frac{1}{q_n^2}\right)}. \tag{4.6}$$

Proof. Using Lemma 2, with the condition $\alpha > 2$, we get

$$\sup_{0\leq x<\infty} \frac{\left|B^*_{n,q_n}\left(t^2;x\right)-x^2\right|}{1+x^\alpha}$$

$$= \sup_{0\leq x<\infty} \frac{1}{1+x^\alpha}\left(\left|\frac{x^2}{q_n\left(1+x\left(1-q_n\right)\right)}+\frac{x^2+x}{q_n^2\left[n\right]_{q_n}\left(1+x\left(1-q_n\right)\right)}-x^2\right|\right)$$

$$\leq \sup_{0\leq x<\infty} \frac{x^2}{1+x^\alpha}\frac{\left(1-q_n\right)}{q_n}\left|1-\frac{x}{\left(1+x\left(1-q_n\right)\right)}\right|$$

$$+ \sup_{0\leq x<\infty} \frac{x^2+x}{1+x^\alpha}\frac{1}{q_n^2\left[n\right]_{q_n}\left(1+x\left(1-q_n\right)\right)}$$

$$\leq \frac{\left(1-q_n\right)}{q_n}+\frac{1}{q_n^2\left[n\right]_{q_n}}.$$

As a consequence of assumptions over the sequences $(q_n)_{n\geq 1}$, the above estimate tends to zero as $n\to\infty$. Thus (4.5) holds on account of Korovkin's theorem (see, e.g., [1, pp.215]).

Now we show that the inequality (4.6) holds. Using the property (4.4) and the Cauchy–Schwarz inequality we get

$$\left|B^*_{n,q_n}\left(f;x\right)-f\left(x\right)\right| \tag{4.7}$$

$$\leq \left(\frac{q^2x}{1+x};\, q\right)_n \sum_{k=0}^{\infty}\left|f\left(x\right)-f\left(\frac{[k]_q}{q^{k+1}\left[n\right]_q}\right)\right|\begin{bmatrix} n+k+1 \\ k\end{bmatrix}_q\left(\frac{q^2x}{1+x}\right)^k$$

$$\leq \omega\left(f;\,\delta\right)\left(\frac{q^2x}{1+x};\, q\right)_n \sum_{k=0}^{\infty}\left(1+\frac{\left|x-\frac{[k]_q}{q^{k+1}[n]_q}\right|}{\delta}\right)\begin{bmatrix} n+k+1 \\ k\end{bmatrix}_q\left(\frac{q^2x}{1+x}\right)^k$$

$$\leq \omega\left(f;\,\delta\right)\left(1+\frac{1}{\delta}\sqrt{B^*_{n,q_n}\left(\left(t-x\right)^2;x\right)}\right).$$

Using Lemma 2, we get

$$B^*_{n,q_n}\left(\left(t-x\right)^2;x\right)$$

$$=\frac{x^2}{q_n\left(1+x\left(1-q_n\right)\right)}+\frac{1}{\left[n\right]_{q_n}}\frac{x^2+x}{q_n^2\left(1+x\left(1-q_n\right)\right)}-\frac{2x^2}{\left(1+x\left(1-q_n\right)\right)}+x^2$$

$$=x^2\frac{\left(1-q_n\right)}{q_n}\frac{1+q_nx}{\left(1+x\left(1-q_n\right)\right)}+\frac{x^2+x}{q_n^2\left[n\right]_{q_n}\left(1+x\left(1-q_n\right)\right)}$$

$$= \frac{x(1+x)}{[n]_{q_n}} \left(\frac{x}{1+x} \frac{(1-q_n^n)}{q_n} \frac{1+q_n x}{(1+x(1-q_n))} + \frac{1}{q_n^2 (1+x(1-q_n))} \right)$$

$$\leq \frac{x(1+x)}{[n]_{q_n}} \left(\frac{1+q_n x}{q_n [n]_{q_n} (1+x)} + \frac{1}{q_n^2 (1+x(1-q_n))} \right)$$

$$\leq \frac{x(1+x)}{[n]_{q_n}} \left(\frac{1}{q_n [n]_{q_n}} + \frac{1}{q_n^2} \right).$$

Using this inequality in (4.7) and choosing $\delta = \sqrt{\frac{x(1+x)}{[n]_{q_n}}}$, then we have desired result. ∎

Remark 3. We note here that, this type theorem does not hold for the operators (1.5).

Remark 4. If the assumption of Theorem 4 holds for the function f, then we have $\lim_{n\to\infty} \omega\left(f; \sqrt{\frac{x(1+x)}{[n]_{q_n}}}\right) = 0$ when x is constant. Thus (4.6) gives us the pointwise rate of convergence of the operators $B^*_{n,q_n}(f;x)$ to $f(x)$. Also this rate of convergence is $\frac{1}{\sqrt{[n]_{q_n}}}$ faster than $\frac{1}{\sqrt{n}}$ which is the rate of convergence of the classical Baskakov operators.

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