DEMONSTRATIO MATHEMATICA
Vol. XLII No 1 2009

Ovidiu T. Pop, Mircea D. Fărcaş

ABOUT THE BIVARIATE OPERATORS OF DURRMEYER-TYPE

Abstract. The aim of this paper is to study the convergence and approximation properties of the bivariate operators and GBS operators of Durrmeyer-type.

1. Introduction

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $m \in \mathbb{N}$ consider the operator $M_m : L_1([0,1]) \to C([0,1])$ defined for any function $f \in L_1([0,1])$ by

$$(M_m f)(x) = (m+1) \sum_{k=0}^{m} p_{m,k}(x) \int_0^1 p_{m,k}(t) f(t) dt, \tag{1.1}$$

for any $x \in [0,1]$, where $p_{m,k}(x)$ are the fundamental polynomials of Bernstein, defined as follows

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}, \tag{1.2}$$

for any $x \in [0,1]$ and any $k \in \{0, 1, \ldots, m\}$. These operators were introduced in 1967 by J.L. Durrmeyer in [10] and were studied in 1981 by M.M. Derriennic in [9].

The purpose of this paper is to give a representation for the bivariate operators and GBS operators of Durrmeyer-type, to establish a convergence theorem for these operators. We also give an approximation theorem for these operators in terms of the first modulus of smoothness and of the mixed modulus of smoothness.

2000 *Mathematics Subject Classification*: 41A10, 41A25, 41A36, 41A63.
Key words and phrases: Linear positive operators, modulus of continuity of the bivariate functions, mixed modulus of smoothness, bivariate operators of Durrmeyer-type.

2. The construct of the bivariate operators of Durrmeyer-type

Let the sets $\Delta_2 = \{(x,y) \in \mathbb{R} \times \mathbb{R} | x, y \geq 0, x+y \leq 1\}$ and $\mathcal{F}(\Delta_2) = \{f | f : \Delta_2 \to \mathbb{R}\}$. For $m \in \mathbb{N}$, let the operator $\mathcal{M}_m : L_1(\Delta_2) \to \mathcal{F}(\Delta_2)$ be defined for any function $f \in L_1(\Delta_2)$ by

(2.1)
$$(\mathcal{M}_m f)(x,y) = (m+1)(m+2) \sum_{\substack{k,j=0 \\ k+j \leq m}} p_{m,k,j}(x,y) \iint_{(\Delta_2)} p_{m,k,j}(s,t) f(s,t) ds dt$$

for any $(x,y) \in \Delta_2$, where

$$p_{m,k,j}(x,y) = \frac{m!}{k!j!(m-k-j)!} x^k y^j (1-x-y)^{m-k-j}, \tag{2.2}$$

for any $k, j \geq 0$, $k+j \leq m$ and any $(x,y) \in \Delta_2$.

The operators defined above are called the bivariate operators of Durrmeyer-type. Clearly, the bivariate operators of Durrmeyer-type are linear and positive. The method was inspired by the construction of Bernstein bivariate operators (see [11] or [14]). For $m \in \mathbb{N}$, the operator $B_m : \mathcal{F}(\Delta_2) \to \mathcal{F}(\Delta_2)$ defined for any function $f \in \mathcal{F}(\Delta_2)$ by

$$(B_m f)(x,y) = \sum_{\substack{k,j=0 \\ k+j \leq m}} p_{m,k,j}(x,y) f\left(\frac{k}{m}, \frac{j}{m}\right) \tag{2.3}$$

for any $(x,y) \in \Delta_2$ is named the Bernstein bivariate operator.

Let the functions $e_{ij} : \Delta_2 \to \mathbb{R}$, $e_{ij}(x,y) = x^i y^j$ for any $(x,y) \in \Delta_2$, where $i, j \in \mathbb{N}_0$. For the following, see [12].

Lemma 2.1. *The operators $(B_m)_{m \geq 1}$ verify for any $(x,y) \in \Delta_2$ the following:*

$$(B_m e_{00})(x,y) = 1, \tag{2.4}$$

$$(B_m e_{10})(x,y) = x, \ (B_m e_{01})(x,y) = y, \tag{2.5}$$

$$\left(B_m(\cdot - x)^2\right)(x,y) = \frac{x(1-x)}{m}, \tag{2.6}$$

$$\left(B_m(* - y)^2\right)(x,y) = \frac{y(1-y)}{m}, \tag{2.7}$$

$$\left(B_m(\cdot - x)^2(* - y)^2\right)(x,y) = \tag{2.8}$$
$$= \frac{3(m-2)}{m^3} x^2 y^2 - \frac{m-2}{m^3}(x^2 y + x y^2) + \frac{m-1}{m^3} xy,$$

$$\left(B_m(\cdot - x)^4(* - y)^2\right)(x,y) = -\frac{5(3m^2-26m+24)}{m^5} x^4 y^2 + \tag{2.9}$$
$$+ \frac{6(3m^2-26m+24)}{m^5} x^3 y^2 - \frac{6(m^2-7m+6)}{m^5} x^3 y -$$

$$-\frac{3m^2-41m+42}{m^5}x^2y^2+\frac{3m^2-26m+24}{m^5}x^4y+$$
$$+\frac{3m^2-17m+14}{m^5}x^2y-\frac{m-2}{m^5}xy^2+\frac{m-1}{m^5}xy$$

and

$$\left(B_m(\cdot-x)^2(*-y)^4\right)(x,y)=-\frac{5(m^2-26m+24)}{m^5}x^2y^4+ \tag{2.10}$$
$$+\frac{6(3m^2-26m+24)}{m^5}x^2y^3-\frac{6(m^2-7m+6)}{m^5}xy^3-$$
$$-\frac{3m^2-41m+42}{m^5}x^2y^2+\frac{3m^2-26m+24}{m^5}xy^4+$$
$$+\frac{3m^2-17m+14}{m^5}xy^2-\frac{m-2}{m^5}x^2y+\frac{m-1}{m^5}xy$$

for any $m\in\mathbb{N}$.

LEMMA 2.2. *The operators* $(B_m)_{m\geq 1}$ *verify for any* $(x,y)\in\Delta_2$ *the following inequalities:*

$$(B_m(\cdot-x)^2)(x,y)\leq\frac{1}{4m}, \tag{2.11}$$

$$(B_m(*-y)^2)(x,y)\leq\frac{1}{4m}, \tag{2.12}$$

for any $m\in\mathbb{N}$,

$$\left(B_m(\cdot-x)^2(*-y)^2\right)(x,y)\leq\frac{9}{4m^2}, \tag{2.13}$$

for any $m\in\mathbb{N}$, $m\geq 2$,

$$\left(B_m(\cdot-x)^4(*-y)^2\right)(x,y)\leq\frac{9}{m^3}, \tag{2.14}$$

$$\left(B_m(\cdot-x)^2(*-y)^4\right)(x,y)\leq\frac{9}{m^3}, \tag{2.15}$$

for any $m\in\mathbb{N}$, $m\geq 8$.

3. Some relations verify by the $(\mathcal{M}_m)_{m\geq 1}$ operators

LEMMA 3.1. *For* $k,j,l\in\mathbb{N}_0$, *we have*

$$I_{k,j}=\int_0^1 s^k(1-s)^j ds=\frac{k!j!}{(k+j+1)!} \tag{3.1}$$

and

$$J_{k,j,l}=\iint\limits_{(\Delta_2)} s^kt^j(1-s-t)^l dsdt=\frac{k!j!l!}{(k+j+l+2)!}. \tag{3.2}$$

Proof. In the following, we use the integration by parts formula. We have $I_{k,j} = \frac{s^{k+1}}{k+1}(1-s)^j\Big|_0^1 + \frac{j}{k+1}\int_0^1 s^{k+1}(1-s)^{j-1}ds = \frac{j}{k+1}I_{k+1,j-1}$ and then $I_{k,j} = \frac{j\cdot(j-1)\cdot\ldots\cdot 2\cdot 1}{(k+1)\cdot(k+2)\cdot\ldots\cdot(k+j)}I_{k+j,0}$. But $I_{k+j,0} = \frac{1}{k+j+1}$ and then we obtain (3.1). For $k,j \in \mathbb{N}_0$ and $s \in [0,1]$, we note $I_{k,j}(s) = \int\limits_0^{1-s} t^k(1-s-t)^j dt$. By the idea above, we have $I_{k,j}(s) = \frac{t^{k+1}}{k+1}(1-s-t)^j\Big|_0^{1-s} + \frac{j}{k+1}\int_0^{1-s} t^{k+1}(1-s-t)^{j-1}dt = \frac{j}{k+1}I_{k+1,j-1}(s)$, from where we obtain $I_{k,j}(s) = \frac{k!j!}{(k+j+1)!}(1-s)^{k+j+1}$. Further on, we have $I_{k,j,l} = \int_0^1\left(\int_0^{1-s} t^j(1-s-t)^l dt\right)ds = \int_0^1 s^k I_{j,l}(s)ds = \frac{j!l!}{(j+l+1)!}\int_0^1 s^k(1-s)^{j+l+1}ds = \frac{j!l!}{(j+l+1)!}I_{k,j+l+1}$, from where (3.2) results. ∎

Lemma 3.2. *The operators $(\mathcal{M}_m)_{m\geq 1}$ verify for any $(x,y) \in \Delta_2$ the following*

$$(3.3)\qquad (\mathcal{M}_m e_{00})(x,y) = 1,$$

$$(3.4)\qquad (\mathcal{M}_m e_{10})(x,y) = \frac{mx+1}{m+3},\ (\mathcal{M}_m e_{01})(x,y) = \frac{my+1}{m+3},$$

$$(3.5)\qquad (\mathcal{M}_m(\cdot - x)^2)(x,y) = \frac{2mx(1-x)+12x^2-8x+2}{(m+3)(m+4)},$$

$$(3.6)\qquad (\mathcal{M}_m(\cdot - y)^2)(x,y) = \frac{2my(1-y)+12y^2-8y+2}{(m+3)(m+4)},$$

$$\begin{aligned}(3.7)\quad &(m+3)(m+4)(m+5)(m+6)(\mathcal{M}_m(\cdot - x)^2(\cdot - y)^2)(x,y) = \\ &(12m^2-372m+360)x^2y^2 + (-4m^2+160m-240)(x^2y+xy^2)+ \\ &+(4m^2-68m+120)xy + (-4m+60)(x^2+y^2)+ \\ &+(4m-24)(x+y)+4,\end{aligned}$$

$$\begin{aligned}(3.8)\quad &(m+3)\cdot(m+4)\cdot\ldots\cdot(m+8)(\mathcal{M}_m(\cdot - x)^4(\cdot - y)^2)(x,y) = \\ &=(-120m^3+1040m^2-60720m+20160)x^4y^2 + (24m^3-2760m^2+ \\ &+22224m-13440)x^4y + (144m^3-12720m^2+75744m- \\ &-26880)x^3y^2 + (-48m^3+4128m^2-25584m+13440)x^3y+ \\ &+(-24m^3+3888m^2-36120m+20610)x^2y^2 + (24m^3-1512m^2+ \\ &+11280m-8064)x^2y + (-144m^2+6480m-8064)xy^2 + (144m^2- \\ &-1968m+2688)xy + (24m^2-1368m+3360)x^4 + (-48m^2+ \\ &+2160m-2688)x^3 + 24(m^2-39m+56)x^2 + 48(-m+28)y^2+ \\ &+48(3m-8)x + 48(m-8)y + 48,\end{aligned}$$

$$(3.9)\quad (m+3)\cdot(m+4)\cdot\ldots\cdot(m+8)(\mathcal{M}_m(\cdot-x)^2(\cdot-y)^4)(x,y)=$$
$$=(-120m^3+1040m^2-60720m+20160)x^2y^4+(24m^3-2760m^2+$$
$$+22224m-13440)xy^4+(144m^3-12720m^2+75744m-$$
$$-26880)x^2y^3+(-48m^3+4128m^2-25584m+13440)xy^3+$$
$$+(-24m^3+3888m^2-36120m+20610)x^2y^2+(24m^3-1512m^2+$$
$$+11280m-8064)xy^2+(-144m^2+6480m-8064)x^2y+(144m^2-$$
$$-1968m+2688)xy+(24m^2-1368m+3360)y^4+(-48m^2+$$
$$+2160m-2688)y^3+24(m^2-39m+56)y^2+48(-m+28)x^2+$$
$$+48(3m-8)y+48(m-8)x+48.$$

Proof. In the following, we take the relations from Lemma 3.1 and Lemma 2.1 into account. We have

$$(\mathcal{M}_m e_{00})(x,y)=(m+1)(m+2)\sum_{\substack{k,j=0\\k+j\le m}} p_{m,k,j}(x,y)\iint_{(\Delta_2)} p_{m,k,j}(s,t)dsdt=$$
$$(m+1)(m+2)\sum_{\substack{k,j=0\\k+j\le m}} p_{m,k,j}(x,y)\frac{m!}{k!j!(m-k-j)!}I_{k,j,m-k-j}=\sum_{\substack{k,j=0\\k+j\le m}} p_{m,k,j}(x,y)=$$
$$(B_m e_{00})(x,y)=1,$$
$$(\mathcal{M}_m e_{10})(x,y)=(m+1)(m+2)\sum_{\substack{k,j=0\\k+j\le m}} p_{m,k,j}(x,y)\iint_{(\Delta_2)} s p_{m,k,j}(s,t)dsdt=$$
$$=(m+1)(m+2)\sum_{\substack{k,j=0\\k+j\le m}} p_{m,k,j}(x,y)\frac{m!}{k!j!(m-k-j)!}I_{k+1,j,m-k-j}=$$
$$=\frac{1}{m+3}\sum_{\substack{k,j=0\\k+j\le m}}(k+1)p_{m,k,j}(x,y)=\frac{m}{m+3}(B_m e_{10})(x,y)+\frac{1}{m+3}(B_m e_{00})(x,y)=$$
$$=\frac{mx+1}{m+3},$$
$$(\mathcal{M}_m(\cdot-x)^2)(x,y)=(\mathcal{M}_m e_{20})(x,y)-2x(\mathcal{M}_m e_{10})(x,y)+x^2(\mathcal{M}_m e_{00})(x,y)=$$
$$=\frac{2mx(1-x)+12x^2-8x+2}{(m+3)(m+4)},\text{ because}$$
$$(\mathcal{M}_m e_{20})(x,y)=(m+1)(m+2)\sum_{\substack{k,j=0\\k+j\le m}} p_{m,k,j}(x,y)\iint_{(\Delta_2)} s^2 p_{m,k,j}(s,t)dsdt=$$
$$=(m+1)(m+2)\sum_{\substack{k,j=0\\k+j\le m}} p_{m,k,j}(x,y)\frac{m!}{k!j!(m-k-j)!}I_{k+2,j,m-k-j}=$$
$$=\frac{1}{m+3}\sum_{\substack{k,j=0\\k+j\le m}}(k+1)(k+2)p_{m,k,j}(x,y)=\frac{m^2}{(m+3)(m+4)}(B_m e_{20})(x,y)+$$
$$+\frac{3m}{(m+3)(m+4)}(B_m e_{10})(x,y)+\frac{2}{(m+3)(m+4)}(B_m e_{00})(x,y)=\frac{m(m-1)x^2+4mx+2}{(m+3)(m+4)}.$$

The other relations from the lemma can be obtain analogously. ∎

LEMMA 3.3. *The operators $(\mathcal{M}_m)_{m\geq 1}$ verify for any $(x,y)\in\Delta_2$ the following estimations*

$$(\mathcal{M}_m(\cdot - x)^2)(x,y) \leq \frac{1}{m+3}, \tag{3.10}$$

$$(\mathcal{M}_m(\cdot - y)^2)(x,y) \leq \frac{1}{m+3}, \tag{3.11}$$

for any $m\in\mathbb{N}$, $m\geq 4$,

$$(\mathcal{M}_m(\cdot - x)^2(\cdot - y)^2)(x,y) \leq \frac{1}{(m+3)^2}, \tag{3.12}$$

for any $m\in\mathbb{N}$, $m\geq 2$,

$$(\mathcal{M}_m(\cdot - x)^4(\cdot - y)^2)(x,y) \leq \frac{1}{(m+3)^3}, \tag{3.13}$$

$$(\mathcal{M}_m(\cdot - x)^2(\cdot - y)^4)(x,y) \leq \frac{1}{(m+3)^3}, \tag{3.14}$$

for any $m\in\mathbb{N}$, $m\geq 100$.

Proof. We use the relations $x,y\geq 0$, $x+y\leq 1$, $x(1-x)\leq 1/4$, $y(1-y)\leq 1/4$, $xy\leq 1/4$ and the results from Lemma 3.2 and we obtain that $(m+3)(m+4)(\mathcal{M}_m(\cdot-x)^2)(x,y)=2mx(1-x)+12x^2-8x+2\leq \frac{m}{2}+6\leq m+4$, for $m\geq 4$; further on, we can write $(m+3)(m+4)(m+5)(m+6)(\mathcal{M}_m(\cdot-x)^2(\cdot-y)^2)(x,y)=4xy(3xy+1-x-y)m^2+4(x+y-x^2-y^2-17xy+40xy(x+y)-93x^2y^2)m+4(1-6(x+y)+15(x^2+y^2)+30xy-60xy(x+y)+90x^2y^2)$. But $4xy(3xy+1-x-y)\leq (x+y)^2\left(\frac{3}{4}(x+y)^2+1-x-y\right)\leq\frac{3}{4}$, $x+y-x^2-y^2-17xy+40xy(x+y)-93x^2y^2=x(1-x)+y(1-y)-17xy+40xy(x+y)-93x^2y^2\leq\frac{1}{2}-17xy+40xy-93x^2y^2=\frac{1}{2}+23xy-93x^2y^2\leq 1$; $1-6(x+y)+15(x^2+y^2)+30xy-60xy(x+y)+90x^2y^2=1-6(x+y)+15(x+y)^2-60xy(x+y)+90x^2y^2=1+3(5(x+y)^2-2(x+y))+30xy(3xy-2(x+y))\leq 10$, because $5(x+y)^2-2(x+y)\leq 3$ and $3xy-2(x+y)\leq 0$. So, we obtain that $(m+3)(m+4)(m+5)(m+6)(\mathcal{M}_m(\cdot-x)^2(\cdot-y)^2)(x,y)\leq\frac{3m^2}{4}+4m+40\leq (m+5)(m+6)$, for $m\geq 2$.
Finally, we have analogously $(m+3)\cdot(m+4)\cdot\ldots\cdot(m+8)(\mathcal{M}_m(\cdot-x)^2(\cdot-y)^4)(x,y)\leq\frac{9(m+4)\cdot(m+5)\cdot\ldots\cdot(m+8)}{(m+3)^2}$, where we can take any $m\in\mathbb{N}$, $m\geq 100$, from where the relation (3.13) results. ∎

4. Approximation and convergence theorems for the bivariate operators of Durrmeyer-type

Let $X, Y \subset \mathbb{R}$ be given intervals, $D \subset X \times Y$ and $f : D \to \mathbb{R}$ be a bounded function. The function $\omega_{total}(f; \cdot, *) : [0,\infty) \times [0,\infty) \to \mathbb{R}$, defined for any $(\delta_1, \delta_2) \in [0,\infty) \times [0,\infty)$ by

$$(4.1) \qquad \omega_{total}(f; \delta_1, \delta_2) = \sup\{|f(x,y) - f(x',y')| : (x,y), (x',y') \in D, \\ |x - x'| \le \delta_1, |y - y'| \le \delta_2\}$$

is called the first order modulus of smoothness of function f or total modulus of continuity of function f.

Remark 4.1. The function from (4.1) is the constraint of the "classical" modulus of smoothness to the domain of definition D of the function f and has the same properties as the "classical" modulus of smoothness.

Remark 4.2. The following theorem can be demonstrated in a similar way as the Shisha-Mond type theorem with the "classical" modulus of smoothness (see for example [5] or [16]).

Theorem 4.1. *Let $L : C(D) \to B(D)$ be a linear positive operator. For any $f \in C(D)$, any $(x,y) \in D$ and any $\delta_1, \delta_2 > 0$, the following inequality*

$$(4.2) \qquad |(Lf)(x,y) - f(x,y)| \le |(Le_{00})(x,y) - 1||f(x,y)| + \Big[(Le_{00})(x,y) + \\ + \delta_1^{-1}\sqrt{(Le_{00})(x,y)(L(\cdot - x)^2)(x,y)} + \delta_2^{-1}\sqrt{(Le_{00})(x,y)(L(\cdot - y)^2)(x,y)} + \\ + \delta_1^{-1}\delta_2^{-1}\sqrt{(Le_{00})^2(x,y)(L(\cdot - x)^2)(x,y)(L(* - y)^2)(x,y)}\Big]\omega_{total}(f; \delta_1, \delta_2)$$

holds, where "$\cdot$" and "$*$" stand for the first and the second variable.

Theorem 4.2. *Let the function $f \in C(\Delta_2)$. Then, for any $(x,y) \in \Delta_2$, any $m \in \mathbb{N}$, $m \ge 4$, we have*

$$(4.3) \quad |(\mathcal{M}_m f)(x,y) - f(x,y)| \\ \le \left(1 + \frac{1}{\delta_1\sqrt{m+3}}\right)\left(1 + \frac{1}{\delta_2\sqrt{m+3}}\right)\omega_{total}(f; \delta_1, \delta_2)$$

for any $\delta_1, \delta_2 > 0$ and

$$(4.4) \qquad |(\mathcal{M}_m f)(x,y) - f(x,y)| \le 4\omega_{total}\left(f; \frac{1}{\sqrt{m+3}}, \frac{1}{\sqrt{m+3}}\right).$$

Proof. We apply Theorem 4.1, where $D = \Delta_2$, Lemma 3.2 and Lemma 3.3. For (4.4), we choose $\delta_1 = \delta_2 = \frac{1}{\sqrt{m+3}}$ in (4.3). ∎

Corollary 4.1. *If $f \in C(\Delta_2)$, then*

$$\lim_{m\to\infty} (\mathcal{M}_m f)(x,y) = f(x,y) \tag{4.5}$$

uniformly on Δ_2.

Proof. It results from (4.4). ∎

5. Approximation and convergence theorems for gbs operators of Durrmeyer-type

Using the idea from section 4, we take the following construction, so the definitions and theorems are constrains from the domain of definition of function. Theorem 5.1 (see [3]) and Theorem 5.2 (see [13]) can be demonstrated in a similar way as in the general case. In the following, let X and Y be real intervals. A function $f : D \to \mathbb{R}$ is called B-continuous (Bögel-continuous) function in $(x_0, y_0) \in D$ if

$$\lim_{(x,y)\to(x_0,y_0)} \Delta f\left((x,y),(x_0,y_0)\right) = 0.$$

Here $\Delta f\left((x,y),(x_0,y_0)\right) = f(x,y) - f(x_0,y) - f(x,y_0) + f(x_0,y_0)$ denotes a so-called mixed difference of f.

A function $f : D \to \mathbb{R}$ is called B-differentiable (Bögel-differentiable) function in $(x_0, y_0) \in D$ if it exists and if the limit is finite:

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\Delta f\left((x,y),(x_0,y_0)\right)}{(x-x_0)(y-y_0)}.$$

The limit is named the B-differential of f in the point (x_0, y_0) and is noted by $D_B f(x_0, y_0)$.

The definition of B-continuity and B-differentiability were introduced by K. Bögel in the papers [6] and [7].

The function $f : D \to \mathbb{R}$ is B-bounded on D if there exists $K > 0$ such that

$$\left|\Delta f\left((x,y),(s,t)\right)\right| \le K$$

for any $(x,y),(s,t) \in D$.

We shall use the function sets $B(D) = \left\{f : D \to \mathbb{R} | f \text{ bounded on } D\right\}$ with the usual sup-norm $\|\cdot\|_\infty$, $B_b(D) = \left\{f : D \to \mathbb{R} | f\ B\text{-bounded on } D\right\}$ and we set $\|f\|_B = \sup_{(x,y),(s,t)\in D} \left|\Delta f\left((x,y),(s,t)\right)\right|$ where $f \in B_b(D)$, $C_b(D) = \left\{f : D \to \mathbb{R} | f\ B\text{-continuous on } D\right\}$ and $D_b(D) = \left\{f : D \to \mathbb{R} | f\ B\text{-differentiable on } D\right\}$.

Let $f \in B_b(D)$. The function $\omega_{\text{mixed}}(f;\cdot,*) : [0,\infty) \times [0,\infty) \to \mathbb{R}$, defined by

$$\omega_{\text{mixed}}(f;\delta_1,\delta_2) = \sup\{|\Delta f((x,y),(s,t))| : (x,y),(s,t) \in D,\ |x-s| \le \delta_1, |y-t| \le \delta_2\} \tag{5.1}$$

for any $(\delta_1,\delta_2) \in [0,\infty) \times [0,\infty)$ is called the mixed modulus of smoothness.

For related topics, see [1], [2], [3] and [4].

Let $L : C_b(D) \to B(D)$ be a linear positive operator. The operator $UL : C_b(D) \to B(D)$ defined for any function $f \in C_b(D)$, any $(x,y) \in D$ by

$$(ULf)(x,y) = (L(f(\cdot,y) + f(x,*) - f(\cdot,*)))(x,y) \tag{5.2}$$

is called GBS operator ("Generalized Boolean Sum" operator) associated to the operator L, where "$\cdot$" and "$*$" stand for the first and second variable.

THEOREM 5.1. *Let $L : C_b(D) \to B(D)$ be a linear positive operator and $UL : C_b(D) \to B(D)$ the associated GBS operator. Then for any $f \in C_b(D)$, any $(x,y) \in D$ and any $\delta_1,\delta_2 > 0$, we have*

$$\begin{aligned}|f(x,y) - (ULf)(x,y)| &\le |f(x,y)|\,|1-(Le_{00})(x,y)| + \\ &+ \Big[(Le_{00})(x,y) + \delta_1^{-1}\sqrt{(L(\cdot-x)^2)(x,y)} + \delta_2^{-1}\sqrt{(L(*-y)^2)(x,y)} + \\ &+ \delta_1^{-1}\delta_2^{-1}\sqrt{(L(\cdot-x)^2(*-y)^2)(x,y)}\Big]\omega_{mixed}(f;\delta_1,\delta_2).\end{aligned} \tag{5.3}$$

THEOREM 5.2. *Let $L : C_b(D) \to B(D)$ be a linear positive operator and $UL : C_b(D) \to B(D)$ the associated GBS operator. Then for any $f \in D_b(D)$ with $D_B f \in B(D)$, any $(x,y) \in D$ and any $\delta_1,\delta_2 > 0$, we have*

$$\begin{aligned}|f(x,y) - (ULf)(x,y)| &\le \\ &\le |f(x,y)||1-(Le_{00})(x,y)| + 3\|D_B f\|_\infty \sqrt{(L(\cdot-x)^2(*-y)^2)(x,y)} + \\ &+ \Big[\sqrt{(L(\cdot-x)^2(*-y)^2)(x,y)} + \delta_1^{-1}\sqrt{(L(\cdot-x)^4(*-y)^2)(x,y)} + \\ &+ \delta_2^{-1}\sqrt{(L(\cdot-x)^2(*-y)^4)(x,y)} + \\ &+ \delta_1^{-1}\delta_2^{-1}\left(L(\cdot-x)^2(*-y)^2\right)(x,y)\Big]\omega_{mixed}(D_B f;\delta_1,\delta_2).\end{aligned} \tag{5.4}$$

If $f \in \mathbb{N}$ and $f \in C_b(\Delta_2)$, then the GBS operator associated to the $\mathcal{M}_m$ operator is defined by

$$\begin{aligned}(U\mathcal{M}_m f)(x,y) &= (m+1)(m+2) \sum_{\substack{k,j=0\\ k+j\le m}} p_{m,k,j}(x,y)\cdot \\ &\cdot \iint_{(\Delta_2)} p_{m,k,j}(s,t)[f(s,y) + f(x,t) - f(s,t)]dsdt,\end{aligned} \tag{5.5}$$

for any $(x,y) \in \Delta_2$.

THEOREM 5.3. *Let the function $f \in C_b(\Delta_2)$. Then, for any $(x,y) \in \Delta_2$, any $m \in \mathbb{N}$, $m \geq 4$, we have*

$$(5.6)\quad |(U\mathcal{M}_m f)(x,y) - f(x,y)| \leq \left(1 + \frac{1}{\delta_1\sqrt{m+3}}\right)\left(1 + \frac{1}{\delta_2\sqrt{m+3}}\right)\omega_{mixed}(f;\delta_1,\delta_2)$$

for any $\delta_1, \delta_2 > 0$ and

$$(5.7)\quad |(U\mathcal{M}_m f)(x,y) - f(x,y)| \leq 4\omega_{mixed}\left(f;\frac{1}{\sqrt{m+3}},\frac{1}{\sqrt{m+3}}\right).$$

Proof. For the first inequality, we apply Theorem 5.1, where $D = \Delta_2$ and Lemma 3.3. The inequality (5.7) is obtained from (5.6) by choosing $\delta_1 = \delta_2 = \frac{1}{\sqrt{m+3}}$. ∎

COROLLARY 5.1. *If $f \in C_b(\Delta_2)$, then*

$$(5.8)\quad \lim_{m\to\infty}(U\mathcal{M}_m f)(x,y) = f(x,y)$$

uniformly on Δ_2.

Proof. It results from (5.7). ∎

THEOREM 5.4. *Let the function $f \in D_b(\Delta_2)$ with $D_B f \in B(\Delta_2)$. Then, for any $(x,y) \in \Delta_2$, any $m \in \mathbb{N}$, $m \geq 100$, we have*

$$(5.9)\quad |(U\mathcal{M}_m f)(x,y) - f(x,y)| \leq \frac{3}{m+3}\|D_B f\|_\infty + \frac{1}{m+3}\left(1 + \frac{1}{\delta_1\sqrt{m+3}}\right)\left(1 + \frac{1}{\delta_2\sqrt{m+3}}\right)\omega_{mixed}(D_B f;\delta_1,\delta_2)$$

for any $\delta_1, \delta_2 > 0$ and

$$(5.10)\quad |(U\mathcal{M}_m f)(x,y) - f(x,y)| \leq \frac{3}{m+3}\|D_B f\|_\infty + \frac{4}{m+3}\omega_{mixed}\left(D_B f;\frac{1}{\sqrt{m+3}},\frac{1}{\sqrt{m+3}}\right).$$

Proof. It results from Theorem 5.2, where $D = \Delta_2$ and Lemma 3.3. ∎

References

[1] I. Badea, *Modul de continuitate în sens Bögel şi unele aplicaţii în aproximarea printr-un operator Bernstein*, Studia Univ. Babeş-Bolyai Math. 18 (2) (1973), 69–78 (Romanian).

[2] C. Badea, I. Badea, H. H. Gonska, *A test function theorem and approximation by pseudopolynomials*, Bull. Austral. Math. Soc. 34 (1986), 55–64.

[3] C. Badea, C. Cottin, *Korovkin-type theorems for generalized boolean sum operators*, Colloq. Math. Soc. Janos Bolyai, 58, Approximation Theory, Kecskemét (Hungary) (1990), 51–67 (1988), 95–108.
[4] C. Badea, I. Badea, C. Cottin, H. H. Gonska, *Notes on the degree of approximation of B-continuous and B-differentiable functions*, J. Approx. Theory Appl. 4 (1988), 95–108.
[5] D. Bărbosu, *Polynomial Approximation by Means of Schurer-Stancu type Operators*, Ed. Universităţii de Nord, Baia Mare, 2006.
[6] K. Bögel, *Mehrdimensionale Differentiation von Funtionen mehrerer Veränderlicher*, J. Reine Angew. Math. 170 (1934), 197–217.
[7] K. Bögel, *Über die mehrdimensionale Differentiation, Integration und beschränkte Variation*, J. Reine Angew. Math. 173 (1935), 5–29.
[8] K. Bögel, *Über die mehrdimensionale Differentiation*, Jber. DMV 65 (1962), 45–71.
[9] M. M. Derriennic, *Sur l'approximation des functions intégrables sur* [0,1] *par des polinômes de Bernstein modifiés*, J. Approx. Theory 31 (1981), 325–343.
[10] J. L. Durrmeyer, *Une formule d'inversion de la transformeé de Laplace: Applications à la théorie des momentes*, Thése de 3e cycle, Faculté des Sciences de l'Université de Paris, 1967.
[11] G. G. Lorentz, *Bernstein Polynomials*, University of Toronto Press, Toronto, 1953.
[12] O. T. Pop, M. D. Fărcaş, *Approximation of B-continuous and B-differentiable functions by GBS operators of Bernstein bivariate polynomials*, J. Inequal. Pure Appl. Math. 7 (2006), 9 pp. (electronic).
[13] O. T. Pop, *Approximation of B-differentiable functions by GBS operators*, Anal. Univ. Oradea, Fasc. Matem. Tom XIV (2007), 15–31.
[14] D. D. Stancu, *Asupra aproximării funcţiilor de două variabile prin polinoame de tip Bernstein. Câteva evaluări asimptotice*, Studii şi Cercetări Matematice 11 (1960), 171–176 (Romanian).
[15] D. D. Stancu, Gh. Coman, Gh. Agratini, Gh. Trîmbiţaş, *Analiză numerică şi teoria aproximării*, I, Presa Universitară Clujeană, Cluj-Napoca, 2001 (Romanian).
[16] A. F. Timan, *Theory of Approximation of Functions of Real Variable*, New York: Macmillan Co. 1963. MR22#8257.

NATIONAL COLLEGE "MIHAI EMINESCU"
5 MIHAI EMINESCU STREET
SATU MARE 440014, ROMANIA
E-mail: ovidiutiberiu@yahoo.com
mirceafarcas2005@yahoo.com

Received October 15, 2007; revised version June 30, 2008.