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EXISTENCE OF BOUNDED SOLUTIONS FOR FOURTH ORDER DIRICHLET PROBLEMS WITH CONVEX-CONCAVE NONLINEARITY

Abstract. We consider the Dirichlet boundary value problem for higher order O.D.E. with nonlinearity being the sum of a derivative of a convex and of a concave function in case when no growth condition is imposed on the concave part.

1. Introduction

We shall consider the higher order ordinary differential equations with Dirichlet boundary type conditions and with nonlinearity being the sum of derivative of a convex and of a concave function in case when no growth condition is imposed on the concave part. In order to demonstrate our approach, we shall show that for any $\sigma \geq 0$ and for λ from a certain interval there exists a bounded solution for the problem

$$\begin{aligned} &\beta\frac{d^4}{dt^4}x(t)+\gamma\frac{d^2}{dt^2}x(t)+\delta x(t)+\sigma G_x(t,x(t))=\lambda F_x(t,x(t)),\\ &x(0)=x(\pi)=\dot{x}(0)=\dot{x}(\pi)=0, \end{aligned} \tag{1.1}$$

where the constants β, γ, δ *are such that* $\beta-\gamma-|\delta|>0$, $\gamma<0$, $\beta>0$; F, F_x, G, $G_x : [0,\pi]\times R \to R$ *are Caratheodory functions with* F *and* G *continuously differentiable and convex with respect to the second variable in* R *for a. e.* $t\in[0,\pi]$ *and with* G_x *continuous on* $[0,\pi]\times R$.

Concerning the growth of F and the properties of G we assume only that

F1 *for any* $d>0$ *there exists a constant* $\alpha>0$ *such that for a. e.* $t\in[0,\pi]$;

$$\max_{x\in[-d,d]}|F_x(t,x)|\leq\alpha \tag{1.2}$$

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F2 $F_x(t,0) \neq 0$, *for a.e.* $t \in [0,\pi]$, *functions* $t \to F(t,0)$, $t \to G(t,0)$ *are integrable on* $[0,\pi]$.

G1 $G_x(t,x)\,x \geq 0$ *for* $x \in R$.

From **F1**, **F2**, **G1** it follows by convexity that integral functionals $x \to \int_0^\pi F(t,x(t))\,dt$, $x \to \int_0^\pi G(t,x(t))\,dt$ are well defined on $L^2(0,\pi)$, see [7].

Our assumptions are not very restrictive. Namely, assumption **G1** provides that G is a coercive function with respect to the second variable while assumption (1.2) reflects the local boundedness of the derivative F_x. For example we may take $F(t,x) = e^x + 4x^4 - 0.25x^2 + f(t)\,x$ and $G(t,x) = g(t)\,x^v$, where $g \in C(0,\pi)$, $f \in L^\infty(0,\pi)$, $g(t) > 0$ for $t \in (0,\pi)$ and where v is any even number. Function F is convex in x on R and F_x is continuous in x. Thus for any $d > 0$ relation (1.2) is satisfied for a certain α.

Higher order problems with both Dirichlet and periodic boundary value have been investigated by a variety of methods and approaches lately, see for example [3], [9], [8] to mention a few works that use either topological or other variational approaches.

We will minimize the Euler action functional

$$J(x) = \int_0^\pi \left(\beta \frac{d^4}{dt^4} x(t) + \gamma \frac{d^2}{dt^2} x(t) + \delta x(t)\right) x(t)\,dt + \sigma \int_0^\pi G(t,x(t))\,dt \\ -\lambda \int_0^\pi F(t,x(t))\,dt$$

over the set

$$X = \left\{ x \in H_0^2(0,\pi) \cap H^4(0,\pi) : \|\dot{x}\|_{L^2(0,\pi)} \leq \frac{d}{\sqrt{\pi}},\ x(t) \in [-d,d] \text{ on } [0,\pi], \right. \\ \left. \|\ddot{x}\|_{L^2(0,\pi)} \leq \sqrt{\frac{(-\gamma + |\delta|)\,d^2}{\beta\pi} + \frac{(\beta - \gamma - |\delta|)\,d^2}{\beta\sqrt{\pi}}} \right\}.$$

We note that J is unbounded in $H_0^2(0,\pi)$. Our main result reads

Theorem 1.1. *Let us assume* **F1**, **F2**, **G1**. *Let* $d > 0$ *be fixed. Let* $\sigma \geq 0$ *and let*

$$0 < \lambda \leq \frac{(\beta - \gamma - |\delta|)\,d}{\alpha\pi}.$$

There exists a solution $x \in X$ *to the Dirichlet Problem* (1.1). *Moreover*

$$\beta \frac{d^4}{dt^4} x(\cdot) + \gamma \frac{d^2}{dt^2} x(\cdot) + \delta x(\cdot) + \sigma G_x(\cdot, x(\cdot)) \in L^\infty(0,\pi).$$

In order to prove Theorem 1.1 we will investigate the abstract realization of (1.1) and provide the abstract variational principle. In case $G_x(t,x) = 0$, the dual variational method was developed in [4]. However now our assumptions are weaker and therefore the construction of the dual variational method substantially differs. Still we may obtain with the approach of the present paper the results from [4] and the present paper advances the construction from [6]. It is worth to be noted that our method applies to a wider class of nonlinear problems than methods from [4], [6].

2. Auxiliary Lemmas

Lemma 2.1. *There exists $x \in X$ such that* $\inf_{u \in X} J(u) = J(x)$.

Proof. X is weakly compact in $H_0^2(0,\pi)$ and J is bounded from below on X. Hence we take a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that $x_n \rightharpoonup x$ in $H^2(0,\pi)$ and $\inf_{u \in X} J(u) = \lim_{n\to\infty} J(x_n)$. Since $\{x_n\}_{n=1}^{\infty}$ contains a subsequence convergent uniformly we get (for this subsequence)

$$\begin{aligned} &\int_0^{\pi} F(t, x_{n_k}(t))\,dt \to \int_0^{\pi} F(t, x(t))\,dt, \\ &\int_0^{\pi} G(t, x_{n_k}(t))\,dt \to \int_0^{\pi} G(t, x(t))\,dt. \end{aligned} \tag{2.1}$$

Thus J weakly l.s.c. on X. Hence $\inf_{u\in X} J(u) = \lim_{n\to\infty} J(x_n) \geq J(x) \geq \inf_{u\in X} J(u)$. ∎

Lemma 2.2. *Assume* **F1**, **F2**, **G1**. *Let* $0 < \lambda \leq \frac{(\beta-\gamma-|\delta|)d}{\alpha\pi}$ *and let* $\sigma \geq 0$. *For each $x \in X$ there exists a solution $u \in X$ to the Dirichlet problem*

$$\begin{aligned} &\beta \frac{d^4}{dt^4} u(t) + \gamma \frac{d^2}{dt^2} u(t) + \delta u(t) + \sigma G_x(t, u(t)) = \lambda F_x(t, x(t)), \\ &u(0) = u(\pi) = \dot{u}(0) = \dot{u}(\pi) = 0. \end{aligned} \tag{2.2}$$

Proof. By (1.2) for a fixed $x \in X$ the function $t \to F_x(t, x(t))$ belongs to $L^{\infty}(0,\pi)$. Since G is convex it follows that the functional

$$\begin{aligned} \widetilde{J}(u) = {} & \frac{\beta}{2} \int_0^{\pi} \left(\frac{d^2}{dt^2} u(t)\right)^2 dt - \frac{\gamma}{2} \int_0^{\pi} \left(\frac{d}{dt} u(t)\right)^2 dt + \frac{\delta}{2} \int_0^{\pi} u^2(t)\,dt \\ & + \sigma \int_0^{\pi} G(t, u(t)) - \int_0^{\pi} \lambda F_x(t, x(t))\, u(t)\,dt \end{aligned}$$

is coercive on $H_0^2(0,\pi)$. Since it is also convex and lower semicontinuous, it follows by a direct method of the calculus of variations that $\widetilde{J}$ has a minimum over $H_0^2(0,\pi)$ which satisfies the Euler-Lagrange equation, i.e. (2.2), compare with [5]. Multiplying (2.2) by u and taking integrals we obtain what follows by the Poincaré inequality, Schwarz inequality and (1.2)

$$(\beta-\gamma-|\delta|)\|\dot{u}\|^2_{L^2(0,\pi)}+\sigma\int_0^\pi G_x(t,u(t))\,u(t)\le\beta\int_0^\pi\left(\frac{d^2}{dt^2}u(t)\right)^2dt$$
$$-\gamma\int_0^\pi\left(\frac{d}{dt}u(t)\right)^2dt+\delta\int_0^\pi u^2(t)\,dt+\sigma\int_0^\pi G_x(t,u(t))\,u(t)$$
$$=\int_0^\pi\lambda F_x(t,x(t))\,u(t)\,dt\le\sqrt{\pi}\lambda\alpha\|\dot{u}\|_{L^2(0,\pi)}.$$

Since $\lambda\le\frac{(\beta-\gamma-|\delta|)d}{\alpha\pi}$ we get

$$(\beta-\gamma-|\delta|)\|\dot{u}\|^2_{L^2(0,\pi)}+\sigma\int_0^\pi G_x(t,u(t))\,u(t)$$
$$\le\sqrt{\pi}\frac{(\beta-\gamma-|\delta|)\,d}{\pi}\|\dot{u}\|_{L^2(0,\pi)}.$$

Since $\int_0^\pi G_x(t,u(t))\,u(t)\ge 0$ we see that $\|\dot{u}\|_{L^2(0,\pi)}\le\frac{d}{\sqrt{\pi}}$. By Sobolev's inequality we get $\max_{t\in[0,\pi]}|u(t)|\le\sqrt{\pi}\|\dot{u}\|_{L^2(0,\pi)}\le d$. By relation

$$\beta\int_0^\pi\left(\frac{d^2}{dt^2}u(t)\right)^2dt-\gamma\int_0^\pi\left(\frac{d}{dt}u(t)\right)^2dt+\delta\int_0^\pi u^2(t)\,dt\le\frac{(\beta-\gamma-|\delta|)\,d^2}{\sqrt{\pi}}$$

and since $\gamma<0$, $\|\dot{u}\|_{L^2(0,\pi)}\le\frac{d}{\sqrt{\pi}}$ we get

$$\|\ddot{u}\|_{L^2(0,\pi)}\le\sqrt{\frac{(-\gamma+|\delta|)\,d^2}{\beta\pi}+\frac{(\beta-\gamma-|\delta|)\,d^2}{\beta\sqrt{\pi}}}.\ \blacksquare$$

3. Existence of solutions for (1.1)

Lemma 2.1 provides the existence of an $x\in X$ which is a candidate for a solution of (1.1). We may not apply the Euler-Lagrange Lemma or the mountain pass geometry to show that it indeed is one. In order to show that x is a solution of (1.1) we introduce a certain abstract dual variational method. Let Y be a separable real Hilbert space with scalar product $\langle\cdot,\cdot\rangle$. We will investigate the existence of solution to equation

$$Lx+G_x(x)=F_x(x),\tag{3.1}$$

where $L:D(L)\subset Y\to Y$ is a densely defined linear operator, i.e. $D(L)$ is a dense subspace of Y. L is a self-adjoint and positive definite linear operator. In that case there exists a densely defined self-adjoint square root operator $S:D(S)\to Y$. We observe that $Sx\in D(S)$ for any $x\in D(L)$ and $S^2=L$, see [2]. On $D(S)$ we use a norm $\|x\|_{D(S)} = \|Sx\|_Y$ which makes it into a complete space.

We assume that

A1 any bounded sequence in $D(S)$ contains a subsequence convergent in Y;
A2 F, $G : Y \to R$ are convex l.s.c. and Gâteaux differentiable functions bounded on bounded sets; $F_x(0) \neq 0$;
A3 there exists a nonempty set $X \subset D(L)$ such that for each $x \in X$ relation

$$L\widetilde{x} + G_x(\widetilde{x}) = F_x(x), \tag{3.2}$$

implies that $\widetilde{x} \in X$; X is weakly compact in $D(S)$;
A4 for any sequence $\{x_k\} \subset X$ strongly convergent in Y to $x \in X$ we have $\lim_{n\to\infty} F(x_n) = F(x)$.

We will investigate on X the action functional $J : D(S) \to R$ defined by

$$J(x) = \frac{1}{2}\langle Sx, Sx\rangle + G(x) - F(x).$$

Theorem 3.1. *We assume* **A1**–**A4**. *There exists $x \in X$ satisfying equation* (3.1) *and such that $J(x) = \inf_{u\in X} J(u)$.*

Proof. J is bounded from below on X. Thus, by the properties of X, see **A3**, we choose a minimizing sequence $\{x_j\}_{j=1}^{\infty}$ weakly convergent in $D(S)$ to a certain $x \in X$. By **A1** it is convergent strongly in Y, possibly up to a subsequence. Since the norm in $D(S)$ is weakly l.s.c. and by **A2** and **A4** we see that functional J is weakly lower-semicontinuous on a sequence $\{x_j\}_{j=1}^{\infty}$. Hence $J(x) = \inf_{u\in X} J(u)$.

Since $x \in X$, there exists $(p, q) \in D(S) \times Y$ such that

$$Sp + q = F_x(x), \tag{3.3}$$

holds. Indeed, by definition of X, there exist $\widetilde{x} \in X$ related to x by (3.2) and such that $S\widetilde{x} = p$ and $q = G_x(\widetilde{x})$. We then have $\langle \widetilde{x}, Sp\rangle - \frac{1}{2}\langle S\widetilde{x}, S\widetilde{x}\rangle = \frac{1}{2}\langle p, p\rangle$ and $\langle q, \widetilde{x}\rangle - G(\widetilde{x}) = G^*(q)$; here G^* denotes the Fenchel-Young transform of a convex functional G, see [1]. Therefore by a direct calculation

$$\begin{aligned} \frac{1}{2}\langle p, p\rangle + G^*(q) &\leq \sup_{u\in X}\left\{\langle u, Sp\rangle - \frac{1}{2}\langle Su, Su\rangle + \langle q, u\rangle - G(u)\right\} \\ &\leq \frac{1}{2}\langle p, p\rangle + G^*(q). \end{aligned} \tag{3.4}$$

By (3.3) we have

$$\langle Sp + q, x\rangle = F(x) + F^*(Sp + q). \tag{3.5}$$

Thus by the Fenchel-Young inequality we obtain

$$\begin{aligned} J(x) &= \frac{1}{2}\langle Sx, Sx\rangle - F(x) + G(x) \\ &= \frac{1}{2}\langle Sx, Sx\rangle - \langle Sp+q, x\rangle + F^*(Sp+q) + G(x) \\ &\geq -\frac{1}{2}\langle p, p\rangle + F^*(Sp+q) - G^*(q). \end{aligned}$$

Next we see by (3.4) that

$$\begin{aligned} J(x) &= \inf_{u\in X} J(u) = \inf_{u\in X}\left\{\frac{1}{2}\langle Su, Su\rangle - F(u) + G(u)\right\} \\ &\leq \inf_{u\in X}\left\{-\langle p, Su\rangle + \frac{1}{2}\langle Su, Su\rangle - \langle q, u\rangle + G(u)\right\} + F^*(Sp+q) \\ &= -\frac{1}{2}\langle p, p\rangle - G^*(q) + F^*(Sp+q). \end{aligned}$$

Hence

$$\frac{1}{2}\langle Sx, Sx\rangle - F(x) + G(x) = -\frac{1}{2}\langle p, p\rangle - G^*(q) + F^*(Sp+q)$$

and by (3.5)

$$\frac{1}{2}\langle Sx, Sx\rangle + \frac{1}{2}\langle p, p\rangle - \langle p, Sx\rangle + G^*(q) + G(x) - \langle q, x\rangle = 0.$$

By the Fenchel-Young inequalities we have actually the equalities

$$\frac{1}{2}\langle Sx, Sx\rangle + \frac{1}{2}\langle p, p\rangle = \langle p, Sx\rangle, \ G^*(q) + G(x) = \langle q, x\rangle.$$

Hence, we get $Sx = p$, $q = G_x(x)$ which inserted into (3.3) show that (3.1) satisfied. ∎

Proof of Theorem 1.1. Let $L = \beta\frac{d^4}{dt^4}x + \gamma\frac{d^2}{dt^2}x + \delta x$ with $D(L) = H_0^2(0,\pi) \cap H^4(0,\pi)$. We see that $D(S) = H_0^1(0,\pi) \cap H^2(0,\pi)$. Thus we have **A1**. **A2** is obviously satisfied. Lemma 2.2 shows that **A3** holds. **A4** holds by first relation in (2.1). The last assertion follows by (1.2). ∎

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