DEMONSTRATIO MATHEMATICA
Vol. XLII No 1 2009

Stanisław Domachowski

LOCAL BOUNDS AND EXISTENCE OF SOLUTIONS TO NON-CONVEX DIFFERENTIAL INCLUSIONS

Abstract. Using a global bifurcation theorem for convex-valued completely continuous mapping we prove an existence theorem for differential inclusions of the form $u'' \in F(t, u, u')$, where F admits a convex-valued, weakly completely continuous selector and u satisfies some nonlinear boundary conditions.

1. Introduction

In this paper we prove an existence theorem for multi-valued boundary value problem

$$(1.1) \qquad \begin{cases} u''(t) \in F(t, u(t), u'(t)) & \text{for a.e. } t \in (a, b) \\ l(u) \in B(u), \end{cases}$$

where $F : [a, b] \times \mathbb{R}^k \times \mathbb{R}^k \to \mathrm{cl}(\mathbb{R}^k)$, $B : C^1([a, b], \mathbb{R}^k) \to \mathrm{cf}(\mathbb{R}^k \times \mathbb{R}^k)$ satisfy suitable assumptions, and $l : C^1([a, b], \mathbb{R}^k) \to \mathbb{R}^k \times \mathbb{R}^k$ represents the Sturm-Lioville boundary conditions. Let us remind that one of the most used methods to get existence results for second order differential equations (inclusions), with the Sturm-Lioville boundary conditions is the topological transversality by A. Granas, for example see [5, 8, 9]. In [12] S.A. Marano considered convex-valued differential inclusions with the Picard boundary conditions. His approach was based on a recent existence theorem for operator inclusions, see [18]. In this paper we consider multi-valued boundary value problems with non-convex multi-valued mappings $F : [a, b] \times \mathbb{R}^k \times \mathbb{R}^k \to \mathrm{cl}(\mathbb{R}^k)$. The assumptions refer to the appropriate asymptotic behaviour of $F(t, x, y)$ for $|x| + |y|$ close to 0 and to $+\infty$, and they are independent of these used in [12]. What is more our boundary conditions are not linear. The approach we present is based on a global bifurcation theorem for convex-valued completely continuous mappings [4].

2000 *Mathematics Subject Classification*: 47H04, 34A60, 34B24.
Key words and phrases: differential inclusions, global bifurcation, selectors, Sturm--Liouville boundary conditions.

The paper will be divided into three sections. Firstly we will state the main existence theorem. In the second section the existence theorem will be proved. Finally some applications of the results given in the previous section, and selector theorems will be provided.

2. Main theorem

Let E be a real Banach space. By $\mathrm{cl}(E)$ we will denote the family of all non-empty, closed and bounded subsets of E. By $\mathrm{cf}(E)$ we will denote the family of all non-empty, closed, bounded and convex subsets of E. For two sets $A, B \in \mathrm{cl}(E)$ we will denote by $\mathrm{D}(A, B)$ the Hausdorff distance between A an B. In particular we put $|A| = \mathrm{D}(A, \{0\})$.

Let E_1, E_2 be two Banach spaces. A multi-valued mapping $\varphi : E_1 \to \mathrm{cl}(E_2)$ is called weakly upper semicontinuous (w-u.s.c.), provided for all sequences $\{x_n\} \subset E_1$ and $\{y_n\} \subset E_2$ the conditions $\{x_n\} \to x$, $\{y_n\} \rightharpoonup y$ and $y_n \in \varphi(x_n)$ for every $n \in \mathbb{N}$ imply $y \in \varphi(x)$ ($\{y_n\} \rightharpoonup y$ denotes the weak convergence).

A multi-valued mapping φ is called weakly completely continuous if φ is w-u.s.c. and for every bounded subset A of E_1 the image $\varphi(A) = \bigcup_{x \in A} \varphi(x)$ is a relatively weakly compact subset of E_2.

In this paper we will need the following notations. For $x = (x_1, \dots, x_k) \in \mathbb{R}^k$ we call x non-negative (and write $x \geq 0$), when $x_i \geq 0$ for $i = 1, \dots, k$. Let $\|\cdot\|_0$ be the supremum norm in $C[a, b]$ and $\|\cdot\|_k$ be the norm in $C^1([a, b], \mathbb{R}^k)$ given by $\|u\|_k = \sum_{i=1}^k (\|u_i\|_0 + \|u_i'\|_0)$ for $u = (u_1, ..., u_k) \in C^1([a, b], \mathbb{R}^k)$. Let $p : \mathbb{R}^k \to \mathbb{R}^k$ be given by $p(x_1, \dots, x_k) = (|x_1|, \dots, |x_k|)$, and let $P : C^1([a, b], \mathbb{R}^k) \to L^1((a, b), \mathbb{R}^k)$ denotes the Nemytskii operator for the mapping p.

Let us remind that a multi-valued mapping $F : [a, b] \times \mathbb{R}^k \times \mathbb{R}^k \to \mathrm{cl}(\mathbb{R}^k)$ is called integrably bounded if for every $R > 0$ there exists a function $m_R \in L^1(a, b)$ such that for every $x, y \in \mathbb{R}^k$ with $|x| + |y| \leq R$ we have $|F(t, x, y)| \leq m_R(t)$ a.e. on $[a, b]$.

In what follows the multi-valued mapping $F : [a, b] \times \mathbb{R}^k \times \mathbb{R}^k \to \mathrm{cl}(\mathbb{R}^k)$ satisfies the condition

(2.1) there exists a weakly completely continuous mapping $\varphi : C^1([a, b], \mathbb{R}^k) \to \mathrm{cf}(L^1((a, b), \mathbb{R}^k))$ such that, for every $v \in C^1([a, b], \mathbb{R}^k)$

$$\varphi(v) \subseteq \{w \in L^1((a, b), \mathbb{R}^k) : \quad w(t) \in F(t, v(t), v'(t)) \text{ a.e. on } [a, b]\}.$$

The mapping $l : C^1([a, b], \mathbb{R}^k) \to \mathbb{R}^k \times \mathbb{R}^k$ is given by

$$l(u_1, ..., u_k) = (l_1(u_1), ..., l_k(u_k))$$

where

$$l_j(u_j) = (-u_j(a) \sin \alpha_j + u_j'(a) \cos \alpha_j, -u_j(b) \sin \beta_j - u_j'(b) \cos \beta_j),$$

with $\alpha_j, \beta_j \in [0, \frac{\pi}{2}]$, $\alpha_j + \beta_j > 0$, $(j = 1, ..., k)$, and $B : C^1([a,b], \mathbb{R}^k) \to \mathrm{cf}(\mathbb{R}^k \times \mathbb{R}^k)$ is completely continuous.

Before we state the existence theorem, we will need some of spectral properties of the linear problem

$$(2.2) \qquad \begin{cases} u''(t) + \lambda u(t) = 0 & \text{for } t \in (a,b) \\ l(u) = 0. \end{cases}$$

It is obvious that $\mu \in \mathbb{R}$ is an eigenvalue of (2.2) if and only if there exists $j \in \{1, ..., k\}$ such that μ is an eigenvalue of the scalar problem

$$(2.2)_j \qquad \begin{cases} u_j''(t) + \lambda u_j(t) = 0 & \text{for } t \in (a,b) \\ l_j(u_j) = 0. \end{cases}$$

It is well known (cf [3, 10]), that there exists exactly one eigenvalue $\mu_j \in \mathbb{R}$ of $(2.2)_j$, with an eigenvector v_{μ_j}, such that $v_{\mu_j}(t) > 0$ for $t \in (a,b)$, and then $\mu_j > 0$. Let us observe that $u_{\mu_j} = (0, ..., v_{\mu_j}, ...0)$ is the eigenvector of (2.2) associated with the eigenvalue μ_j. The set of eigenvalues μ_i of (2.2), for which there exists non-negative eigenvector u_{μ_i}, is non-empty and contains at most k-elements. Let us denote this set by $\Lambda = \{\mu_i : i = 1, 2, \ldots, N\}$, where $N \leq k$.

Theorem 1. *Let $F : [a,b] \times \mathbb{R}^k \times \mathbb{R}^k \to \mathrm{cl}(\mathbb{R}^k)$ be an integrably bounded mapping satisfying* (2.1), *and for every $\varepsilon > 0$ there exists $R > 0$ such that*

$$(2.3) \quad D(F(t,x,y), \{-m_2 p(x)\}) \leq \varepsilon(|x| + |y|) \text{ for } t \in [a,b] \quad |x| + |y| \geq R;$$

for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(2.4) \quad D(F(t,x,y), \{-m_1 p(x)\}) \leq \varepsilon(|x| + |y|) \text{ for } t \in [a,b] \quad |x| + |y| \leq \delta$$

with constants $m_1, m_2 > 0$ such that

$$\min\{m_1, m_2\} < \min \Lambda \leq \max \Lambda < \max\{m_1, m_2\}.$$

In addition assume that a completely continuous mapping $B : C^1([a,b], \mathbb{R}^k) \to \mathrm{cf}(\mathbb{R}^k \times \mathbb{R}^k)$ satisfies

$$(2.5) \qquad \forall_{\varepsilon>0} \exists_{\delta>0} \forall_{u \in C^1([a,b], \mathbb{R}^k)} \|u\|_k \leq \delta \Rightarrow |B(u)| \leq \varepsilon \|u\|_k;$$

$$(2.6) \qquad \forall_{\varepsilon>0} \exists_{R>0} \forall_{u \in C^1([a,b], \mathbb{R}^k)} \|u\|_k \geq R \Rightarrow |B(u)| \leq \varepsilon \|u\|_k.$$

Then there exists at least one non-trivial solution of boundary value problem (1.1).

3. Proof of Theorem 1

We need some notations to prove Theorem 1. Let $\Psi : (0, \infty) \times C^1([a,b], \mathbb{R}^k) \to \mathrm{cf}(C^1([a,b], \mathbb{R}^k))$ be a completely continuous mapping such that $0 \in \Psi(\lambda, 0)$ for every $\lambda \in (0, \infty)$. Let $f : (0, \infty) \times C^1([a,b], \mathbb{R}^k) \to$

$\mathrm{cf}(C^1([a,b],\mathbb{R}^k))$ be given by

$$f(\lambda,u)=u-\Psi(\lambda,u). \tag{3.1}$$

We call $(\mu,0)\in(0,\infty)\times C^1([a,b],\mathbb{R}^k)$ a bifurcation point of f if for each neibourghood U of $(\mu,0)$ in $(0,\infty)\times C^1([a,b],\mathbb{R}^k)$ there exists a point $(\lambda,u)\in U$ such that $u\neq 0$ and $0\in f(\lambda,u)$. Let us denote the set of all bifurcation points of f by $\mathcal{B}_f$. Let $\mathcal{R}_f\subset(0,\infty)\times C^1([a,b],\mathbb{R}^k)$ be a closure (in $(0,\infty)\times C^1([a,b],\mathbb{R}^k)$) of the set of non-trivial solutions of the inclusion $0\in f(\lambda,u)$, i.e.

$$\mathcal{R}_f=\overline{\{(\lambda,u)\in(0,\infty)\times C^1([a,b],\mathbb{R}^k):\ u\neq 0\ \wedge\ 0\in f(\lambda,u)\ \}}.$$

For each λ satisfying $(\lambda,0)\notin\mathcal{B}_f$ there exists $r_0>0$, such that for $\|u\|_k=r\in(0,r_0]$ the relation $u\notin\Psi(\lambda,u)$ holds, so the value $\deg(f(\lambda,\cdot),B(0,r),0)$ is defined (where $B(0,r)$ denotes an open ball with the centre at 0 and a radius $r>0$).

Assume that for an interval $[a_1,b_1]\subset(0,\infty)$ there exists $\delta>0$ such that

$$\Big(([a_1-\delta,a_1)\cup(b_1,b_1+\delta])\times\{0\}\Big)\cap\mathcal{B}_f=\emptyset.$$

Then we may define the bifurcation index $s[f,a_1,b_1]$ of the mapping f, with respect to the interval $[a_1,b_1]$ as

$$s[f,a_1,b_1]=\lim_{\lambda\to b_1^+}\deg(f(\lambda,\cdot),B(0,r),0)-\lim_{\lambda\to a_1^-}\deg(f(\lambda,\cdot),B(0,r),0),$$

where $r=r(\lambda)>0$ is small enough.

The main tool used in this section is a global bifurcation theorem for convex-valued completely continuous mappings called Theorem A, which is a consequence of the generalization of the Rabinowitz global bifurcation alternative (see [17]).

Theorem A. [4] *Let $f:(0,\infty)\times C^1([a,b],\mathbb{R}^k)\to\mathrm{cf}(C^1([a,b],\mathbb{R}^k))$ be given by (3.1). Assume that there exists an interval $[a_1,b_1]\subset(0,\infty)$ such that $\mathcal{B}_f\subset[a_1,b_1]\times\{0\}$ and $s[f,a_1,b_1]\neq 0$. Then there exists a non-compact component $\mathcal{C}\subset\mathcal{R}_f$ satisfying $\mathcal{C}\cap\mathcal{B}_f\neq\emptyset$.*

It is well known that with a boundary value problem

$$\begin{cases} u''(t)=h(t) & \text{for a.e. } t\in(a,b)\\ l(u)=0, \end{cases} \tag{3.2}$$

we may associate continuous mappings $P_1:C^1([a,b],\mathbb{R}^k)\to C^1([a,b],\mathbb{R}^k)$, $i:\mathbb{R}^k\times\mathbb{R}^k\to C^1([a,b],\mathbb{R}^k)$ and $T:L^1((a,b),\mathbb{R}^k)\to C^1([a,b],\mathbb{R}^k)$ given by formulas

$$P_1(u)(t)=u(a)+u'(a)(t-a) \tag{3.3}$$

$$i(x,y)(t)=x+y(t-a) \tag{3.4}$$

$$T(u)(t) = \int_a^t \int_a^s u(\tau) d\tau ds. \tag{3.5}$$

Let us observe that $u = P_1(u) + i(l(u)) + T(h)$ iff $u \in C^1([a,b], \mathbb{R}^k)$, $u' : [a,b] \to \mathbb{R}^k$ is absolutely continuous and u is a solution of (3.2).

We will also need the following results.

(3.6) (Maximum principle cf [14]) If the mappings $u \in C^2([a,b], \mathbb{R}^k)$ and $h \in C([a,b], \mathbb{R}^k)$ satisfy

$$\begin{cases} u''(t) = h(t) & \text{for } t \in (a,b) \\ l(u) = 0 \end{cases}$$

and $h \leq 0$ then $u \geq 0$.

According to the property of Green's function for (3.2) (cf [3, 10]) we obtain the following property.

(3.7) Let us assume that $\mu_i \in \Lambda$ and let u_{μ_i} be an eigenvector of (2.2) associated with μ_i such that $u_{\mu_i} \geq 0$, moreover assume that $\lambda > \max \Lambda$ then for $\tau \in (0,1]$ the problem

$$\begin{cases} u''(t) + \lambda u(t) + \tau \mu_i u_{\mu_i}(t) = 0 & \text{for } t \in (a,b) \\ l(u) = 0 \\ u \geq 0 \end{cases}$$

has no solutions.

Proof of Theorem 1. Let us denote $m = \min\{m_1, m_2\}$ and $M = \max\{m_1, m_2\}$. Let $\nu > \frac{\max \Lambda}{m}$ be fixed constant. Let $q_1, q_2 : (0, +\infty) \to [0,1]$ be continuous mappings forming the partition of unity associated with the open cover $\{(0, 2\nu), (\nu, +\infty)\}$ of the interval $(0, +\infty)$. By assumption (2.1) there exists a weakly completely continuous multi-valued mapping $\varphi : C^1([a,b], \mathbb{R}^k) \to \mathrm{cf}(L^1((a,b), \mathbb{R}^k))$ such that

$$\varphi(u) \subseteq \{w \in L^1((a,b), \mathbb{R}^k) : \quad w(t) \in F(t, u(t), u'(t)) \text{ a.e. on } [a,b]\} \tag{3.8}$$

for each $u \in C^1([a,b], \mathbb{R}^k)$. In virtue of (2.1), and the integrably bounded of F the composition $T \circ \varphi : C^1([a,b], \mathbb{R}^k) \to \mathrm{cf}(C^1([a,b], \mathbb{R}^k))$ is completely continuos (cf [16]). Let $f : (0, \infty) \times C^1([a,b], \mathbb{R}^k) \to \mathrm{cf}(C^1([a,b], \mathbb{R}^k))$ be given by the formula

$$f(\lambda, u) = u - P_1(u) - i(l(u)) + \lambda q_1(\lambda) i(B(u)) - T(\lambda q_1(\lambda) \varphi(u) - \lambda q_2(\lambda) m_2 P(u)).$$

Since $\nu > 1$ then if $0 \in f(1, u)$ then u is a solution to (1.1). So it is enough to show that there exists $u \in C^1([a,b], \mathbb{R}^k)$ such that $0 \in f(1, u)$. To prove this we apply Theorem A.

The proof will be given in three steps.

Step 1. We are going to show that $\mathcal{B}_f \subset \{(\frac{\mu_i}{m_1}, 0) : \mu_i \in \Lambda\}$. Let us take a sequence $\{(\lambda_n, u_n)\} \subset (0, +\infty) \times C^1([a,b], \mathbb{R}^k)$ of non-trivial solutions to the inclusion

$$u_n \in P_1(u_n) + i(l(u_n)) - \lambda_n q_1(\lambda_n) i(B(u_n)) + \lambda_n q_1(\lambda_n) T(\varphi(u_n)) \\ -\lambda_n q_2(\lambda_n) m_2 T(P(u_n))$$

such that $\lambda_n \to \lambda_0 \in [0, +\infty)$ and $u_n \to 0$. Thus we have

$$u_n \in P_1(u_n) + i(l(u_n)) - \lambda_n q_1(\lambda_n) i(B(u_n)) + \\ + \lambda_n q_1(\lambda_n) T(\varphi(u_n) + m_1 P(u_n)) - \lambda_n (m_1 q_1(\lambda_n) + m_2 q_2(\lambda_n)) TP(u_n).$$

Let us denote $v_n = \frac{u_n}{\|u_n\|_k}$, therefore we have

$$v_n \in P_1(v_n) + i(l(v_n)) - \lambda_n q_1(\lambda_n) i\left(\frac{B(u_n)}{\|u_n\|_k}\right) + \lambda_n q_1(\lambda_n) T \frac{\varphi(u_n) + m_1 P(u_n)}{\|u_n\|_k} - \\ -\lambda_n (m_1 q_1(\lambda_n) + m_2 q_2(\lambda_n)) TP(v_n)).$$

By (2.4) and (2.5) we obtain $|\frac{\varphi(u_n) + m_1 P(u_n)}{\|u_n\|_k}| \to 0$, and $|\frac{B(u_n)}{\|u_n\|_k}| \to 0$. Since the sequence $\{(m_1 q_1(\lambda_n) + m_2 q_2(\lambda_n)) P(v_n)\}$ is bounded there exists a subsequence of $\{v_n\}$ convergent to $v_0 \in C^1([a,b], \mathbb{R}^k)$, where $\|v_0\|_k = 1$. So letting $n \to +\infty$ we obtain

$$v_0 = P_1(v_0) + i(l(v_0)) - \lambda_0((m_1 q_1(\lambda_0) + m_2 q_2(\lambda_0)) TP(v_0))$$

and

$$\begin{cases} v_0''(t) + \lambda_0 (m_1 q_1(\lambda_0) + m_2 q_2(\lambda_0)) p(v_0(t)) = 0 & \text{for a.e. } t \in (a,b) \\ l(v_0) = 0, \end{cases}$$

Since $\lambda_0 (m_1 q_1(\lambda_0) + m_2 q_2(\lambda_0)) p(v_0(t)) \geq 0$, then by (3.6) $v_0 \geq 0$ in consequence $(m_1 q_1(\lambda_0) + m_2 q_2(\lambda_0)) \lambda_0 \in \Lambda$. No matter what is the value of λ_0 we have $m_1 q_1(\lambda_0) + m_2 q_2(\lambda_0) \in [m, M]$, so

$$\lambda_0 \leq \frac{\max \Lambda}{m} < \nu,$$

that implies $m_1 \lambda_0 \in \Lambda$ completing the proof of Step 1.

Step 2. We will now show that $s[f, \frac{\min \Lambda}{m_1}, \frac{\max \Lambda}{m_1}] = -1$. First let us observe that for $\lambda \notin \{\frac{\mu_i}{m_1} : \mu_i \in \Lambda\}$ there exists $r > 0$ such that, according to (2.4), (2.5) the mapping $f(\lambda, \cdot) : \overline{B(0,r)} \to \mathrm{cf}(C^1([a,b], \mathbb{R}^k))$ is homotopic to a mapping $\bar{f}(\lambda, \cdot) : \overline{B(0,r)} \to C^1([a,b], \mathbb{R}^k))$ given by

$$\bar{f}(\lambda, u) = u - P_1(u) - i(l(u)) + \lambda (m_1 q_1(\lambda) + m_2 q_2(\lambda)) TP(u).$$

We can also see, that the mapping $\bar{f}(\lambda, \cdot) : \overline{B(0,r)} \to C^1([a,b], \mathbb{R}^k)$ for $\lambda \geq \nu$, may be joined by homotopy with a mapping $f_0(\lambda, \cdot) : \overline{B(0,r)} \to C^1([a,b], \mathbb{R}^k)$ given by

$$f_0(\lambda, u) = u - P_1(u) - i(l(u)) + \lambda m_1 TP(u).$$

Let the homotopy $h : [0,1] \times \overline{B(0,r)} \to C^1([a,b],\mathbb{R}^k)$ be given by

$h(\tau,u) = u - P_1(u) - i(l(u)) + \lambda(\tau m_1 q_1(\lambda) + \tau m_2 q_2(\lambda) + (1-\tau)m_1)TP(u).$

Similarly to what we showed in Step 1 of this proof, for any non-trivial zero of the homotopy h there must be

$$\lambda(\tau m_1 q_1(\lambda) + \tau m_2 q_2(\lambda) + (1-\tau)m_1) \in \Lambda,$$

that implies $\lambda \le \frac{\max\Lambda}{m}$ and contradicts $\lambda \ge \nu$. On the other hand for $\lambda < \nu$ we have $\bar{f}(\lambda,\cdot) = f_0(\lambda,\cdot)$. Let $r > 0$ and $\lambda_0 \in (0, \frac{\min\Lambda}{m_1})$ be fixed, then the mapping $f_0(\lambda_0,\cdot) : \overline{B(0,r)} \to C^1([a,b],\mathbb{R}^k)$ may be joined by homotopy with the identity mapping. Hence, by the homotopy property of the topological degree, we have $\deg(f_0(\lambda_0,\cdot), B(0,r), 0) = 1$. Assume now that $\lambda_0 \in (\frac{\max\Lambda}{m_1}, +\infty)$. Choose any $i \in \{1,...,N\}$ and denote by u_{μ_i} a non-trivial mapping, such that

$$u_{\mu_i} = P_1(u_{\mu_i}) + il(u_{\mu_i}) - \mu_i T u_{\mu_i}$$

and $u_{\mu_i}(t) \ge 0$, for $t \in (a,b)$. We will show that the mapping $f_0(\lambda_0,\cdot)$ may be joined by homotopy on $\overline{B(0,r)}$ with $f_1 : \overline{B(0,r)} \to C^1([a,b],\mathbb{R}^k)$, given by

$$f_1(u) = f_0(\lambda_0,u) + \mu_i T u_{\mu_i}.$$

A homotopy $h : [0,1] \times \overline{B(0,r)} \to C^1([a,b],\mathbb{R}^k)$ is given by

$$h(\tau,u) = f_0(\lambda_0,u) + \tau\mu_i T u_{\mu_i}.$$

Assume now that for $\|u\|_k \le r$ and $\tau \in (0,1]$ the equality $h(\tau,u) = 0$ holds, so we have

$$\begin{cases} u''(t) + \lambda_0 m_1 p(u(t)) + \tau\mu_i u_{\mu_i}(t) = 0 & \text{for a.e. } t \in (a,b) \\ l(u) = 0. \end{cases}$$

Then by (3.6) and (3.7) we obtain contradiction, so for $\tau \in (0,1]$ and $\|u\|_k \le 0$ $h(\tau,u) \neq 0$. If $\tau = 0$, then $h(0,u) = 0$ if and only if $f_0(\lambda_0,u) = 0$. Since $m_1\lambda_0 \notin \Lambda$, the condition $f_0(\lambda_0,u) = 0$ implies $u = 0$. Hence the homotopy h has no non-trivial zeroes, and $h(1,\cdot)$ has no zeroes at all and that is why

$$\deg(f_0(\lambda_0,\cdot), B(0,r), 0) = 0.$$

Step 3. Let us observe that by Theorem A there exists a non-compact component $\mathcal{C} \subset \mathcal{R}_f$. Now we are going to show that there exists a sequence $\{(\lambda_n,u_n)\} \subset \mathcal{C}$, such that $\|u_n\|_k \to +\infty$ and $\lambda_n \to \lambda_0 \in \{\frac{\mu_i}{m_2} : \mu_i \in \Lambda\}$.

Since the set $\mathcal{C}$ is not compact there exists a sequence $\{(\lambda_n,u_n)\} \subset \mathcal{C}$ such that $\lambda_n \to 0$ or $\lambda_n \to +\infty$ or $\|u_n\|_k \to +\infty$. We are going to show that there must be $\|u_n\|_k \to +\infty$. First let us assume that $\lambda_n \to 0$ and $\{\|u_n\|_k\}$ is bounded. Since $T\varphi$, B are completely continuous, and $\lambda_n \to 0$ there exists a subsequence of $\{u_n\}$ convergent to zero in the space $C^1([a,b],\mathbb{R}^k)$. In this case, as we showed in Step 1 $u_n \to 0$ and $\lambda_n \to \lambda_0$ implies that $\lambda_0 \in \{\frac{\mu_i}{m_1} : \mu_i \in \Lambda\}$ what contradicts $\lambda_n \to 0$. Now let us consider the case $\lambda_n \to +\infty$. Then for almost all $n \in \mathbb{N}$ there must be $q_2(\lambda_n) = 1$, $u_n \neq 0$ and

$$u_n = P_1(u_n) + il(u_n) - \lambda_n T m_2 P(u_n).$$

By (3.6) $u_n \geq 0$, so we have $\lambda_n \in \{\frac{\mu_i}{m_2} : \mu_i \in \Lambda\}$ what contradicts $\lambda_n \to +\infty$.

Then we may assume that $\|u_n\|_k \to +\infty$ and $\lambda_n \to \lambda_0 \in (0,+\infty)$. We can see that

$$v_n \in P_1(v_n) + i(l(v_n)) - \lambda_n q_1(\lambda_n) i\left(\frac{B(u_n)}{\|u_n\|_k}\right) + \lambda_n q_1(\lambda_n) T \frac{\varphi(u_n) + m_2 P(u_n)}{\|u_n\|_k} -$$
$$-\lambda_n m_2 T P(v_n)),$$

where $v_n = \frac{u_n}{\|u_n\|_k}$. From (2.3), (2.6) and integrably bounded of the mapping F there exists a subsequence of $\{v_n\}$ convergent to $v_0 \in C^1([a,b],\mathbb{R}^k)$, where $\|v_0\|_k = 1$. So letting $n \to +\infty$ we obtain

$$v_0 = P_1(v_0) + il(v_0) - \lambda_0 T m_2 P(v_0),$$

what results in $\lambda_0 \in \{\frac{\mu_i}{m_2} : \mu_i \in \Lambda\}$. As a consequence of Step 1 and Step 3 of this proof, we can see that the connected set $\mathcal{C}$ contains pairs (λ_1, u) and (λ_2, u) with $\lambda_1 < 1$ and $\lambda_2 > 1$. By connectedness of $\mathcal{C}$ there exists u with $(1,u) \in \mathcal{C}$. For such solution of inclusion $0 \in f(\lambda, u)$ there must be $u \neq 0$, because $(1,0) \notin \mathcal{R}_f$. So the proof is completed. ∎

4. Examples

In the first part of this section we will give a class of multi-valued mappings which admits a convex-valued weakly comletely continuous selectors. The problem concerning the existence of a continuous selector and a weakly completely continuous selector have been studied by many authors for example see: Antosiewicz and Cellina[1], Łojasiewicz [11], Pliś [13], Pruszko [15, 16], Fryszkowski [7], Bressan and Colombo [2], Frigon and Granas [6].

In what follows we will consider integrably bounded mappings $F : [a,b] \times \mathbb{R}^k \to \mathrm{cl}(\mathbb{R}^n)$ satisfying some of the following properties:

$$\text{(4.1)} \quad \begin{cases} F : [a,b] \times \mathbb{R}^k \to \mathrm{cl}(\mathbb{R}^n) & \text{is } \mathcal{L} \otimes B \text{ measurable} \\ F(t,\cdot) : \mathbb{R}^k \to \mathrm{cl}(\mathbb{R}^n) & \text{is l.s.c. for a.e. } t \in [a,b]. \end{cases}$$

(Let us recall that $A \subseteq [a,b] \times \mathbb{R}^k$ is $\mathcal{L} \otimes B$ measurable if A belongs to the σ–algebra generated by all sets of the form $N \times B$ where N is Lebesgue measurable in $[a,b]$ and B is Borel measurable in $\mathbb{R}^k$).

$$\text{(4.2)} \quad \begin{cases} F(\cdot,x) : [a,b] \to \mathrm{cl}(\mathbb{R}^n) & \text{is measurable for all } x \in \mathbb{R}^k \\ F(t,\cdot) : \mathbb{R}^k \to \mathrm{cl}(\mathbb{R}^n) & \text{is continuous for a.e. } t \in [a,b]. \end{cases}$$

$$F : [a,b] \times \mathbb{R}^k \to \mathrm{cl}(\mathbb{R}^n) \quad \text{is l.s.c.} \tag{4.3}$$

$$\begin{cases} F(\cdot, x) : [a,b] \to \mathrm{cf}(\mathbb{R}^n) & \text{is measurable for all } x \in \mathbb{R}^k \\ F(t, \cdot) : \mathbb{R}^k \to \mathrm{cf}(\mathbb{R}^n) & \text{is u.s.c. for a.e. } t \in [a,b]. \end{cases} \tag{4.4}$$

Let us recall that with the mapping $F : [a,b] \times \mathbb{R}^k \to \mathrm{cl}(\mathbb{R}^n)$ we can associate the Niemytskii operator $\mathcal{F} : C([a,b], \mathbb{R}^k) \to \mathrm{cl}(L^1((a,b), \mathbb{R}^n))$ given by

$$\mathcal{F}(v) = \{w \in L^1((a,b), \mathbb{R}^n) \quad w(t) \in F(t, u(t)) \quad \text{for a.e. } t \in (a,b)\}.$$

Now we state without proof the following Proposition, and next applying Theorems 1 we would obtain the existence theorems for boundary value problems.

Proposition 1. (cf. [1, 2, 6, 7, 11, 13, 15, 16].) *If $F : [a,b] \times \mathbb{R}^k \to \mathrm{cl}(\mathbb{R}^n)$ is an integrably bounded multi-valued mapping satisfying one of the conditions* (4.1), (4.2), (4.3), *or* (4.4) *then the Nemytskii operator $\mathcal{F} : C([a,b], \mathbb{R}^k) \to \mathrm{cl}(L^1((a,b), \mathbb{R}^n))$, associated with F, admits a convex-valued weakly completely continuous selector*

From Proposition 1 and Theorem 1 we obtain the followin theorem.

Theorem 2. *Let $B : C^1([a,b], \mathbb{R}^k) \to \mathrm{cf}(\mathbb{R}^k \times \mathbb{R}^k)$ be a completely continuous mapping satisfying* (2.5) *and* (2.6) *and let $F : [a,b] \times \mathbb{R}^k \times \mathbb{R}^k \to \mathrm{cl}(\mathbb{R}^k)$ be an integrably bounded multi-valued mapping such that one of the hypotheses* (4.1), (4.2), (4.3), *or* (4.4) *holds. If, moreover F satisfies* (2.3) *and* (2.4) *with constants $m_1, m_2 > 0$ such that $\min\{m_1, m_2\} < \min \Lambda \leq \max \Lambda < \max\{m_1, m_2\}$, then there exists at least one non-trivial solution of boundary value problem* (1.1).

In [12] the author proved that if $F : [0,1] \times \mathbb{R}^k \times \mathbb{R}^k \to \mathrm{cf}(\mathbb{R}^k)$ is the Caratheodory multi- valued mapping and satisfies suitable assumptions then the problem

$$\begin{cases} u''(t) \in F(t, u(t), u'(t)) & \text{for a.e. } t \in (a,b) \\ u(0) = u(1) = 0, \end{cases} \tag{4.5}$$

has at least one solution (see Theorem 2.1 [12]).
Below we will give an exampel of the Picard problem, which does not satisfy all assumptions of Theorem 2.1 in [12], but all assumptions of Theorem 1 are satisfied.

Example. Let a mapping $F : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathrm{cl}(\mathbb{R})$ be given by the formula

$$F(t,x,y) = \begin{cases} -9|x| & \text{if } |x| \leq 1 \\ -18|x| + 9 & \text{if } |x| > 1. \end{cases}$$

Then there exists at least one non-trivial solution of problem (4.5).

Let us observe that the mapping F dose not satisfy all assumptions of Theorem 2.1 in [12], but assumptions of Theorem 1 are satisfied.

Acknowledgement. The author is grateful to Professor Tadeusz Pruszko for the inspiration and help during preparation of this article.

References

[1] H. A. Antosiewicz, A. Cellina, *Continuous selectors and differential relations*, J. Differential Equations 19 (1975), 386–398.
[2] A. Bressan, G. Colombo, *Extensions and selections of maps decomposable values*, Stud. Math. 90 (1988), 70–85.
[3] E. Coddington, N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.
[4] S. Domachowski, J. Gulgowski, *A global bifurcation theorem for convex-valued differential inclusions*, Zeitschrift für Analysis und ihre Anwendungen 23 (2004), 275–292.
[5] L. H. Erbe, W. Krawcewicz, *Nonlinear boundary value problems for differential inclusions* $y'' \in F(t, y, y')$, Ann. Polon. Math. 54 (1991), 195–226.
[6] M. Frigon, A. Granas, *Théorèmes d'existence pour des inclusions différentiielles sans convexité*, C.R. Acad. Sci. Paris, Serie I Math. 310 (1990), 819–822.
[7] A. Fryszkowski, *Continuous selections for a class of non-convex multi-valued maps*, Stud. Math. 76 (1983), 163–174.
[8] A. Granas, R. B. Guenther, J. W. Lee, *On a theorem of S. Bernstein*, Pacific Math. 73 (2) (1977), 67–82.
[9] A. Granas, R. B. Guenther, J. W. Lee, *Nonlinear boundary value problems for some classes of ordinary differential equations, Dissertationes Math.* 244 (1985).
[10] P. Hartman, *Ordinary Differential Equations*, Birkhauser, Boston-Basel-Stuttgart, 1982.
[11] S. Łojasiewicz (Jr), *The existence of solutions for lower semicontinuous orientor fields*, Bull. Acad. Polon. Sci. 9–10 (1980), 483–487.
[12] S. A. Mararano, *Existence theorems for a multi-valued boundary value problem*, Bull. Austral. Math. Soc. 45 (1992), 249–260.
[13] A. Pliś, *Measurable orientor fields*, Bull. Acad. Polon. Sci. 8 (1965), 857–859.
[14] M. H. Protter, H. F. Weinberger, *Maximum Principles in Differential Equations*, Springer-Verlag, New York, 1984.
[15] T. Pruszko, *Topological degree methods in multi-valued boundary value problems*, J. Nonlinear Anal. 9 (1981), 959–973.
[16] T. Pruszko, *Some applications of the topological degree theory to multi-valued boundary value problems*, Dissertationes Math. 229 (1984), 1–52.
[17] P. Rabinowitz, *Some global results for nonlinear eigenvalue problems*, J. of Funct. Anal. 7 (1971), 487–513.
[18] O. N. Ricceri, B. Ricceri, *An existence theorem for inclusions of the type* $\Psi(u)(t) \in F(t, \Phi(u)(t))$ *and application to a multi-valued boundary value problem*, Appl. Anal. 38 (1990), 259–270.

INSTITUTE OF MATHEMATICS
UNIVERSITY OF GDAŃSK
ul. Wita Stwosza 57
80-952 GDAŃSK, POALND
E-mail: mdom@math.univ.gda.pl

Received January 1, 2008; revised version September 8, 2008.