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# EXISTENCE OF POSITIVE SOLUTIONS TO BVPS OF HIGHER ORDER DELAY DIFFERENTIAL EQUATIONS

**Abstract.** The paper is concerned with the existence of positive solutions for the nonlinear eigenvalue problem with singularity and the superlinear semipositone problem of higher order delay differential equations. The main results are obtained by using Guo-Krasnoselskii's fixed point theorem in cones. These results extend some of the existing literature.

## 1. Introduction

Boundary-value problems (BVPs) for higher order delay differential equations arise in a variety of areas of applied mathematics, physics and variational problems of control theory. The theory of BVPs of higher order delay differential equations provides a general framework for mathematical modelling of many real world phenomena. In recent years, remarked progress has been made in the theory of BVPs of second-order delay differential equations by the development of the theory of functional differential equations, see, for example [2–8, 19] and the references therein. However, there is only a small amount of work dedicated to the theory of BVPs for higher order delay differential equations.

In this paper, we considered the existence of positive solutions for the following boundary-value problem of the higher order delay differential equation(BVPs)

$$u^{(n)}(t) + \lambda g(t, u(t-\tau)) = 0, \qquad 0 < t < 1, \quad \tau > 0, \tag{1.1}$$

with the boundary conditions

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$$(1.2)\qquad \begin{cases} u(t) = u'(t) = \cdots = u^{(n-3)}(t) = u^{(n-2)}(t) = 0, & -\tau \le t \le 0, \\ u^{(n-2)}(1) = 0, \end{cases}$$

where $\lambda$ is a positive real parameter.

Throughout the paper we assume that $n \ge 3$ is an integer.

For the case $\tau = 0$, the problem (1.1) and (1.2) is related to multi-point BVPs of ordinary differential equations and was studied by Graef and Yang in [18]. Particularly, in the case $\tau = 0$ and $n = 2$, the existence of positive solutions for BVP (1.1) and (1.2) with singularity has been widely studied by many authors, such as Ha and Lee [9] by using the method of upper and lower solutions, and Fink et al [10] by using the shooting method.

Here, we should also mention the recent work by Bai and Xu [19]. In [19], the authors considered the case $n = 2$ for BVP (1.1) and (1.2) and obtained the existence of positive solutions to BVP (1.1) and (1.2).

In present paper, we consider the more general BVP for the higher order($n \ge 3$) differential equations (1.1) and (1.2).

Define $G_2 : [0,1] \times [0,1] \longrightarrow [0,\infty)$ by

$$(1.3)\qquad G_2(t,s) = \begin{cases} (1-t)s, & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1. \end{cases}$$

Note that $G_2(t,s) > 0$ for $t, s \in (0,1)$. For $n \ge 3$, we define

$$(1.4)\qquad G_n(t,s) = \int_0^t G_{n-1}(v,s)dv.$$

Then $G_n(t,s)$ is the Green's function for the problem (1.1) and (1.2). Moreover, solving the BVP (1.1)-(1.2) is equivalent to finding a solution to the integral equation

$$(1.5)\qquad u(t) = \begin{cases} 0, & -\tau \le t \le 0, \\ \lambda \int_0^1 G_n(t,s) g(s, u(s-\tau))ds, & 0 \le t \le 1. \end{cases}$$

In section 2 of the paper, we shall present some sufficient conditions with $\lambda$ belonging to an open interval of eigenvalues to ensure the existence of positive solutions to BVP (1.1) and (1.2) by the well-known Guo-Krasnoselskii fixed point theorem in cones [11]. We assume that

($A_1$) $0 < \tau < 1$;

($A_2$) $g(t,u) = a(t)f(t,u)$, $a : (0,1) \to [0,\infty)$ is continuous and $f : [0,1] \times [0,\infty)$ is continuous;

($A_3$) $\int_0^1 s(1-s)a(s)ds < \infty$, $\exists \theta \in [\frac{\tau}{2}, \frac{1}{2})$ such that $\int_\tau^{1-2\theta+\tau} a(s)ds > 0$.

Here $f$ is neither superlinear or sublinear. Especially, we allow that $a(t)$ has some suitable singularity at the ends of (0,1).

In section 3 of the paper, we also consider the existence of positive solutions with $g$ regular.

We need the following assumptions

($B_1$) $g : [0, 1] \times [0, \infty) \to R$ is continuous;

($B_2$) there exists a positive constant $M > 0$ such that $g(t, u) \geq -M$ for every $t \in [0, 1]$ and $u \geq 0$;

($B_3$) there exist $0 < \alpha < \beta < 1$ such that $\beta + \tau < 1$;

($B_4$) $\lim_{u\to\infty} \frac{g(t,u)}{u} = \infty$ uniformly for $t \in [\alpha, \beta] \subset (0, 1)$.

The condition ($B_1$) shows that $g(t, 0)$ need not be non-negative (semipositone).

The existence of positive solutions for semipositone problems has been extensively studied by Shivaji and co-authors. We refer readers to [12–17] and the references therein.

The main tool of this paper is the following Guo-Kranoselskii fixed point theorem in cones [11].

**Theorem K.** *Let $E$ be a Banach space and $K \subset E$ be a cone. Assume $\Omega_1, \Omega_2$ are open disks of $E$ with $\overline{\Omega}_1 \subset \Omega_2$, and let $T : K \bigcap(\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that*

(i) $\|Tu\| \leq \|u\|$ *if* $u \in K \bigcap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$ *if* $u \in K \bigcap \partial\Omega_2$; *or*
(ii) $\|Tu\| \geq \|u\|$ *if* $u \in K \bigcap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$ *if* $u \in K \bigcap \partial\Omega_2$.

*Then $T$ has a fixed point in $K \bigcap(\overline{\Omega}_2 \backslash \Omega_1)$.*

## 2. Eigenvalue problem with singularity

In this section, we will establish some existence results for the nonlinear eigenvalue problem (1.1) and (1.2) with singularity. Firstly, we give the following definition of positive solution of BVP (1.1) and (1.2).

**Definition 2.1.** $u(t)$ is called a solution of BVP (1.1) and (1.2) if it satisfies the following

(1) $u \in C[-\tau, 1] \bigcap C^n(0, 1)$;
(2) $u(t) > 0$ for all $t \in (0, 1)$ and satisfies conditions (1.2);
(3) $u^{(n)}(t) = -\lambda g(t, u(t - \tau))$ for $t \in (0, 1)$.

Let

$$E = \Big\{ u \in C[-\tau, 1] : \quad u(t) = u'(t) = \cdots = u^{(n-3)}(t) = u^{(n-2)}(t) = 0, \quad -\tau \le t \le 0, \quad u^{(n-2)}(1) = 0, \Big\} \tag{2.1}$$

with the norm $\|\cdot\|$ given by $\|u\| = \sup\{| u(t) | : -\tau \le t \le 1\}$, then $(E, \|\cdot\|)$ is a Banach space. It is obvious that $\|\cdot\|_{[0,1]} = \|\cdot\|$ for $u \in E : u \ge 0$. Here $\|\cdot\|_{[0,1]}$ stands for the sup-norm of $C[0,1]$. One can find that

$$G_n(t,s) \le G_2(s,s) = s(1-s), \qquad (t,s) \in [0,1] \times [0,1] \tag{2.2}$$

and

$$\frac{G_n(t,s)}{G_2(s,s)} = \frac{G_n(t,s)}{s(1-s)} \ge \theta^{n-1}, \qquad t \in [\theta, 1-\theta]. \tag{2.3}$$

Define a cone $K \subset E$ by

$$K = \{u \in E : u(t) \ge 0, \forall t \in [0,1], \min_{\theta \le t \le 1-\theta} u(t) \ge \theta^{n-1}\|u\|\}. \tag{2.4}$$

Let

$$\min f_\infty := \liminf_{u \to \infty} \min_{s \in [0,1]} \frac{f(s,u)}{u},$$
$$\max f_0 := \limsup_{u \to 0^+} \max_{s \in [0,1]} \frac{f(s,u)}{u},$$
$$\min f_0 := \liminf_{u \to 0^+} \min_{s \in [0,1]} \frac{f(s,u)}{u},$$
$$\max f_\infty := \limsup_{u \to \infty} \max_{s \in [0,1]} \frac{f(s,u)}{u}.$$

**THEOREM 2.1.** *Let* $(A_1) - (A_3)$ *hold and* $\min f_\infty > 0, \max f_0 < \infty$. *Then there exists at least one positive solution to BVP* (1.1) *and* (1.2) *for*

$$\lambda \in \left( \frac{1}{\theta^{n-1} \min f_\infty \cdot \sup\limits_{t \in [0,1]} \int_\tau^{1-2\theta+\tau} G_n(t,s)a(s)ds}, \frac{1}{\max f_0 \cdot \int_\tau^1 G_2(s,s)a(s)ds} \right). \tag{2.5}$$

**Proof.** Define the integral operator $T$ by

$$Tu(t) = \begin{cases} 0, & -\tau \le t \le 0, \\ \lambda \int_0^1 G_n(t,s)a(s)f(s,u(s-\tau))ds, & 0 \le t \le 1 \end{cases}$$

for each $u \in K$. It can be verified that for each $u \in K$, $Tu \in K$ by (2.3) and $T$ is a completely continous operator by the Arzela-Ascoli Theorem. Now we prove that $T$ has a fixed point in $K$ by using Theorem $K$. By (2.5), there exists a $\varepsilon > 0$ such that

$$\frac{1}{\theta^{n-1}(\min f_\infty - \varepsilon)\cdot \sup\limits_{t\in[0,1]} \int_\tau^{1-2\theta+\tau} G_n(t,s)a(s)ds} \le \lambda \le \frac{1}{(\max f_0 + \varepsilon)\cdot \int_\tau^1 G_2(s,s)a(s)ds}. \tag{2.6}$$

Let $\varepsilon$ be fixed. By $\max f_0 < \infty$, there exists an $H_1 > 0$ such that for $u : 0 \le u \le H_1$,

$$f(s,u) \le (\max f_0 + \varepsilon)u. \tag{2.7}$$

Let $\Omega_1 = \{u \in E : \|u\| < H_1\}$, then for $u \in K \bigcap \partial\Omega_1$, we have by (2.6) and (2.7)

$$\begin{aligned}
\|Tu\| &\le \lambda \int_0^1 G_2(s,s)a(s)f(s,u(s-\tau))ds \\
&\le \lambda \int_0^1 G_2(s,s)a(s)(\max f_0 + \varepsilon)u(s-\tau)ds \\
&= \lambda(\max f_0 + \varepsilon) \int_\tau^1 G_2(s,s)a(s)u(s-\tau)ds \\
&= \lambda(\max f_0 + \varepsilon) \int_0^{1-\tau} G_2(s+\tau, s+\tau)a(s+\tau)u(s)ds \\
&\le \lambda(\max f_0 + \varepsilon) \int_0^{1-\tau} G_2(s+\tau, s+\tau)a(s+\tau)ds\|u\| \\
&= \lambda(\max f_0 + \varepsilon) \int_\tau^1 G_2(s,s)a(s)ds\|u\| \le \|u\|.
\end{aligned}$$

Next, by $\min f_\infty > 0$, there exists an $\overline{H}_2 > 0$ such that $f(s,u) \ge (\min f_\infty - \varepsilon)u$ for $u > \overline{H}_2$. Take $H_2 = \max\{\overline{H}_2, 2H_1\}$ and set $\Omega_2 = \{u \in E : \|u\| < H_2\}$. Then for $u \in K \bigcap \partial\Omega_2$, we have by (2.4) and (2.6)

$$\begin{aligned}
\|Tu\| &= \lambda \sup_{t\in[0,1]} \int_0^1 G_n(t,s)a(s)f(s,u(s-\tau))ds \\
&\ge \lambda \sup_{t\in[0,1]} \int_\tau^1 G_n(t,s)a(s)(\min f_\infty - \varepsilon)u(s-\tau)ds
\end{aligned}$$

$$= \lambda(\min f_\infty - \varepsilon) \sup_{t\in[0,1]} \int_\tau^1 G_n(t,s)a(s)u(s-\tau)ds$$

$$= \lambda(\min f_\infty - \varepsilon) \sup_{t\in[0,1]} \int_0^{1-\tau} G_n(t,s+\tau)a(s+\tau)u(s)ds$$

$$\geq \lambda(\min f_\infty - \varepsilon) \sup_{t\in[0,1]} \int_0^{1-2\theta} G_n(t,s+\tau)a(s+\tau)u(s)ds$$

$$\geq \lambda\theta^{n-1}(\min f_\infty - \varepsilon) \sup_{t\in[0,1]} \int_0^{1-2\theta} G_n(t,s+\tau)a(s+\tau)ds\|u\|$$

$$= \lambda\theta^{n-1}(\min f_\infty - \varepsilon) \sup_{t\in[0,1]} \int_\tau^{1-2\theta+\tau} G_n(t,s)a(s)ds\|u\| \geq \|u\|.$$

Therefore, by the first part of Theorem $K$, $T$ has a fixed point $u \in K\bigcap(\overline{\Omega}_2\backslash\Omega_1)$, and $u(t)$ is a positive solution of BVP (1.1) and (1.2). The proof is complete.

**THEOREM 2.2.** *Let* $(A_1) - (A_3)$ *hold and* $\min f_0 > 0, \max f_\infty < \infty$. *Then there exists at least one positive solution to BVP* (1.1) *and* (1.2) *for*

$$\lambda \in \left(\frac{1}{\theta^{n-1}\min f_0 \cdot \sup_{t\in[0,1]} \int_\tau^{1-2\theta+\tau} G_n(t,s)a(s)ds}, \frac{1}{\max f_\infty \cdot \int_\tau^1 G_2(s,s)a(s)ds}\right). \tag{2.8}$$

**Proof.** Suppose that $\lambda$ satisfies (2.8). Let $\varepsilon > 0$ be such that

$$\frac{1}{\theta^{n-1}(\min f_0 - \varepsilon)\cdot \sup_{t\in[0,1]} \int_\tau^{1-2\theta+\tau} G_n(t,s)a(s)ds} \leq \lambda \leq \frac{1}{(\max f_\infty + \varepsilon)\cdot \int_\tau^1 G_2(s,s)a(s)ds}. \tag{2.9}$$

By $\min f_0 > 0$, there exists a $H_1 > 0$ such that for $0 \leq u \leq H_1$,

$$f(s,u) \geq (\min f_0 - \varepsilon)u. \tag{2.10}$$

Let $\Omega_1 = \{u \in E : \|u\| < H_1\}$, then for $u \in K\bigcap\partial\Omega_1$, we have by (2.4), (2.9) and (2.10)

$$\|Tu\| = \lambda \sup_{t\in[0,1]} \int_0^1 G_n(t,s)a(s)f(s,u(s-\tau))ds$$

$$\geq \lambda \sup_{t\in[0,1]} \int_0^1 G_n(t,s)a(s)(\min f_0 - \varepsilon)u(s-\tau)ds$$

$$
\begin{aligned}
&= \lambda(\min f_0 - \varepsilon) \sup_{t\in[0,1]} \int_{\tau}^{1} G_n(t,s)a(s)u(s-\tau)ds \\
&= \lambda(\min f_0 - \varepsilon) \sup_{t\in[0,1]} \int_{0}^{1-\tau} G_n(t,s+\tau)a(s+\tau)u(s)ds \\
&\geq \lambda(\min f_0 - \varepsilon) \sup_{t\in[0,1]} \int_{0}^{1-2\theta} G_n(t,s+\tau)a(s+\tau)u(s)ds \\
&\geq \lambda\theta^{n-1}(\min f_0 - \varepsilon) \sup_{t\in[0,1]} \int_{0}^{1-2\theta} G_n(t,s+\tau)a(s+\tau)ds\|u\| \\
&= \lambda\theta^{n-1}(\min f_0 - \varepsilon) \sup_{t\in[0,1]} \int_{\tau}^{1-2\theta+\tau} G_n(t,s)a(s)ds\|u\| \geq \|u\|.
\end{aligned}
$$

Again by $\max f_\infty < \infty$, there exists a $\overline{H}_2 > 0$ such that for $u > \overline{H}_2$,

$$
f(s,u) \leq (\max f_\infty + \varepsilon)u. \tag{2.11}
$$

There are two cases: (i) $f$ is bounded, and (ii) $f$ is unbounded.

For case (i), we can choose $N > 0$ such that $f(s,u) \leq N$ for $s \in [0,1]$ and $0 \leq u < \infty$. Let $H_2 = \max\{2H_1, \lambda N \int_0^1 G_2(s,s)a(s)ds\}$ and $\Omega_2 = \{u \in E : \|u\| < H_2\}$. Then for $u \in K \bigcap \partial\Omega_2$, we have

$$
\begin{aligned}
\|Tu\| &\leq \lambda \int_0^1 G_2(s,s)a(s)f(s,u(s-\tau))ds \\
&\leq \lambda N \int_0^1 G_2(s,s)a(s)ds \leq H_2 = \|u\|.
\end{aligned}
$$

For case (ii), we can choose $H_2 > \max\{2H_1, \overline{H}_2\}$ such that $f(s,u) \leq f(s,H_2)$ for $s \in [0,1]$ and $0 < u < H_2$. Let $\Omega_2 = \{u \in E : \|u\| < H_2\}$, then for $u \in K \bigcap \partial\Omega_2$, by (2.11), we have

$$
\begin{aligned}
\|Tu\| &\leq \lambda \int_0^1 G_2(s,s)a(s)f(s,u(s-\tau))ds \\
&\leq \lambda \int_0^1 G_2(s,s)a(s)f(s,H_2)ds \\
&\leq \lambda(\max f_\infty + \varepsilon) \int_0^1 G_2(s,s)a(s)dsH_2 \leq H_2 = \|u\|.
\end{aligned}
$$

Thus by the second part of the theorem $K$, we deduce that $T$ has a fixed point $u \in K \cap (\overline{\Omega}_2 \backslash \Omega_1)$ and it is a positive solution of BVP (1.1) and (1.2), completing the proof of Theorem 2.2.

## 3. Semipositone problem

In this section we consider the existence of positive solutions for BVP (1.1) and (1.2) with $g$ superlinear and semipositone. We still denote that

$$(3.1)\qquad E=\left\{u\in C[-\tau,1]:\ \begin{array}{ll} u(t)=u'(t)=\ldots=u^{(n-3)}(t)=u^{(n-2)}(t)=0, & -\tau\le t\le 0,\\ u^{(n-2)}(1)=0, & \end{array}\right\}$$

with the norm $\|\cdot\|$ given by $\|u\|=\sup\{|u(t)|:-\tau\le t\le 1\}$.

**THEOREM 3.1.** *Let* $(B_1)-(B_4)$ *hold. Then there exists at least one positive solution of BVP* (1.1) *and* (1.2) *for* $\lambda>0$ *sufficiently small.*

In order to prove Theorem 3.1, we need the following lemmas that provide us with some useful information.

**LEMMA 3.1.** *Let* $u$ *satisfy*

$$\begin{aligned} &-u^{(n)}(t)=h(t),\quad 0<t<1,\tau>0,\\ &u(t)=u'(t)=\ldots=u^{(n-3)}(t)=u^{(n-2)}(t)=0,\quad -\tau\le t\le 0,\\ &u^{(n-2)}(1)=0, \end{aligned}$$

*where* $h\in C[-\tau,1]$, $h\ge 0$. *Then*

$$u(t)\ge\|u\|_{[0,1]}\cdot q(t),t\in[0,1],$$

*here* $q(t)=\frac{t^{n-2}}{(n-2)!}-\frac{t^{n-1}}{(n-1)!}$, $\quad t\in[0,1]$.

**Proof.** It's obvious that $G_2(t,s)\ge\min\{t,1-\frac{1}{t}\}G_2(s,s)$, then

$$\begin{aligned} G_3(t,s)&=\int_0^t G_2(v,s)dv\ge(t-\frac{t^2}{2})G_2(s,s),\\ G_4(t,s)&=\int_0^t G_3(v,s)dv\ge G_2(s,s)\int_0^t\left(v-\frac{v^2}{2}\right)dv\ge\left(\frac{t^2}{2}-\frac{t^3}{6}\right)G_2(s,s),\\ \cdots&=\cdots\\ G_n(t,s)&=\int_0^t G_{n-1}(v,s)dv\ge\left[\frac{t^{n-2}}{(n-2)!}-\frac{t^{n-1}}{(n-1)!}\right]G_2(s,s), \end{aligned}$$

so we have

$$u(t)=\int_0^1 G_n(t,s)h(s)ds\ge q(t)\int_0^1 G_2(s,s)h(s)ds\ge\|u\|_{[0,1]}\cdot q(t).$$

**Lemma 3.2.** *Let $\overline{\omega}$ be the solution of*

$$\begin{aligned}&-u^{(n)}(t)=1, 0<t<1, \tau>0,\\&u(t)=u'(t)=\ldots=u^{(n-3)}(t)=u^{(n-2)}(t)=0, \quad -\tau\le t\le 0,\\&u^{(n-2)}(1)=0.\end{aligned}$$

*Then* $\overline{\omega}(t)\le q(t), t\in[0,1]$.

**Proof.** In fact, for $t\in[0,1]$, $\overline{\omega}(t)=\int_0^1 G_n(t,s)ds$. For $n=2$, we have

$$\overline{\omega}(t)=\int_0^1 G_2(t,s)ds=\frac{1}{2}t(1-t)<q(t).$$

If for $n=k$, we have

$$\overline{\omega}(t)=\int_0^1 G_k(t,s)ds\le q(t)=\frac{t^{k-2}}{(k-2)!}-\frac{t^{k-1}}{(k-1)!},$$

then when $n=k+1$, we can get

$$\begin{aligned}\overline{\omega}(t)&=\int_0^1 G_{k+1}(t,s)ds=\int_0^1\left(\int_0^t G_k(v,s)dv\right)ds\\&=\int_0^t\left(\int_0^1 G_k(v,s)ds\right)dv\le\int_0^t\left[\frac{v^{k-2}}{(k-2)!}-\frac{v^{k-1}}{(k-1)!}\right]dv\\&=\frac{t^{k-1}}{(k-1)!}-\frac{t^k}{k!}.\end{aligned}$$

So for $t\in[0,1]$, $\overline{\omega}(t)\le q(t)$ and $\|\omega\|_{[0,1]}\le 1$.

Let $g_1(t,u)=g(t,u)+M, \omega(t)=\lambda M\overline{\omega}(t)$.

**Lemma 3.3.** *$u(t)$ is a positive solution to BVP (1.1) and (1.2) if and only if $\widetilde{u}=u+\omega$ is a solution of*

$$u^{(n)}(t)+\lambda\widetilde{g}(t,u(t-\tau)-\omega(t-\tau))=0, \quad 0<t<1, \quad \tau>0, \tag{3.2}$$

$$\begin{aligned}&u(t)=u'(t)=\ldots=u^{(n-3)}(t)=u^{(n-2)}(t)=0, \quad -\tau\le t\le 0,\\&u^{(n-2)}(1)=0,\end{aligned} \tag{3.3}$$

*with $\widetilde{u}(t)>\omega(t)$, $t\in(0,1)$. Here*

$$\widetilde{g}(t,u)=\begin{cases}g_1(t,u), & u\ge 0,\\ g_1(t,0), & u\le 0.\end{cases} \tag{3.4}$$

**Proof of Theorem 3.1.** Let $\lambda$ satisfy

$$0<\lambda<\min\left\{\frac{1}{C},\frac{1}{M}\right\}, \tag{3.5}$$

where $C = \sup\{g_1(s,u) : 0 \le s \le 1, 0 \le u \le 1\}$. BVP (3.2)-(3.3) is equivalent to

$$u(t) = \begin{cases} 0, & -\tau \le t \le 0, \\ \lambda \int_0^1 G_n(t,s)\widetilde{g}(s, u(s-\tau) - \omega(s-\tau))ds, & 0 \le t \le 1, \end{cases}$$
$$=: Tu(t).$$

Define a cone $K \subset E$ by

$$K = \{u \in E : u(t) \ge \|u\|_{[0,1]} \cdot q(t), t \in [0,1]\}. \tag{3.6}$$

It can be verified that $TK \subset K$. In fact, for $u \in K$, $\|Tu\| = \|Tu\|_{[0,1]}$, $\|Tu\|_{[0,1]} \le \lambda \int_0^1 G_2(s,s)\widetilde{g}(s, u(s-\tau) - \omega(s-\tau))ds$, which implies that

$$\begin{aligned} Tu(t) &\ge \lambda \left[ \frac{t^{n-2}}{(n-2)!} - \frac{t^{n-1}}{(n-1)!} \right] \int_0^1 G_2(s,s)\widetilde{g}(s, u(s-\tau) - \omega(s-\tau))ds \\ &\ge q(t)\|Tu\|_{[0,1]}, \qquad \forall t \in [0,1]. \end{aligned}$$

On the other hand, one can find that $T$ is a completely continuous operator by the Arzela-Ascoli Theorem. We shall prove that $T$ has a fixed point in $K$ by using Theorem $K$.
Let $\Omega_1 = \{u \in E : \|u\| < 1\}$. For $u \in K \bigcap \partial\Omega_1$,

$$\begin{aligned} Tu(t) &= \lambda \int_0^1 G_n(t,s)\widetilde{g}(s, u(s-\tau) - \omega(s-\tau))ds \\ &\le \lambda C \int_0^1 G_n(t,s)ds \\ &= \lambda C \overline{\omega}(t) \le 1, \qquad t \in [0,1]. \end{aligned}$$

Since $0 \le u - \omega \le u \le 1$. Thus $\|Tu\| \le \|u\|$ for $u \in K \cap \partial\Omega_1$.

Now we choose a constant $\widetilde{M} > 0$ such that

$$\frac{1}{2}\lambda\delta\widetilde{M} \sup_{0 \le t \le 1} \int_{\alpha+\tau}^{\beta+\tau} G_n(t,s)ds > 1, \tag{3.7}$$

where $\delta = \min\{q(t) : \alpha \le t \le \beta\}$. By $(B_4)$, there is a constant $L > 0$, such that

$$\widetilde{g}(t,u) \ge \widetilde{M}u, \qquad \forall u \ge L, t \in [\alpha, \beta]. \tag{3.8}$$

Set

$$R = 1 + \max\left\{2\lambda M, \frac{2L}{\delta}\right\} \tag{3.9}$$

and define $\Omega_2 = \{u \in E : \|u\| < R\}$. For $u \in K \bigcap \partial\Omega_2$, we have from (3.6) and Lemma 3.2 that

$$\omega(t) = \lambda M\overline{\omega}(t) \leq \lambda M q(t) \leq \frac{\lambda M}{R}u(t), \qquad t \in [0,1],$$

which implies that

$$(3.10) \qquad u(t) - \omega(t) \geq \left(1 - \frac{\lambda M}{R}\right)u(t) \geq \frac{1}{2}u(t) \geq \frac{1}{2}q(t)\|u\| \geq \frac{1}{2}Rq(t) \geq \frac{1}{2}R\delta, \qquad t \in [\alpha,\beta].$$

Therefore by (3.8)-(3.10), for $u \in K\bigcap\partial\Omega_2$, we have

$$(3.11) \qquad \widetilde{g}(t, u(t-\tau) - \omega(t-\tau)) \geq \widetilde{M}(u(t-\tau) - \omega(t-\tau)) \geq \frac{1}{2}\widetilde{M}R\delta, \quad t \in [\alpha+\tau, \beta+\tau].$$

Combining (3.7) and (3.11) ,we get

$$\begin{aligned}\|Tu\| &= \lambda \sup_{0\leq t\leq 1}\int_0^1 G_n(t,s)\widetilde{g}(s, u(s-\tau) - \omega(s-\tau))ds \\ &\geq \lambda \sup_{0\leq t\leq 1}\int_{\alpha+\tau}^{\beta+\tau} G_n(t,s)\widetilde{g}(s, u(s-\tau) - \omega(s-\tau))ds \\ &\geq \frac{1}{2}\lambda\delta\widetilde{M}R \sup_{0\leq t\leq 1}\int_{\alpha+\tau}^{\beta+\tau} G_n(t,s)ds \geq R = \|u\|\end{aligned}$$

for each $u \in K\bigcap\partial\Omega_2$.

Thus by the first part of Theorem $K$, $T$ has a fixed point $\widetilde{u}$ with $1 \leq \|\widetilde{u}\| \leq R$. It follows that $\widetilde{u}(t) \geq q(t) \geq \lambda Mq(t) \geq \omega(t)$, $t \in (0,1)$, and so $u = \widetilde{u} - \omega$ is a positive solution of BVP (1.1) and (1.2), completing the proof of Theorem 3.1.

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