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ON $\mathcal{E}$-CONTINUITY AND $\mathcal{E}$-MINIMALITY OF MULTIFUNCTIONS OF TWO VARIABLES

Abstract. In this article we formulate and prove some sufficient conditions for the $l - \mathcal{E}_{X\times Y}$-continuity and the $\mathcal{E}_{X\times Y}$- minimality of multifunctions of two variables.

1. Introduction

Let (X, T_X), (Y, T_Y) and (Z, T_Z) be topological spaces. In many papers the words mapping and function have the same meaning and both mean a single-valued mappings which are denote by f, g, h etc. The multifunctions have the same meaning as multi-valued mappings and are denote by F, G, H etc. Instead of $F : X \to 2^Y \setminus \{\emptyset\}$ or $F : X\times Y \to 2^Z \setminus \{\emptyset\}$ we write respectively $F : X \mapsto Y$ or $F : (X \times Y) \mapsto Z$.

The continuity of multifunctions and their applications are well described in [3]. Very general notion of quasicontinuities of functions and multifunctions is studed in literature. Motivations for these investigations come from linear programming, stochastic processes, probability, statistics, differential inclusions, differentiation in Banach spaces and dynamical systems (compare [5]).

In [5] M. Matejdes introduces the notion of cluster systems $\mathcal{E}$ and investigates the notions of lower and upper $\mathcal{E}$-continuities and $\mathcal{E}$-minimalty of multifunctions of one variable.

Definition. ([5]) By a multifunction $F : X \mapsto Y$ we understand a subset of cartesian product $X \times Y$ with the non-empty values $\{y \in Y : (x, y) \in F\}$ $=: F(x)$ for $x \in X$.

The set $cl(F)$ is the multifunction defined as $cl(F) = \{y \in Y : (x, y) \in cl(Gr(F))\}$, where $Gr(F) = \{(x, y) \in X \times Y : y \in F(x)\}$ denotes the graph

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of F. Observe that a function $f : X \to Y$ as a single-valued mapping is the multifunction with values $\{f(x)\}$, $x \in X$.

Any nonempty system $\mathcal{E} \subset 2^X \setminus \{\emptyset\}$ is called a cluster system. In [5] for some spacial cluster systems are use special denotations, for example:

$\mathcal{O} = \{O :\ O \text{ is an open non} - \text{empty}\}$,

$\mathcal{B} = \{B :\ B \text{ is of second category with the Baire property}\}$,

$\mathcal{A} = \{A :\ A \text{ is not nowhere dense}\}$.

Definition. A multifunction $F : X \mapsto Z$ is said $l - \mathcal{E}$-continuous (resp. $u - \mathcal{E}$-continuous) at a point $x \in X$ if for all sets $U \in T_X$ and $V \in T_Z$ with $x \in U$ and $V \cap F(x) \neq \emptyset$ (resp. $F(x) \subset V$) there is a set $E \in \mathcal{E}$ contained in U and such that for each point $e \in E$ the intersection $F(e) \cap V \neq \emptyset$ (resp. $F(e) \subset V$). If F is $l - \mathcal{E}$-continuous and $u - \mathcal{E}$-continuous at every point then F is $\mathcal{E}$-continuous.

Definition. A multifunction $F : X \mapsto Z$ is $\mathcal{E}$-minimal at x if for all sets $U \in T_X$ and $V \in T_Z$ with $x \in U$ and $V \cap F(x) \neq \emptyset$ there is a set $E \in \mathcal{E}$ contained in U and such that for every point $e \in E$ the value $F(e) \subset V$. If F is $\mathcal{E}$-minimal at every point then F is $\mathcal{E}$-minimal.

Observe that for single-valued multifunctions the notions of $l - \mathcal{E}$-continuity, of $u - \mathcal{E}$-continuity and of $\mathcal{E}$-minimality are equivalent and that each $\mathcal{E}$-minimal at x multifunction $F : X \mapsto Z$ is also $l - \mathcal{E}$-continuous and $u - \mathcal{E}$-continuous at x.

Examples. Using this concept, the $l-\mathcal{B}$-continuity and $u-\mathcal{B}$-continuity is called respectively lower and upper Baire continuity, and the $l-\mathcal{O}$-continuity and $u - \mathcal{O}$-continuity denote respectively lower quasicontinuity and upper quasicontinuity of multifunctions ([5, 4, 6]).

Let $X = Z = \mathbb{R}$ and let $T_X = T_Z = T_e$, where T_e denotes the natural topology in $\mathbb{R}$. Then

(1) unilaterally continuous functions (also discontinuous) $f : \mathbb{R} \to \mathbb{R}$ are $\mathcal{O}$-minimal,
(2) the function of Dirichlet is $l - \mathcal{A}$ and $u - \mathcal{A}$-continuous at each point, $l - \mathcal{B}$ and $u - \mathcal{B}$-continuous at each irrational point and is not either $l - \mathcal{O}$ or $u - \mathcal{O}$-continuous at any point;
(3) the multifunction $F(x) = \{0\}$ for irrational x and $F(x) = [0,1]$ for rational x is $u - \mathcal{O}$-continuous and $l - \mathcal{O}$-discontinuous at each rational x and $l - \mathcal{O}$-continuous and $u - \mathcal{O}$-discontinuous at each irrational x.

2. Multifunctions of two variables

Now let $\mathcal{E}_X \subset 2^X \setminus \{\emptyset\}$ and $\mathcal{E}_Y \subset 2^Y \setminus \{\emptyset\}$ be cluster systems in X and respectively in Y and let

$$\mathcal{E}_{X\times Y} = \{A \times B : A \in \mathcal{E}_X,\ B \in \mathcal{E}_Y\}.$$

Similarly as in Part **I** we define the $l - \mathcal{E}_{X\times Y}$-continuity and the $u - \mathcal{E}_{X\times Y}$-continuity.

By Kempisty theorem ([4, 6]) each function $f : \mathbb{R}^2 \to \mathbb{R}$ separately quasicontinuous (i.e. with quasocintinuous horizontal and vertical sections) is quasicontinuous as the function of two variable. This theorem is true for $X = Y = Z = \mathbb{R}$ and $T_X = T_Y = T_Z = T_e$ and for very abstract topological spaces satisfying some additional conditions. Moreover it is not true for arbitrary topological spaces.

Example 4. Let T_d denotes the density topology in $\mathbb{R}$ ([1, 8]). In [2], Example 4) it is shown a construction of function $f : \mathbb{R}^2 \to \mathbb{R}$ such that the vertical sections $f_x(t) = f(x,t)$, $t, x \in \mathbb{R}$, and the horizontal sections $f^y(t) = f(t,y)$, $t, y \in \mathbb{R}$, are T_d-continuous and f is not $(T_d \times T_d)$-quasicontinuous.

Example 5. Let $X = Y = Q$, where Q denotes the set of all rationals considered with the natural metric $\rho(x,y) = |x-y|$ and let $Z = \mathbb{R}$ (also with the natural metric). Enumerate all points of the cartesien product $Q \times Q$ in a sequence (A_k) where $A_k = (x_k, y_k)$ such that $A_k \neq A_m$ for $k \neq m$. There are two disjoint sets $A, B \subset Q \times Q$ dense in $Q \times Q$ and such that the vertical and horizontal sections of the sets A and B are empty or contain only one element.

For the point $A_1 = (x_1, y_1)$ we define the straight lines p_1, q_1 defined by the formules $y = y_1$ and $x = x_1$ and find a continuous functions $g_1 : p_1 \to [0,1]$ and $h_1 : q_1 \to [0,1]$ such that $g_1(A \cap p_1) = \{1\}$, $g_1(B \cap p_1) = \{0\}$, $h_1(A \cap q_1) = \{1\}$, $h_1(B \cap q_1) = \{0\}$ and $h_1(p_1 \cap q_1) = g_1(p_1 \cap q_1)$.

Proceeding similarly for $n > 1$ we consider the point $A_n = (x_n, y_n)$, the straight lines p_n, q_n defined by formules $y = y_n$ and $x = x_n$ and find continuous functions $g_n : p_n \to [0,1]$ and $h_n : q_n \to [0,1]$ such that:

- if $p_i = p_n$ for some $i < n$ then $g_n = g_i$;
- if $(x,y) \in p_n \cap q_i$ for some $i < n$ then $g_n(x,y) = h_i(x,y)$;
- $g_n(A \cap p_n) = \{1\}$ and $g_n(B \cap p_n) = \{0\}$;
- if $q_i = q_n$ for some $i < n$ then $h_n = h_i$;
- if $(x,y) \in p_i \cap q_n$ for some $i \leq n$ then $h_n(x,y) = g_i(x,y)$; and
- $h_n(A \cap q_n) = \{1\}$ and $h_n(B \cap q_n) = \{0\}$.

Now we put $f(x_n, y_n) = g_n(x,y) = h_n(x,y)$ for $n \geq 1$. Observe that the vertical and horizontal sections of the function $f : (Q \times Q) \to [0,1]$ are continuous, but is not quasicontinuous at any point $(x, y) \in Q \times Q$, because the preimages $f^{-1}(0)$ and $f^{-1}(1)$ are dense in $Q \times Q$.

3. The main results

In this part we formulate and prove sufficient conditions for $l - \mathcal{E}_{X\times Y}$-continuity and $\mathcal{E}_{X\times Y}$-minimality of multifunctions $F : (X \times Y) \mapsto Z$.

Theorem 1. *Assume that (X, T_X) is a Baire space such that for each point $x \in X$ there is a set $U(x) \in T_X$ containing x having a countable basis of open sets and that for each point $y \in Y$ there are a set $V(y) \in T_Y$ containing y and a countable subfamily $\mathcal{A}(y) \subset \mathcal{E}_Y$ such that for each set $E \subset V(y)$ belonging to $\mathcal{E}_Y$ there is a set $G \in \mathcal{A}(y)$ contained in E. Moreover, assume that each set belonging to $\mathcal{E}_X$ has the Baire property and is of the second category and that for every nonempty set $P \in T_Z$ there is a sequence of sets $P_n \in T_Z$ such that the closures $cl(P_n) \subset P$ and $P = \bigcup_{n=1}^{\infty} P_n$. Let $F : (X \times Y) \mapsto Z$ be a multifunction such that the sections $F_x(v) = F(x, v)$, $x \in X$, are $l - \mathcal{E}_Y$-continuous and the sections $F^y(u) = F(u, y)$, $y \in Y$, are $l - \mathcal{E}_X$-continuous. Then F is $l - \mathcal{E}_{X\times Y}$-continuous.*

Proof. Fix a point $(a, b) \in X \times Y$ and sets $U \in T_X$, $V \in T_Y$, $W \in T_Z$ such that $(a, b) \in U \times V$ and $F(a, b) \cap W \neq \emptyset$. Without loss of the generality we can assume that in U there is a countable basis $\mathcal{B}(U)$ of nonempty open sets belonging to T_X. There are also a set $V(b) \in T_Y$ containing b and a countable family $\mathcal{A}(b) \subset \mathcal{E}_Y$ such that for each set $E \subset V(b)$ belonging to $\mathcal{E}_Y$ there is a set $G \in \mathcal{A}(b)$ contained in E. Let $V_1 = V \cap V(b)$. Moreover, let $W_n \in T_Z$ be any family of nonempty sets such that $cl(W_n) \subset W$ and $W = \bigcup_{n=1}^{\infty} W_n$. Since the section F^b is $l - \mathcal{E}_X$- continuous, there is a set $U_1 \in \mathcal{E}_X$ contained in U such that $F(u, b) \cap W \neq \emptyset$ for each $u \in U_1$. From the $l - \mathcal{E}_Y$- continuity of the sections F_x at b it follows that for each point $u \in U_1$ there are a positive integer $n(u)$ and a set $A(u) \in \mathcal{A}(b)$ such that $F(u, v) \cap W_{n(u)} \neq \emptyset$ for all $v \in A(u)$. Since the set U_1 is of the second category and the family $\mathcal{A}(b) \times N$ (N denotes the set of all positive integers) is countable, there are a set $E_2 \in \mathcal{A}(b)$ and a positive integer k such that the set $B = \{u \in U_1; A(u) = E_2, \quad n(u) = k\}$ is of the second category. There is a nonempty set $U_2 \subset U_1$ belonging to T_X such that every subset $H \subset U \setminus B$ with the Baire property is of the first category. Let $E_1 \subset U_2$ be a set belonging to $\mathcal{E}_X$. Obviously, $E_1 \times E_2 \in \mathcal{E}_{X\times Y}$ and $E_1 \times E_2 \subset U \times V$.

We will prove that $F(u, v) \cap cl(W_k) \neq \emptyset$ for all $(u, v) \in E_1 \times E_2$. For this assume to a contradiction that there is a point $(c, d) \in E_1 \times E_2$ with $F(c, d) \cap cl(W_k) = \emptyset$. From the $l - \mathcal{E}_X$-continuity of the section F^d at c it follows that there is a set $E_3 \in \mathcal{E}_X$ contained in U_2 such that $F(u, d) \cap (Z \setminus cl(W_k)) \neq \emptyset$ for every point $u \in E_3$. Since $E_3 \subset U_2 \subset U$ has the Baire property and is of the second category, there is a point $z \in B \cap E_3$. Consequently we obtain a contradiction $F(z,d) \cap W_k \neq \emptyset$ and $F(z,k) \subset Z \setminus cl(W_k)$. So $F(u,v) \cap W \supset F(u,v) \cap cl(W_k) \neq \emptyset$ for all $(u,v) \in E_1 \times E_2$ and the proof is completed. ∎

Theorem 1 is not true for the $u - \mathcal{E}_{X\times Y}$-continuity.

Example 6. Let $X = Y = \mathbb{R}$, $Z = \mathbb{R}^2$, $T_X = T_Y = T_e$, $T_Z = T_{X\times Y} = T_e \times T_e$, $\mathcal{E}_X = \mathcal{E}_Y = T_e \setminus \{\emptyset\}$ and $\mathcal{E}_{X\times Y} = T_e \times T_e \setminus \{\emptyset\}$. Suppose that $A \subset \mathbb{R}^2$ is a dense set such that the vertical and horizontal sections of A are empty or are singletons (see for example [7]). Then the multifunction

$$F(x,y) = \{0\} \times [0,1] \text{ for } (x,y) \in A \text{ and}$$
$$F(x,y) = \{(0,0)\} \text{ for } (x,y) \in \mathbb{R}^2 \setminus A,$$

is not $u - \mathcal{E}_{X\times Y}$-continuous, but the vertical and horizontal sections F_x and F^y, $x, y \in \mathbb{R}$, are $u - \mathcal{E}_{\mathbb{R}}$-continuous.

However, the respective version of Theorem 1 for the $\mathcal{E}_{X\times Y}$-minimality is true.

Theorem 2. *Assume that (X, T_X) is a Baire space such that for each point $x \in X$ there is a set $U(x) \in T_X$ containing x having a countable basis of open sets and that for each point $y \in Y$ there are a set $V(y) \in T_Y$ containing y and a countable subfamily $\mathcal{A}(y) \subset \mathcal{E}_Y$ such that for each set $E \subset V(y)$ belonging to $\mathcal{E}_Y$ there is a set $G \in \mathcal{A}(y)$ contained in E. Moreover, assume that each set belonging to $\mathcal{E}_X$ has the Baire property and is of the second category and that for every nonempty set $P \in T_Z$ there is a sequence of sets $P_n \in T_Z$ such that the closures $cl(P_n) \subset P$ and $P = \bigcup_{n=1}^{\infty} P_n$. Let $F : (X \times Y) \mapsto Z$ be a multifunction such that the sections $F_x(v) = F(x,v)$, $x \in X$, are $l - \mathcal{E}_Y$-minimal and the sections $F^y(u) = F(u,y)$, $y \in Y$, are $l - \mathcal{E}_X$-minimal. Then F is $l - \mathcal{E}_{X\times Y}$-minimal.*

Proof. The proof is completely similar to that of Theorem 1. ∎

Conclusion

It is evident that in Theorems 1 and 2 the hypotheses that (X, T_X) is a Baire space is important, because arbitrary nonempty open set in X must be of the second category. Example 5 confirms it. Example 4 confirms the importance of other hypotheses.

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