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MAXIMAL CLASSES FOR THE FAMILY OF QUASI-CONTINUOUS FUNCTIONS WITH CLOSED GRAPH

Abstract. In this paper we consider classes of functions $f : \mathbb{R} \to \mathbb{R}$. The maximal *additive* class for the family $\mathcal{QU}$ of quasi-continuous functions with closed graph is equal to the class of all continuous functions. We also show that the maximal *multiplicative* class for $\mathcal{QU}$ is equal to a class of continuous functions, which fulfil an extra condition.

1. Introduction

Through out this paper $\mathbb{R}$ denotes the set of all real numbers, and we consider $\mathbb{R}$ and $\mathbb{R}\times\mathbb{R}$ endowed with their natural topologies. The symbol $\mathbb{R}^{\mathbb{R}}$ stands for the set of all functions $f : \mathbb{R} \to \mathbb{R}$, and the symbols $\mathcal{C}$, $Const$, $\mathcal{Q}$, $\mathcal{D}$, $\mathcal{B}_1$ and $\mathcal{U}$ denote the subsets of $\mathbb{R}^{\mathbb{R}}$ consisting of all continuous, constant, quasi-continuous, Darboux, Baire-one and functions with closed graph, respectively. Moreover, we set

$$\mathcal{C}^* = \{f \in \mathcal{C} \ : \ f \equiv 0 \ or \ f(x) \neq 0, \ for \ all \ x \in \mathbb{R}\}.$$

We will also use the following abbreviations.
For $\mathcal{F}$ and $\mathcal{G}$ nonempty subsets of $\mathbb{R}^{\mathbb{R}}$, the symbol $\mathcal{FG}$ denotes the set $\mathcal{F} \cap \mathcal{G}$, and the sets

$$\mathcal{M}_a(\mathcal{F}) = \{g \in \mathbb{R}^{\mathbb{R}} \ : \ (\forall f \in \mathcal{F}) \ g + f \in \mathcal{F}\},$$
$$\mathcal{M}_m(\mathcal{F}) = \{g \in \mathbb{R}^{\mathbb{R}} \ : \ (\forall f \in \mathcal{F}) \ g \cdot f \in \mathcal{F}\},$$
$$\mathcal{M}_{max}(\mathcal{F}) = \{g \in \mathbb{R}^{\mathbb{R}} \ : \ (\forall f \in \mathcal{F}) \ max\{g, f\} \in \mathcal{F}\},$$
$$\mathcal{M}_{min}(\mathcal{F}) = \{g \in \mathbb{R}^{\mathbb{R}} \ : \ (\forall f \in \mathcal{F}) \ min\{g, f\} \in \mathcal{F}\},$$

are called *maximal additive, multiplicative, maximum and minimum classes for the family of functions $\mathcal{F}$, respectively.*
In 1986 Grande and Sołtysik [4] showed that

$$\mathcal{M}_a(\mathcal{Q}) = \mathcal{C}, \tag{1}$$

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and in 2003 Szczuka [9] proved, that if $\mathcal{F}$ denotes the set of functions that fulfil the so-called Świątkowski condition (or the strong Świątkowski condition), then $\mathcal{M}_a(\mathcal{F}) = \mathcal{M}_m(\mathcal{F}) = \mathcal{M}_{max}(\mathcal{F}) = Const$. It was also shown that

- $\mathcal{M}_a(\mathcal{D}) = Const$ (Radakovič [8], 1931),
- $\mathcal{M}_a(\mathcal{DB}_1) = \mathcal{C}$ (Bruckner [2], 1978),
- $\mathcal{M}_a(\mathcal{DQ}) = Const$ (Natkaniec [7], 1992),
- $\mathcal{M}_a(\mathcal{DB}_1\mathcal{Q}) = \mathcal{C}$ (Banaszewski [1], 1992), (see also [5]).

In 1987 Menkyna [6] considered real functions on a locally compact normal space X and obtained two results, which, for the particular case $X = \mathbb{R}$, take the form

$$\mathcal{M}_a(\mathcal{U}) = \mathcal{C}, \tag{2}$$

$$\mathcal{M}_m(\mathcal{U}) = \mathcal{C}^*. \tag{3}$$

In this paper, we prove the following theorem that supplements the above listed results.

Theorem. *With the above notations, we have*

$$\mathcal{M}_a(\mathcal{QU}) = \mathcal{C}, \tag{a}$$

$$\mathcal{M}_m(\mathcal{QU}) = \mathcal{C}^*, \tag{b}$$

$$\mathcal{M}_{max}(\mathcal{QU}) = \mathcal{M}_{min}(\mathcal{QU}) = \emptyset. \tag{c}$$

Our terminology and notations are standard. The symbols $\mathcal{F}+\mathcal{G}$ and $\mathcal{F}\cdot\mathcal{G}$ denote the respective sets $\{f+g \,:\, f \in \mathcal{F},\ g \in \mathcal{G}\}$ and $\{f\cdot g \,:\, f \in \mathcal{F},\ g \in \mathcal{G}\}$. Notice that $\mathcal{F} \cdot \mathcal{G} \neq \mathcal{FG}$. For $f \in \mathbb{R}^{\mathbb{R}}$ the symbol $C(f)$ denotes the set of all *continouity points* of f and $G(f) \subset \mathbb{R} \times \mathbb{R}$ denotes the graph of f.

2. Definitions and useful lemmas

Definition 1. A function $f \in \mathbb{R}^{\mathbb{R}}$ is *quasi-continuous at* $x_0 \in \mathbb{R}$ if for every neighbourhood (a, b) of x_0 and every $\epsilon > 0$, there is an interval $(c, d) \subset (a, b)$ such that $\mid f(x) - f(y) \mid < \epsilon$, for each $y \in (c, d)$.

In the proof of the Theorem we will use the following characterization of quasi-continouity (see [3], p. 526)

Lemma 1. *A function $f \in \mathbb{R}^{\mathbb{R}}$ is quasi-continuous if and only if for every $x_0 \in \mathbb{R}$ there is a sequence $(x_n) \subset C(f)$ with $\lim_{n\to\infty} x_n = x_0$, such that $\lim_{n\to\infty} f(x_n) = f(x_0)$.*

We will also use the following properties (their easy proofs are left to the reader):

$$\text{(4)} \qquad if\ 0 \in \mathcal{F}\ then\ \mathcal{M}_a(\mathcal{F}) \subset \mathcal{F},$$

$$\text{(5)} \qquad if\ 1 \in \mathcal{F}\ then\ \mathcal{M}_m(\mathcal{F}) \subset \mathcal{F}.$$

$$\text{(6)} \qquad \mathcal{C} \cdot \mathcal{Q} \subset \mathcal{Q},$$

$$\text{(7)} \qquad \mathcal{C} + \mathcal{U} \subset \mathcal{U}.$$

For $g \in \mathcal{QU}$ discontinuous at some $x_0 \in \mathbb{R}$ we have

$$\text{(8)} \qquad \begin{array}{c} \lim_{x\to x_0^-} g(x) = g(x_0) \text{ and } \mid \lim_{x\to x_0^+} g(x) \mid = +\infty, \text{ or} \\ \lim_{x\to x_0^+} g(x) = g(x_0) \text{ and } \mid \lim_{x\to x_0^-} g(x) \mid = +\infty. \end{array}$$

3. The proof of the Theorem

Part (a) By (1) and (7) we obtain $\mathcal{C} + \mathcal{QU} \subset \mathcal{QU}$, so $\mathcal{C} \subset \mathcal{M}_a(\mathcal{QU})$. Now we shall show this inclusion can be reversed. Let $g \in \mathcal{M}_a(\mathcal{QU})$ be arbitrary fixed. We claim that $g \in \mathcal{C}$. If this were not so, g would be discontinuous at some $x_0 \in \mathbb{R}$. Notice that $g \in \mathcal{QU}$. Consider the first condition in (8). For the function $f_1 : \mathbb{R} \to \mathbb{R}$ defined by the formula

$$f_1(x) = \begin{cases} \frac{1}{x_0 - x} & ; x < x_0, \\ 0 & ; x \geqslant x_0, \end{cases}$$

we have $f_1 \in \mathcal{QU}$ and $|\lim_{x\to x_0^-}(g + f_1)(x)| = +\infty = |\lim_{x\to x_0^+}(g + f_1)(x)|$. Hence, by Lemma 1, function $g + f_1$ is not quasi-continuous, a contradiction. For the second condition in (8) we use similar arguments and we obtain a contradiction too. Thus $g \in \mathcal{C}$, as claimed.

Part (b) By (3) and (6) we obtain $\mathcal{C}^* \cdot \mathcal{QU} \subset \mathcal{QU}$, so $\mathcal{C}^* \subset \mathcal{M}_m(\mathcal{QU})$. Now we shall show this inclusion can be reversed. Let $g \in \mathcal{M}_m(\mathcal{QU})$ be arbitrary fixed. We claim that $g \in \mathcal{C}^*$. Assume this is not so. We shall prove first that g is continuous. In the opposite case there is $x_0 \notin C(g)$. Observe that $g \in \mathcal{QU}$. We will consider the first condition in (8). Let us assume that $\lim_{x\to x_0^+} f(x) = +\infty$. Let $x_1 > x_0$ be a point such that $f(x) > 0$, for every $x \in (x_0, x_1]$. We define $f_2 : \mathbb{R} \to \mathbb{R}$ as a function given by the formula

$$f_2(x) = \begin{cases} \frac{1}{x - x_0} & ; x < x_0, \\ 0 & ; x = x_0, \\ \frac{1}{g(x)} & ; x \in (x_0, x_1), \\ \frac{1}{g(x_1)} & ; x \geqslant x_1. \end{cases}$$

Of course $f_2 \in \mathcal{QU}$ and $(x_0, 1) \in cl(G(g\cdot f_3))\setminus G(g\cdot f_3)$. Hence $g \notin \mathcal{M}_m(\mathcal{QU})$, a contradiction. By similar arguments, we obtain a contradiction for the case with $\lim_{x\to x_0^+} f(x) = -\infty$, and for the second case of (8). Hence $g \in \mathcal{C}$. Now we have to show that $g \in \mathcal{C}^*$. If this were not so, there would exist

- a point $x_0 \in g^{-1}(0) \setminus int(g^{-1}(0))$ with $g(x_0) = 0$, and
- $\delta > 0$ such that $g(x) \neq 0$, for every $x \in [x_0 - \delta, x_0)$ (or $x \in (x_0, x_0 + \delta]$).

Let $I = [x_0 - \delta, x_0)$ and $a = g(x_0 - \delta)$. We define $f_3 : \mathbb{R} \to \mathbb{R}$ as a function given by the formula

$$f_3(x) = \begin{cases} \frac{1}{g(x)} & ; x \in I, \\ \frac{1}{a} & ; x < x_0 - \delta, \\ 1 & ; x \geqslant x_0, \end{cases}$$

Of course $f_3 \in \mathcal{QU}$ and $(x_0, 1) \in cl(G(g\cdot f_3))\setminus G(g\cdot f_3)$. Hence $g \notin \mathcal{M}_m(\mathcal{QU})$, a contradiction. The contradiction shows that we must have $g \in \mathcal{C}^*$, as claimed.

Part (c) Let $g : \mathbb{R} \to \mathbb{R}$. Choose $x_0 \in \mathbb{R}$ for which there is a sequence $(x_n)_n$ such that $x_n \nearrow x_0$ and $\lim_{n\to\infty} g(x_n) = g(x_0)$ (the set of the points $x_0 \in \mathbb{R}$ for which there is no sequence with the latter property is countable). We define $f_4 : \mathbb{R} \to \mathbb{R}$ as a function given by the formula

$$f_4(x) = \begin{cases} \frac{1}{x - x_0} & ; x < x_0, \\ g(x_0) + 1 & ; x \geqslant x_0. \end{cases}$$

Of course $f_4 \in \mathcal{QU}$ and $h = \max(f_4, g) \notin \mathcal{U}$ (because $(x_0, g(x_0)) \in cl(G(h))\setminus G(h)$.) Hence $\mathcal{M}_{max}(\mathcal{QU}) = \emptyset$.

With similar argumentation we can show that $\mathcal{M}_{min}(\mathcal{QU}) = \emptyset$. ■

References

[1] D. Banaszewski, *On some subclasses of $\mathcal{DB}_1$ functions*, Problemy Mat. 13 (1993), 33–41.

[2] A. M. Bruckner, *Differentiation of Real Functions*, Lecture Notes in Math. no. 659, Springer Verlag, Berlin-Heidelberg-New York (1978).

[3] Z. Grande, T. Natkaniec, *Lattices generated by $\mathcal{F}$-quasi-continuous functions*, Polish Acad. Sci. Math. 34 (1986), 525–530.

[4] Z. Grande, L. Sołtysik, *Some remarks on quasi-continuous real functions*, Problemy Mat. 10 (1990), 79–86.

[5] J. M. Jastrzębski, J. M. Jędrzejewski, T. Natkaniec, *On some subclasses of Darboux functions*, Fund. Math. 138 (1991), no. 3, 165-173.

[6] R. Menkyna, *The maximal additive and multiplicative families for functions with closed graph*, Act. Math. Univ. Com. 52/53 (1987), 149–152.

[7] T. Natkaniec, *On quasi-continuous functions having Darboux property*, Math. Pannon. 3 (1992), 81–96.

[8] T. Radakovič, *Über Darbouxsche und stetige Funktionen*, Monatsh. Math. Phys. 38 (1931), 117–122.
[9] P. Szczuka, *Maximal classes for the family of strong Świątkowski functions*, Real Anal. Exchange 28 (2) (2003), 429–438.

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