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SAMENESS BETWEEN BASED UNIVERSAL ALGEBRAS

Abstract. This is the continuation of the paper "Transformations between Menger systems". To define when two universal algebras with bases "are the same", here we propose a universal notion of transformation that comes from a triple characterization concerning three representation facets: the determinations of the *Menger system, analytic monoid* and *endomorphism representation* corresponding to a basis.

Hence, this notion consists of three equivalent definitions. It characterizes another technical variant and also the universal version of the very semi-linear transformations *that were coordinate-free.*

Universal transformations allow us to check the *actual* invariance of general algebraic constructions, contrary to the seeming invariance of representation-free thinking. They propose a new interpretation of free algebras as superpositions of "analytic spaces" and deny that our algebras differ from vector spaces at fundamental stages.

Contrary to present beliefs, even the foundation of abstract Linear Algebra turns out to be incomplete.

0. Introduction

0.0 The sameness problem. To understand what a universal algebra is requires to be able to define when two of them "are the same". Isomorphisms and general isomorphisms [3] are examples of such definitions. They seem to work when we view such algebras from the outside, viz. through notions mainly derived from the homomorphisms between such algebras (and in particular homomorphisms from algebras of terms) as in Birkhoff's theorem and in applications to Logic.

This very view focusses on free algebras, viz. the ones that have bases. Yet, based algebras open a new *intrinsic* view, which can be relevant to Computer Science applications as the ones in **0.4** of [7]. In fact, **0.5** of [9] shows that we can define bases both by the above-mentioned homomor-

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phisms (conventional abstract definition) and by the endomorphisms *alone* (representation dependent definition of **0.2** of [11]).

These two views are different. Counterexample **3.6** of [11] proved that, in spite of the seeming invariance of representation-free thinking, the abstract representation-free Algebra of the past century turns out unable to check the actual invariance of an *elementary* notion (about bases) that also concern the outside view. On the contrary, the representation of endomorphism application by basis dependent Menger systems provided such algebras with a transformation notion, the "descriptions", able to perform that check.

(Ironically, this restriction of the homomorphism category to the endomorphism monoid made a groupoid, viz. a category, replace the automorphism group, viz. a monoid.)

Then, descriptions might be able to define algebra sameness. Yet, their (generalized) Menger systems merely were one of three set-theoretical facets of endomorphism representation. Two others were the *analytic monoid* and the very *representation function* corresponding to a basis.

Moreover, [11] hints at a further (algebraic) facet: the generalization of scalars or dilatations from vector spaces to based universal algebras. Such dilatations form the intersection of two well-known structures of Universal Algebra: the endomorphism monoid and the clone of elementary functions. As recalled in **0.2** scalars provided Linear Algebra with the sameness notion rising from semi-linear transformations: why not to generalize it?

Therefore, we can conceive too many ways to compare based algebras from the inside. One might well fear that no single intrinsic sameness exists.

0.1 The solution, this paper presents for the problem of intrinsic sameness, concerns the class of based algebras, where the above-mentioned abstract treatments fail. Hence, it also concerns all free algebras, but for a new interpretation of them shown in **3.6**.

(For the class of all universal algebras this merely is a negative hint: as free algebras are algebras, Abstract Algebra cannot define such sameness nor algebraic invariance. Some affirmative hints might come from providing general endomorphism monoids with concrete characterizations, a yet unsolved problem [2, 4].)

The three previous set-theoretical facets of endomorphism representation are the three structures that directly rise from the choice of a basis as in **0.2** of [11]. Each of them has its general definition of transformation corresponding to the structure purposes. Each of them also shares with the semi-linear transformations, we know from general vector spaces, the splitting into two component bijections, one of which is between carriers.

After the descriptions, the transformations introduced in [11] for Menger systems, in **1.5** we define transformations for analytic monoids. We only require to preserve the units and that the other component bijection, which is between universal matrices, can determine the one between carriers.

A first property of such transformations is their characterization in **1.6** (A) by a reduced monoid composition involving the generator of constants of section **1** of [11]. Another is the preservation of scalar monoids. It universalizes the preservation of scalar fields that semi-linear transformations assume by definition.

The third structure is the representation of the endomorphism monoid. As it concerns (general) dilatations, its transformations in **2.2** require both a preservation of endomorphisms and a *full preservation* of dilatations. This means that also the "amounts" of dilatations, which come from elements called their indicators, are preserved. Clearly, even semi-linear transformations did require this, but for the formulation, because their dilatations were algebra operations preserved by the field isomorphism.

From the proof in **3.0** that these three universal transformations are the same we get two immediate consequences: a preservation of universal flocks and a characterization of the "representation-free" universal transformations, called *renamings,* that transform matrices columnwise. After universalizing the semi-linear transformations as below, we use this triple characterization also to prove that they are equivalent to the others.

0.2 Semi-linear transformations provide vector spaces with a general sameness notion that differs from the abstract one of an isomorphism. (**3.5** (A) will recall their technical details). Isomorphisms (linear transformations) are able to formalize sameness only in a proper subclass of such spaces, corresponding to certain underlying fields, as the real, rational and some Galois ones.

With one of such fields we can identify the transformations that formalize sameness either by basis transitions or by carrier bijections (the isomorphisms), since the former determine the latter and conversely. With other fields, as the complex one recalled in **3.5** of [11], also some bijections that are not isomorphisms for vector spaces work as transformations, provided that they are coupled with some field auto(/iso)morphism.

Then, basis transitions cannot identify transformations anymore. One transition can have two transformations: this transformation couple and its induced isomorphism, which again corresponds to another couple with the identity as field isomorphism. Such couples, called *semi-linear transformations,* replace isomorphisms when comparing general vector spaces.

This failure of isomorphisms did not weaken the abstract approach of the past century both in Linear Algebra and in Universal Algebra. It merely fuelled the idea that vector spaces are fairly peculiar cases of universal algebras, so that one might split their two theories. The "generalized conception of space" and the "uniform method" of A.N. Whitehead (preface of [12]) seemed naive wishes.

In fact, it turned out that even such general carrier bijections were some abstract isomorphisms (between such remarkable algebras as Abelian groups) and that no reference frame was necessary. Moreover, in the universal case, the general isomorphisms (that Marczewski's caution called weak) generalized semi-linear transformations [3], albeit not formally.

On the contrary our "Segre descriptions", which rely on generalized scalars, in **3.3** *formally* generalize semi-linear transformations to *any based universal algebra* and in **3.4** become equivalent to the previous descriptions. Also, they show why abstract notions work in vector spaces while fail in general: within such spaces scalars are representation-free contrary to the general case.

While this denies any transformation peculiarity to vector spaces, their natural characterizations as universal algebras are simpler than their conventional definitions. E.g., in [10] they merely come out as "dilatation complete" Abelian groups with dilatable bases.

Then, Whitehead was not so naive. (Also, his treatment of Linear Algebra in [12] was representation dependent.) This also hints that some other abstract beliefs and notions that appear sound and crystal clear might deserve some check. For instance, as **3.5** (A) will show, we still need some statements that Linear Algebra failed to state and prove about the very semi-linear transformations, on which the "first fundamental theorem of projective geometry" [1] relies.

1. Analytic transformations

1.0 Definition. While the transformations in [11] concerned two Menger systems, in **1.5** they will concern two analytic monoids denoted as in **1.2** ibid.. Here, we introduce some preliminary notions and results.

Given a bijection $t: A^X \rightarrowtail\!\!\!\twoheadrightarrow B^Y$, consider the relation $g \subseteq A \times B$ defined for all $a \in A$ and $b \in B$ by $\langle a, b \rangle \in g$ iff

$$(0) \qquad t(L \circ \boldsymbol{k}_a) = t(L) \diamond \boldsymbol{\kappa}_b \;\text{ for all } L: X \to A\;,$$

namely g relates a and b when bijection t isomorphically relates the two unary operations on A^X and B^Y of "right product by the corresponding constant": $\chi'_a: A^X \to A^X$ and $\xi'_b: B^Y \to B^Y$ such that $\chi'_a(L) = L \circ \boldsymbol{k}_a$ and $\xi'_b(M) = M \diamond \boldsymbol{\kappa}_b$ for all $L: X \to A$ and $M: Y \to B$. Such a right product also occurred in the axioms of definition **1.0** in [11] for analytic monoids.

If g relates any $a \in A$ with some $b \in B$ and, conversely, any b with some a, then we say that t *totally induces* g *from* A *to* B or that g is the *relation* *totally induced by t from A to B.* Since our functions $\chi' : A \to (A^X)^{A^X}$ and $\xi' : B \to (B^Y)^{B^Y}$ define *unary* algebras on A^X and B^Y respectively, total induction requires that they are *generally* isomorphic algebras as in **2** of [3].

When $X, Y \neq \emptyset$, t determines A and B. Then, we merely say that g is the *relation totally induced by* t and we write $g = G_t$, where we denote the function relating the t's to the g's by $G \subseteq (B^Y)^{A^X} \times P(A \times B)$.

The requirement that t is such a bijection again implies that singleton carriers coexist as in (31) of [11]. Then, both analytic monoids are trivial and total induction by (0) defines $g = A \times B$, where by **1.0** ibid. A and B are only required to be either both empty or both nonempty: e.g. it prevents that $A^X = \emptyset^\emptyset$ and $B^Y = 1^2$, as **3.1** (A) ibid. did.

This agrees with the behavior of trivial analytic monoids in **1.0** of [11], whereas it disagrees with the premise $g : A \mapsto\!\!\twoheadrightarrow B$ of **3.1** (A) ibid.. Yet, the corresponding conclusions still hold. Anyway, if one of the monoid carriers is not singleton, then both $X, Y \neq \emptyset$.

1.1 Lemmata. *If* $t : A^X \mapsto\!\!\twoheadrightarrow B^Y$ *totally induces a relation from* A *to* B, $g \subseteq A \times B$, *then*

(A) t *retypes* $\boldsymbol{K}$ *as in* **1.6** *of* [11] *and,*

(B) *when* $X, Y \neq \emptyset$, *the induced relation is a bijection,* $g = G_t : A \mapsto\!\!\twoheadrightarrow B$.

Proof. (A) When the carriers are singleton, it follows from $t = t \cdot \boldsymbol{i}_C$ as in **1.6** of [11]. Otherwise, the dimensions are not trivial and by **1.0** ibid. $A, B \neq \emptyset$. Since for every $a \in A$ there is some $b \in B$ that satisfies (0), from axiom (4) of [11], (0) and (8) ibid. we get for all $a \in A$ that $t(\boldsymbol{k}_a) = t(U \circ \boldsymbol{k}_a) = t(U) \diamond \boldsymbol{\kappa}_b = \boldsymbol{\kappa}_{\xi_b(t(U))} = \boldsymbol{\kappa}_{b'}$ for some $b' = \xi_b(t(U))$. Conversely, for all $b \in B$ from axiom (11) ibid., (0) and (7) ibid. we get $\boldsymbol{\kappa}_b = V \diamond \boldsymbol{\kappa}_b = t(t^{-1}(V)) \diamond \boldsymbol{\kappa}_b = t(t^{-1}(V) \circ \boldsymbol{k}_a) = t(\boldsymbol{k}_{\chi_a(t^{-1}(V))}) = t(\boldsymbol{k}_{a'})$ for some $a' = \chi_a(t^{-1}(V))$ with $\langle a, b \rangle \in g$.

(B) Let us show that the induced relation is a function, $g = G_t : A \to B$. Let $\langle a, b' \rangle, \langle a, b'' \rangle \in g$. In (0) take $L = t^{-1}(V)$. Then, $\boldsymbol{\kappa}_{b'} = V \diamond \boldsymbol{\kappa}_{b'} = t(L) \diamond \boldsymbol{\kappa}_{b'} = t(L \circ \boldsymbol{k}_a) = t(L) \diamond \boldsymbol{\kappa}_{b''} = V \diamond \boldsymbol{\kappa}_{b''} = \boldsymbol{\kappa}_{b''}$, because of axiom (11) of [11]. Hence, (9) ibid. gets $b' = b''$.

Since $g : A \twoheadrightarrow B$ comes from the total induction assumption, now we only have to show that g^{-1} too is a function, $g^{-1} : B \to A$. This, follows from (4) ibid. and (2) ibid. by the converse of the preceding reasoning. In fact, $\langle a', b \rangle, \langle a'', b \rangle \in g$ implies $t(\boldsymbol{k}_{a'}) = t(U \circ \boldsymbol{k}_{a'}) = t(U) \diamond \boldsymbol{\kappa}_b = t(U \circ \boldsymbol{k}_{a''}) = t(\boldsymbol{k}_{a''})$ by (0) with $L = U$. Since t is one to one, $\boldsymbol{k}_{a'} = \boldsymbol{k}_{a''}$, whence $a' = a''$. ∎

1.2 Corollary. *If* $t\colon A^X \mapsto\!\!\!\twoheadrightarrow B^Y$ *totally induces the bijection g of* **1.1** (B) *for* $X, Y \neq \emptyset$ *and preserves the unit,*

$$(1) \qquad t(U) = V,$$

then g is the $\boldsymbol{K}$*–induced bijection in* **1.8** *of* [11]*: for all* $a \in A$ *and every* $y \in Y$, $g(a) = G_t(a) = t(\boldsymbol{k}_a)(y)$, *namely* $t(\boldsymbol{k}_a) = \boldsymbol{\kappa}_{g(a)}$.

Proof. Take $L = U$ in (0). Then, for every $a \in A$ by (4) of [11], (1) and (11) ibid. get $t(\boldsymbol{k}_a) = t(U \circ \boldsymbol{k}_a) = V \diamond \boldsymbol{\kappa}_b = \boldsymbol{\kappa}_b$, where by **1.1** (B) $b = g(a)$. Hence, by (9) ibid. $t(\boldsymbol{k}_a)(y) = \boldsymbol{\kappa}_{g(a)}(y) = g(a)$ for every $y \in Y$. ∎

1.3 Definition. Given two analytic monoids as in **1.2** of [11], we say that $t\colon A^X \mapsto\!\!\!\twoheadrightarrow B^Y$ *preserves* $\boldsymbol{K}$*–restricted products* when

$$(2) \qquad t(L \circ \boldsymbol{k}_a) = t(L) \diamond t(\boldsymbol{k}_a) \quad \text{for all } L\colon X \to A \text{ and } a \in A\ .$$

1.4 Lemma. *If* $t\colon A^X \mapsto\!\!\!\twoheadrightarrow B^Y$ *totally induces a relation from A to B and preserves the unit, then it preserves* $\boldsymbol{K}$*–restricted products.*

Proof. Trivial for singleton carriers. Otherwise $X, Y \neq \emptyset$. Then, start from (0) and use **1.1** (B) and **1.2**: for all $L\colon X \to A$ and $a \in A$, $t(L \circ \boldsymbol{k}_a) = t(L) \diamond \boldsymbol{\kappa}_b = t(L) \diamond \boldsymbol{\kappa}_{g(a)} = t(L) \diamond t(\boldsymbol{k}_a)$. ∎

1.5 Definitions. Consider a bijection between the carriers of our two analytic monoids, $t\colon A^X \mapsto\!\!\!\twoheadrightarrow B^Y$. The conditions of total induction and unit preservation are enough to get the preservation of other features between such analytic monoids, as we have just shown and we will also find in **1.6**. Hence, we will say that t is an *analytic transformation from* the former monoid *to* the latter when it totally induces g from A to B as in **1.0** and preserves the unit as in (1). However, even the two preservation properties, we have shown in the preceding lemmata, are enough and will allow us to use the following characterization **1.6** (A).

When $X, Y \neq \emptyset$, the two analytic monoids identify the two Menger systems in **1.2** of [11], while t can be the subject of the depiction property (25) ibid. Then, we say that g, the bijection $\boldsymbol{K}$-induced by t as in **1.2**, is the *analytic description of* χ *by* ξ or *from* the former monoid *onto* the latter.

When $Y = \emptyset$, both the expression of G in **1.2** and the one of T in (32) of [11] fail to express g and t respectively, though both **1.2** and **3.1** (C) ibid. are true. Yet, contrary to matrix transformations, analytic descriptions are not defined, because of the set-theoretical reason in the note of **1.6** ibid..

1.6 Theorems.

(A) *When both dimensions are not trivial,* $t\colon A^X \mapsto\!\!\!\twoheadrightarrow B^Y$ *is an analytic transformation iff it retypes* $\boldsymbol{K}$ *as in* **1.6** *of* [11] *and preserves* $\boldsymbol{K}$*-restricted products as in (2).*

(B) *An analytic transformation t is a monoid isomorphism, namely it preserves the units, $t(U) = V$, and the matrix product, $t(M \circ L) = t(M) \diamond t(L)$ for all $L, M : X \to A$.*

Proofs. (A) (Only if) $\boldsymbol{K}$–retyping comes from **1.1** (A), while the other preservation comes from **1.4**. (Hence, this holds even for $X = \emptyset$ or $Y = \emptyset$.)

(If) From (2) and **1.7** (A) of [11] we get $t(L \circ \boldsymbol{k}_a) = t(L) \diamond t(\boldsymbol{k}_a) = t(L) \diamond \boldsymbol{\kappa}_{g(a)}$ for some $g : A \mapsto\!\!\!\twoheadrightarrow B$ and all $L : X \to A$ and $a \in A$. Hence, t totally induces some relation, which by (0) and **1.1** (B) is this g. For $L = U$ this also implies that, for all $b = g(a) \in B$, $\boldsymbol{\kappa}_b = t(\boldsymbol{k}_a) = t(U \circ \boldsymbol{k}_a) = t(U) \diamond \boldsymbol{\kappa}_b$ by (4) ibid.. Then, by (8) ibid. and (9) ibid. $\xi_b(t(U)) = b$ for all $b \in B$, which by **1.4** ibid. and (13) ibid. states that $t(U) = \boldsymbol{r}''_V(\boldsymbol{i}_B)$ by **0.2** ibid., namely $t(U) = V$.

(Notice that, when some dimension is trivial, say $X = \emptyset$, the preservation of the unit still comes from (31) of [11] as observed in **1.0**, whereas total induction fails for $A = \emptyset$ and $B \neq \emptyset$.)

(B) t preserves the units by definition. It also trivially preserves the matrix product in the singleton carrier case. Hence, we can assume $X, Y \neq \emptyset$ and, for all $L, M : X \to A$ and $y \in Y$, in order to prove $(t(M \circ L))_y = (t(M) \diamond t(L))_y$, we prove $\boldsymbol{\kappa}_{(t(M \circ L))_y} = \boldsymbol{\kappa}_{(t(M) \diamond t(L))_y}$ because of (24) of [11].

In fact, we use (10) of [11], **1.1** (A), **1.2**, **1.4**, (5) ibid., **1.4**, **1.2**, (10) ibid., **1.1** (A), (2), (10) ibid., (12) ibid. and (10) ibid. to get, $\boldsymbol{\kappa}_{(t(M \circ L))_y} = t(M \circ L) \diamond \boldsymbol{\kappa}_{V(y)} = t(M \circ L) \diamond t(\boldsymbol{k}_{V'(y)}) = t((M \circ L) \circ \boldsymbol{k}_{V'(y)}) = t(M \circ (L \circ \boldsymbol{k}_{V'(y)})) = t(M \circ t^{-1}(t(L \circ \boldsymbol{k}_{V'(y)}))) = t(M \circ t^{-1}(t(L) \diamond \boldsymbol{\kappa}_{V(y)}))) = t(M \circ t^{-1}(\boldsymbol{\kappa}_{(t(L))_y})) = t(M) \diamond t(t^{-1}(\boldsymbol{\kappa}_{(t(L))_y})) = t(M) \diamond \boldsymbol{\kappa}_{(t(L))_y} = t(M) \diamond (t(L) \diamond \boldsymbol{\kappa}_{V(y)}) = (t(M) \diamond t(L)) \diamond \boldsymbol{\kappa}_{V(y)} = \boldsymbol{\kappa}_{(t(M) \diamond t(L))_y}$. ■

1.7 Corollaries.

(A) *An analytic transformation t preserves the scalars in both ways: for all $S : X \to A$*

(3) $\quad t(S) \diamond \boldsymbol{\kappa}_b = \boldsymbol{\kappa}_b \diamond t(S)$ *for all* $b \in B$ *iff* $S \circ \boldsymbol{k}_a = \boldsymbol{k}_a \circ S$ *for all* $a \in A$,

according to characterization **2.4** (C) *of* [11]. *Hence, $t \cdot \boldsymbol{i}_F : F \mapsto\!\!\!\twoheadrightarrow G$ is an isomorphism between scalar monoids by* **2.4** (F) *ibid. and* **1.6** (B) .

(B) *When $X, Y \neq \emptyset$, t and its analytic description g preserve the derived Menger systems as in (36) of* [11].

Proofs. (A) In case of singleton carriers, the unit scalar is the only matrix and the statement is obvious. Otherwise, we only have to prove (3) for $X, Y \neq \emptyset$. By **1.6** (B) $S \circ \boldsymbol{k}_a = \boldsymbol{k}_a \circ S$ for all $a \in A$ iff $t(S) \diamond t(\boldsymbol{k}_a) = t(\boldsymbol{k}_a) \diamond t(S)$ for all $a \in A$ and by **1.2** iff $t(S) \diamond \boldsymbol{\kappa}_{g(a)} = \boldsymbol{\kappa}_{g(a)} \diamond t(S)$ for all $a \in A$, which by **1.1** (B) is equivalent to $t(S) \diamond \boldsymbol{\kappa}_b = \boldsymbol{\kappa}_b \diamond t(S)$ for all $b \in B$.

(B) Take some $y \in Y$. By (9) of [11], **1.2**, (7) ibid., **1.6** (A), **1.2**, (8) ibid. and (9) ibid. we get $g(\chi_a(L)) = \boldsymbol{\kappa}_{g(\chi_a(L))}(y) = t(\boldsymbol{k}_{\chi_a(L)})(y) = t(L \circ \boldsymbol{k}_a)(y) = (t(L) \diamond t(\boldsymbol{k}_a))(y) = (t(L) \diamond \boldsymbol{\kappa}_{g(a)})(y) = \boldsymbol{\kappa}_{\xi_{g(a)}(t(L))}(y) = \xi_{g(a)}(t(L))$ for all $a \in A$ and $L : X \to A$. ∎

1.8 Lemmata. *Let g be the analytic description for t as in* **1.5** *and consider the two derived Menger systems, then*

(A) *g is a centralizer bijection: for all $e : A \to A$ and $f : B \to B$ such that $g \cdot e = f \cdot g$,*

(4) $$e \in \mathcal{E} \text{ iff } f \in \mathcal{F}\ ;$$

(B) *$c \in A$ is a dilatation indicator (in the former Menger system) iff $g(c) \in B$ is (in the latter).*

Proofs. (A) By (14) of [11] and (15) ibid. we can prove that, when

(5) $$g(e(a)) = f(g(a)) \text{ for all } a \in A\,,$$

for each $L : X \to A$, such that

(6) $$e(a) = \chi_a(L) \text{ for all } a \in A\,,$$

there is an $M : Y \to B$, such that

(7) $$\xi_b(M) = f(b) \text{ for all } b \in B\,,$$

and — conversely — for each such an M there is such an L. Since $g : A \rightarrowtail B$, we can replace (6) by $g(e(a)) = g(\chi_a(L))$ for all $a \in A$. Since $g : A \twoheadrightarrow B$, we can replace (7) by $\xi_{g(a)}(M) = f(g(a))$ for all $a \in A$.

Therefore, because of (5), we only have to prove that for each L there is an M and for each M there is an L such that $g(\chi_a(L)) = \xi_{g(a)}(M)$ for all $a \in A$. This is what our relation $t : A^X \rightarrowtail\!\!\!\twoheadrightarrow B^Y$ does by **1.7** (B), when we set $M = t(L)$ in (36) of [11].

(B) By **2.3** of [11] any c is a dilatation indicator iff there is $L : X \to A$ such that $\chi_c(\boldsymbol{k}_a) = \chi_a(L)$ for all $a \in A$. As $g : A \rightarrowtail\!\!\!\twoheadrightarrow B$, this occurs iff $g(\chi_c(\boldsymbol{k}_a)) = g(\chi_a(L))$ for all $a \in A$, namely by **1.7** (B) and **1.2** iff $\xi_{g(c)}(\boldsymbol{\kappa}_{g(a)}) = \xi_{g(a)}(t(L))$ for all $a \in A$. Since both $t : A^X \rightarrowtail\!\!\!\twoheadrightarrow B^Y$ and $g : A \twoheadrightarrow B$, we can set $M = t(L)$ and $b = g(a)$ to rewrite it as $\xi_{g(c)}(\boldsymbol{\kappa}_b) = \xi_b(M)$ for all $b \in B$.

Hence, c is a dilatation indicator iff there is $M : Y \to B$ such that the last condition holds. By **2.3** of [11] this occurs iff $d = g(c) \in B$ is a dilatation indicator. ∎

2. Geometric descriptions and transformations

2.0 Definition. Consider two representations for based algebras as in (13) of [11] that derive our Menger systems χ and ξ by **0.2** ibid.. Given any $g : A \to B$, let $\widetilde{g}$ denote the function that indexes relations by endomorphisms,

$\widetilde{g}:\mathcal{E}\to P(B\times B)$, defined for all $e\in\mathcal{E}\subseteq A\times A$ by

(8) $$\widetilde{g}_e=\{\langle g(a'),g(a'')\rangle \mid \langle a',a''\rangle\in e\}.$$

Namely, $\widetilde{g}_e$ is the "image" of e under g. We will call it the *g–image of e*.

2.1 Lemmata. *If $g:A\twoheadrightarrow B$, then*

(A) $\widetilde{g}:\mathcal{E}\rightarrowtail B^B$,

(B) *every g–image of an endomorphism is its "g–transformed", i.e. $\widetilde{g}_e = g\cdot e\cdot g^{-1}$, for all $e\in\mathcal{E}$, which implies that*

(C) $\widetilde{g}_e(g(a))=g(e(a))$ *for all $a\in A$, and that*

(D) *g–images preserve compositions, $\widetilde{g}_{e''\cdot e'}=\widetilde{g}_{e''}\cdot\widetilde{g}_{e'}$, for all $e',e''\in\mathcal{E}$,*

(E) *and identities, $\widetilde{g}_{\boldsymbol{i}_A}=\boldsymbol{i}_B$.*

Proofs. (B and A) As $e:A\to A$, set $a'=a$ in (8) to rewrite it as

$$\widetilde{g}_e=\{\langle g(a),g(e(a))\rangle \mid a\in A\}.$$

It follows that $\widetilde{g}_e\cdot g=g\cdot e$. Since $g^{-1}:B\twoheadrightarrow A$, we get $g\cdot e\cdot g^{-1}=\widetilde{g}_e\cdot(g\cdot g^{-1})=\widetilde{g}_e$. Hence, $\widetilde{g}_e:B\to B$ for all $e\in\mathcal{E}$, because compositions of functions are functions. This also shows that $\widetilde{g}$ has to be one to one, because $\widetilde{g}_{e'}=\widetilde{g}_{e''}$ by the bijectivities of g and g^{-1} implies $e'=e''$.

(C) It follows from $\widetilde{g}_e:B\to B$ and from $\widetilde{g}_e\cdot g=g\cdot e$ as above.

(D) Trivial computations: $\widetilde{g}_{e''\cdot e'}=g\cdot e''\cdot e'\cdot g^{-1}=g\cdot e''\cdot g^{-1}\cdot g\cdot e'\cdot g^{-1}=\widetilde{g}_{e''}\cdot\widetilde{g}_{e'}$ for all $e',e''\in\mathcal{E}$.

(E) Immediate from $g:A\twoheadrightarrow B$ and (8). ∎

2.2 Definitions. We say that a bijection $g:A\twoheadrightarrow B$ *fully preserves dilatations* when the g–images preserve all dilatations in both ways, $\widetilde{g}_e$ is a dilatation of ξ iff e is of χ, while g preserves the "amount" of the dilatation involved by preserving the indicators in both ways, viz. $\chi_c\cdot\boldsymbol{k}=e\in\mathcal{E}$ iff $\widetilde{g}_e=\xi_{g(c)}\cdot\boldsymbol{\kappa}\in\mathcal{F}$.

We say that $g:A\twoheadrightarrow B$ is a *geometric description of χ by ξ* or *from the representation of $\mathcal{E}$ by U to the one of $\mathcal{F}$ by V*, when g fully preserves dilatations, $\chi_c\cdot\boldsymbol{k}\in\mathcal{E}$ iff $\xi_{g(c)}\cdot\boldsymbol{\kappa}=\widetilde{g}_{\chi_c\cdot\boldsymbol{k}}\in\mathcal{F}$, while the g–images preserve all endomorphisms in both ways, $\widetilde{g}:\mathcal{E}\twoheadrightarrow\mathcal{F}$. The adjective "geometric" refers to the next property **2.4** and to its corollaries **3.1** (C) and (D) (used in **3.2** to show that in vector spaces descriptions induce projectivities).

In such a case $\widetilde{g}:\mathcal{E}\twoheadrightarrow\mathcal{F}$ by **2.1** (A). We call it a *geometric transformation from* the representation of $\mathcal{E}$ by U *to* the one of $\mathcal{F}$ by V, as in (13) of [11]. As shown in **6.8** (D) of [6], it is not necessary to assume two algebras. We can well start only from two composition submonoids on certain $\mathcal{E}\subseteq A^A$ and $\mathcal{F}\subseteq B^B$.

2.3 Corollary. *Let $g: A \Vdash\!\!\twoheadrightarrow B$ be a geometric description as above. Then, the two trivial dimensions must coexist, $X = \emptyset$ iff $Y = \emptyset$, or both A and B are singleton. Hence, (31) of* [11] *holds and, when the sets of matrices are singleton, every bijection from A onto B is such a g.*

Proof. The coexistence of singletons comes from $g: A \Vdash\!\!\twoheadrightarrow B$. Then, consider dimension triviality without singletons. Since by (13) of [11] $\boldsymbol{r}''_V \cdot \widetilde{g} \cdot \boldsymbol{r}'^{-1}_U : A^X \Vdash\!\!\twoheadrightarrow B^Y$, trivial dimensions must coexist. In this case, any $g: A \Vdash\!\!\twoheadrightarrow B$ is a geometric description, because the only dilatations and endomorphisms are the two identities, both with or both without indicators as in **2.0** ibid. or **2.2** ibid. respectively. ∎

2.4 Theorem. *A geometric description preserves flock combiners in both ways: $c \in \Phi'_U$ iff $g(c) \in \Phi''_V$.*

Proof. Let c be a flock combiner of χ, $\chi_c \cdot \boldsymbol{k} = \boldsymbol{i}_A$. As $\boldsymbol{i}_A \in \mathcal{E}$, $\widetilde{g}_{\boldsymbol{i}_A} \in \mathcal{F}$ is a dilatation of ξ and $g(c)$ is one of its indicator, because g fully preserves dilatations. By **2.1** (E) it has to be an indicator of the identity, $\xi_{g(c)} \cdot \boldsymbol{\kappa} = \boldsymbol{i}_B$. Hence, $g(c)$ is a flock combiner of ξ. Conversely, since $g: A \Vdash\!\!\twoheadrightarrow B$, we can start with any such flock combiner $g(c)$ and, since $\widetilde{g}: \mathcal{E} \Vdash\!\!\twoheadrightarrow \mathcal{F}$, we can reverse the above passages to get that c is a flock combiner of χ. By **2.1** (C) of [11] we can also says that g preserves reference flocks in both ways. ∎

3. The triple characterization

3.0 Theorem. *When the bases or units are not trivial, all three notions of description, as well as of transformation, are the same, namely g is a description iff it is analytic and iff it is geometric, while the corresponding matrix and analytic transformations t are the same and correspond to the geometric one: $t(e \cdot U) = \widetilde{g}_e \cdot V$, for all $e \in \mathcal{E}$. In the trivial case this holds for the two descriptions and for the three transformations.*

Proofs. At first, we consider $X, Y \neq \emptyset$.

(description $\Rightarrow$ analytic) Let us show that, given a description $g: A \Vdash\!\!\twoheadrightarrow B$, its matrix transformation t is an analytic transformation between the derived analytic monoids. We use characterization **1.6** (A). By **3.1** (B) of [11] it is a bijection $t: A^X \Vdash\!\!\twoheadrightarrow B^Y$. It also retypes $\boldsymbol{K}$ by **1.7** (A) ibid.. In fact, by **3.1** (C) ibid., **3.4** (A) ibid., (26) ibid. and (9) ibid. $t(\boldsymbol{k}_a)(y) = g(\chi_{V'(y)}(\boldsymbol{k}_a)) = g(a) = \boldsymbol{\kappa}_{g(a)}(y)$, for all $y \in Y$ and $a \in A$, i.e. (25) ibid. holds. Moreover, this shows that t $\boldsymbol{K}$–induces our description.

Lastly, the preservation of $\boldsymbol{K}$–restricted multiplications comes from properties and equations of [11]: (Monoid to Menger) and (Menger loop) in **1.4**, (7), (25), (36), (8) and (25) again. In fact, for all $a \in A$ and $L: X \to A$, $t(L \circ \boldsymbol{k}_a) = t(\boldsymbol{k}_{\chi_a(L)}) = \boldsymbol{\kappa}_{g(\chi_a(L))} = \boldsymbol{\kappa}_{\xi_{g(a)}(t(L))} = t(L) \diamond \boldsymbol{\kappa}_{g(a)} = t(L) \diamond t(\boldsymbol{k}_a)$.

As t is an analytic transformation that $\boldsymbol{K}$–induces g, a description has to be an analytic one.

(analytic $\Rightarrow$ geometric) Since the units are not trivial, by **1.2** of [11] there only is one pair of based algebras χ with basis U and ξ with V, which are derived from the two analytic monoids as Menger systems. Keep g and notice that by **2.1** (B) we can rewrite the premise $g \cdot e = f \cdot g$ in lemma **1.8** (A) as $\widetilde{g}_e = f$, since $g: A \Vdash\!\!\twoheadrightarrow B$. Hence this lemma tells us that $\widetilde{g}$ preserves all endomorphisms in both ways.

To check the full preservation of dilatations, let us start with a dilatation $e = \chi_c \cdot \boldsymbol{k} : A \to A$ of χ, for any dilatation indicator $c \in A$. Consider $\widetilde{g}_{\chi_c \cdot \boldsymbol{k}} = f$. Since $g: A \Vdash\!\!\twoheadrightarrow B$, by **2.1** (B) $g \cdot \chi_c \cdot \boldsymbol{k} = f \cdot g$. Then, for all $a \in A$, $f(g(a)) = (f \cdot g)(a) = (g \cdot \chi_c \cdot \boldsymbol{k})(a) = g(\chi_c(\boldsymbol{k}_a)) = \xi_{g(c)}(t(\boldsymbol{k}_a)) = \xi_{g(c)}(\boldsymbol{\kappa}_{g(a)})$ because of **1.7** (B) and **1.2**. As $g: A \twoheadrightarrow B$, we can rewrite it as $f(b) = \xi_{g(c)}(\boldsymbol{\kappa}_b) = (\xi_{g(c)} \cdot \boldsymbol{\kappa})(b)$ for all $b \in B$. Hence, $f = \xi_{g(c)} \cdot \boldsymbol{\kappa}$, where $g(c) = d$ has to be a dilatation indicator because of **1.8** (B), namely f is a dilatation of ξ.

Conversely, given any dilatation $f = \xi_d \cdot \boldsymbol{\kappa}$ of ξ, we can set $d = g(c)$, since $g: A \twoheadrightarrow B$. By reversing the above passages we can use **1.8** (B) again to find that c is a dilatation indicator defining the above dilatation e of χ with $\widetilde{g}_e = f$.

Lastly, let us check that the geometric transformation $\widetilde{g}: \mathcal{E} \Vdash\!\!\twoheadrightarrow \mathcal{F}$, we found, is the one corresponding to our starting analytic transformation $t: A^X \Vdash\!\!\twoheadrightarrow B^Y$. Namely, when e denotes the endomorphism of χ corresponding to a matrix $L = e \cdot U : X \to A$, i.e. by (14) of [11] $e(a) = \chi_a(L)$ for all $a \in A$, the endomorphism $\widetilde{g}_e$ of ξ has to correspond to $t(L)$, i.e. $t(L) = \widetilde{g}_e \cdot V$ or by (15) ibid. $\widetilde{g}_e(b) = \xi_b(t(L))$ for all $b \in B$. This immediately comes from **1.7** (B). In fact, by (14) ibid. and (36) ibid. $\widetilde{g}_e(b) = g(e(g^{-1}(b))) = g(\chi_{g^{-1}(b)}(L)) = \xi_{g(g^{-1}(b))}(t(L)) = \xi_b(t(L))$ for all $b \in B$.

(geometric $\Rightarrow$ description) Keep g and the derived Menger systems. By **2.1** (C) of [11] and **2.4** we only have to show that g totally induces some t and that $\widetilde{g}$ corresponds to t. We can do it first by defining a $t' : A^X \twoheadrightarrow B^Y$, such that it corresponds to $\widetilde{g}$, and then by checking that $t' \subseteq t$ (which implies $t' = t$ by **3.1** (B) ibid.).

This correspondence is $t'(e \cdot U) = \widetilde{g}_e \cdot V$, for all $e \in \mathcal{E}$. As $\widetilde{g}_e \in \mathcal{F}$, this serves to define a $t' : A^X \to B^Y$, since we can rewrite it as $t'(\boldsymbol{r}'_U(e)) = \widetilde{g}_e \cdot V = \boldsymbol{r}''_V(\widetilde{g}_e)$ for all $e \in \mathcal{E}$ and get $t' = \boldsymbol{r}''_V \cdot \widetilde{g} \cdot \boldsymbol{r}'^{-1}_U$ by (13) of [11].

By that (13) and **2.2** this $t' = \boldsymbol{r}''_V \cdot \widetilde{g} \cdot \boldsymbol{r}'^{-1}_U$ is a bijection onto B^Y, $t' : A^X \Vdash\!\!\twoheadrightarrow B^Y$. Let us show that all pairs $\langle L, M \rangle \in t'$ satisfy (29) of [11]. Any such a pair is in t' when there is an $e \in \mathcal{E}$, such that $L = e \cdot U$ and $M = \widetilde{g}_e \cdot V$. By (14) ibid. and (15) ibid. it satisfies (29) ibid. when such an e and $\widetilde{g}_e$ satisfy $g(e(a)) = \widetilde{g}_e(g(a))$ for all $a \in A$. They do by **2.1** (C).

(Trivial case) To check that a geometric description is a description between the derived Menger systems and conversely, note that the derivation of a Menger system from a based algebra preserves the trivialities $X = \emptyset$ or $Y = \emptyset$ and the carriers in both ways. Then, both **3.3** of [11] and **2.3** give us the same set of all bijections $g : A \twoheadrightarrow B$. (Note also that the above proof (geometric $\Rightarrow$ description) holds even for such dimensions.)

To check the three transformations, note that all derivations preserve any dimension triviality, each of which implies singletons as in (31) of [11], that **1.5** and **3.3** ibid. give us the same $t = \{\langle U, V\rangle\}$ by **3.1** (A) ibid. and **1.0** and that $t(e \cdot U) = \widetilde{g}_e \cdot V$, for all $e \in \mathcal{E} = \{\boldsymbol{i}_A\}$, by **2.3** and (geometric $\Rightarrow$ description). This also ensures that the two descriptions corresponds to their two transformations. ∎

3.1 Corollaries.

(A) *A description is a renaming iff the former basis is the converse basis,* $U = V'$.

(B) *When the former basis and the converse one are co-indexed,* $X = Y$, *the converse basis is a basis,* $\boldsymbol{r}'_{V'} : \mathcal{E} \twoheadrightarrow A^X$.

(C) *A geometric description preserves flocks in both ways:* $a \in \Phi'_L$ *iff* $g(a) \in \Phi''_{t(L)}$ *for all* $a \in A$ *and* $L : X \to A$.

(D) *A matrix transformation induces a flock inclusion isomorphism, namely it preserves inclusion among flocks in both ways: for all* $L, M : X \to A$, $\Phi'_L \subseteq \Phi'_M$ *iff* $\Phi''_{t(L)} \subseteq \Phi''_{t(M)}$.

Proofs. (A) (If) Trivial for $X = Y = \emptyset$. Otherwise, assume that $U = g^{-1} \cdot V = V' : X \to A$. Then, by (32) of [11] and (16) ibid. $t(M)(x) = g(\chi_{g^{-1}(V(x))}(M)) = g(\chi_{U(x)}(M)) = g(M(x)) = (g \cdot M)(x)$ for all $M : X \to A$ and $x \in X$. Hence, (33) ibid. holds.

(Only if) From **3.0**, **1.5** and (33) of [11] $V = t(U) = n \cdot U$. This implies $U = n^{-1} \cdot V = V'$, since $n : A \twoheadrightarrow B$.

(B) Consider the function $\mathbf{B}_{g^{-1}} : B^X \to A^X$ such that $\mathbf{B}_{g^{-1}}(M) = g^{-1} \cdot M$ for all $M : X \to B$. From $g^{-1} : B \twoheadrightarrow A$ we easily get $\mathbf{B}_{g^{-1}} : B^X \twoheadrightarrow A^X$. Then, for all $e \in \mathcal{E}$, $\boldsymbol{r}_{V'}(e) = e \cdot g^{-1} \cdot V = g^{-1} \cdot (g \cdot e \cdot g^{-1}) \cdot V = g^{-1} \cdot \boldsymbol{r}''_V(\widetilde{g}_e) = \mathbf{B}_{g^{-1}}(\boldsymbol{r}''_V(\widetilde{g}_e)) = (\mathbf{B}_{g^{-1}} \cdot \boldsymbol{r}''_V \cdot \widetilde{g})(e)$, namely $\boldsymbol{r}'_{V'} = \mathbf{B}_{g^{-1}} \cdot \boldsymbol{r}''_V \cdot \widetilde{g} : \mathcal{E} \twoheadrightarrow A^X$ as in (0) of [11], since by **3.0** it is a composition of bijections also because of (13) ibid. and **2.2**.

(C) Let $a \in \Phi'_L$, namely $a = \chi_c(L)$ for some $c \in \Phi'_U$. Then, by **3.0** and (36) of [11] $g(a) = g(\chi_c(L)) = \xi_{g(c)}(t(L))$, namely by **2.4** $g(a) \in \Phi''_{t(L)}$. Clearly, we can reverse this implication by **3.4** (D) (Symmetry) ibid..

(D) (Only if) By (C) we can merely show that $g(a) \in \Phi''_{t(M)}$ for all $a \in \Phi'_L$. Since $\Phi'_L \subseteq \Phi'_M$, $a \in \Phi'_M$ and by (C) $g(a) \in \Phi''_{t(M)}$ for all such a's. (If) By symmetry: by (C) the premise becomes $g(a) \in \Phi''_{t(M)}$ for all $a \in \Phi'_L$, while the conclusion $a \in \Phi'_M$ again follows from (C) for all such a's. ∎

3.2 Example. Within vector spaces descriptions share some properties of the semi-linear transformations, which we will recall in **3.5** (A). Here, we recall that a *projectivity* is an inclusion isomorphism $p : \mathcal{S} \rightarrowtail\!\!\twoheadrightarrow \mathcal{T}$ between the sets of subspaces $\mathcal{S}$ and $\mathcal{T}$ of two vector spaces,

$$(9) \qquad A' \subseteq A'' \text{ iff } p(A') \subseteq p(A'') , \quad \text{for all } A', A'' \in \mathcal{S} .$$

Let A and B respectively denote the carriers of the vector spaces. Then, we say that a bijection $g : A \rightarrowtail\!\!\twoheadrightarrow B$ *induces* p, when p is the corresponding restriction of the image function of g,

$$(10) \qquad p(A') = \{g(a) \mid a \in A'\} , \quad \text{for all } A' \in \mathcal{S} \subseteq PA .$$

We prove that, *when the Menger systems or analytic monoids come from vector spaces, any description $g : A \rightarrowtail\!\!\twoheadrightarrow B$ induces a projectivity.*

Proof. Consider the vector-space flocks, defined as in **I.1** of [1], that are not the whole space. By the lemma in **VII.7** ibid. such proper flocks are all and only our flocks with respect to its Menger system (its vector times matrix multiplication), since a vector space has one dimension only. By the recalled definition the proper subspaces are all and only the flocks containing **0**. Therefore, given χ and ξ, we can define an injection $p : \mathcal{S} \rightarrowtail PB$ from $g : A \rightarrowtail B$ by (10) and also get $p : \mathcal{S} \rightarrowtail\!\!\twoheadrightarrow \mathcal{T}$.

In fact, by **3.1** (C) $p(A') = \Phi''_{t(L)}$ for all $L : X \to A$ such that $A' = \Phi'_L$ and for all $A' \in \mathcal{S} \setminus \{A\}$. The flock $\Phi''_{t(L)}$ must contain **0**, because by **3.0** and **1.8** (A) g commutes with the two null endomorphisms, the only constant valued ones in vector spaces. ($A' = A$ is a trivial case.) Conversely, by symmetry for every $B' \in \mathcal{T} \setminus \{B\}$ we get an $M : Y \to B$ with $B' = \Phi''_M$ and an L with $M = t(L)$, such that $B' = p(A')$ for some $A' \in \mathcal{S} \setminus \{A\}$ by **3.1** (B) of [11].

Finally, we get (9) by restricting **3.1** (D) to $\mathcal{S}$. ∎

An immediate corollary of this statement is that *within vector spaces descriptions preserve subspace dimensions,* since projectivities do.

3.3 Definitions. Let two algebras with our bases U and V define the Menger systems χ and ξ respectively. A bijection $\varsigma' : A \rightarrowtail\!\!\twoheadrightarrow B$ is called a *Segre description* between our Menger systems or analytic monoids or based algebras, when it is a centralizer one as in **1.8** (A) that preserves the reference flocks and there is a surjection $\varsigma'' : F \twoheadrightarrow G$ between scalars such that

$$(11) \qquad \varsigma'(\chi_a(S)) = \xi_{\varsigma'(a)}(\varsigma''(S)) \quad \text{for all } a \in A \text{ and } S \in F.$$

The requirement $\varsigma'' : F \twoheadrightarrow G$ is equivalent to merely require a relation $\rho \subseteq F \times G$ with $\{e \mid \langle e, f\rangle \in \rho\} = F$ and $\{f \mid \langle e, f\rangle \in \rho\} = G$ that contains ς''. Anyway, as the next proof will show, ς' and the bases make ς'' an isomorphism between scalar monoids, which we call the *scalar isomorphism.*

Such descriptions involve both centralizer notions: the bijection one in **1.8** (A) and the sub-monoid one in **2.4** (C,F) of [11]. Note the definition symmetry: ς' is a Segre description between χ and ξ with ρ iff ς'^{-1} is a Segre description between ξ and χ with ρ^{-1}.

3.4 Theorem. *Any description is a Segre description and conversely.*

Proof. (description $\Rightarrow$ Segre) Take $\varsigma'' = t \cdot \boldsymbol{i}_F$, which by **3.0**, **1.7** (A) and **1.6** (B) is an isomorphism between scalar monoids. Then, (11) with $\varsigma' = g$ is a restriction of (36) of [11]. Hence, any description (which preserves the reference flocks) is a Segre description, because by **3.0** and **1.8** (A) $\varsigma' = g$ is the required centralizer.

(Segre $\Rightarrow$ description) The same triviality cases for a geometric description in **2.3** also occur for a Segre description. In fact, in that proof we only have to disregard dilatation indicators outside the reference flocks. Then, our statement is obvious and we assume $X, Y \neq \emptyset$.

Any Segre description with non trivial dimensions is a geometric description, because a centralizer bijection $\varsigma' = g$ preserves all endomorphisms, $\widetilde{\varsigma'}: \mathcal{E} \twoheadrightarrow \mathcal{F}$, as we observed in the proof (analytic $\Rightarrow$ geometric) of **3.0**, and because (11) and the reference flock preservation imply the full preservation of dilatations, as we are going to show.

In fact, for each $e \in \Delta$ we have $S = e \cdot U \in F$, $\varsigma''(S) \in G$ and the corresponding dilatation $f \in \Gamma$, $\boldsymbol{r}''_V(f) = \varsigma''(S)$, such that by (11), (14) of [11], and (15) ibid. $\varsigma'(e(a)) = f(\varsigma'(a))$ for all $a \in A$, namely $\varsigma' \cdot e = f \cdot \varsigma'$. Conversely, since $\varsigma'': F \twoheadrightarrow G$, given any $f \in \Gamma$ we have such an $e \in \Delta$. Hence, by **2.1** (A)

$$\widetilde{\varsigma'} \cdot \boldsymbol{i}_\Delta : \Delta \rightarrowtail\!\!\twoheadrightarrow \Gamma \quad \text{and} \quad \boldsymbol{r}''_V \cdot \widetilde{\varsigma'} \cdot \boldsymbol{i}_\Delta = \varsigma'' \cdot \boldsymbol{r}'_U \cdot \boldsymbol{i}_\Delta. \tag{12}$$

Moreover, for each $a \in A$ consider the endomorphism $h_a \in \mathcal{E}$ defined by (14) of [11] as $h_a(c) = \chi_c(\boldsymbol{k}_a)$ for all $c \in A$. Since ς' is a centralizer bijection, $\widetilde{\varsigma'}: \mathcal{E} \rightarrowtail\!\!\twoheadrightarrow \mathcal{F}$, for each a by (15) ibid. there is an endomorphism $\ell_a \in \mathcal{F}$ and a matrix $M_a : Y \to B$ such that $\varsigma'(\chi_c(\boldsymbol{k}_a)) = \varsigma'(h_a(c)) = \ell_a(\varsigma'(c)) = \xi_{\varsigma'(c)}(M_a)$ for all $c \in A$. Given any $y \in Y$, take $c = \varsigma'^{-1}(V_y)$ and get $\chi_c(\boldsymbol{k}_a) = a$ for each $a \in A$, since $c \in \Phi'_U$ by **2.1** (A) of [11] and the preservation of reference flocks. Then, by (11) and (19) ibid. $\varsigma'(a) = \varsigma'(\chi_c(\boldsymbol{k}_a)) = \xi_{V(y)}(M_a) = M_a(y)$ for each $a \in A$ and every $y \in Y$, namely by (9) ibid. $M_a = \boldsymbol{\kappa}_{\varsigma'(a)}$.

Since M_a is constant with respect to any c, we got that $\varsigma'(\chi_c(\boldsymbol{k}_a)) = \xi_{\varsigma'(c)}(\boldsymbol{\kappa}_{\varsigma'(a)})$ for all $c \in A$. By the former of (12) this implies the full preservation of dilatations: $\widetilde{\varsigma'}(\chi_c \cdot \boldsymbol{k}) = \xi_{\varsigma'(c)} \cdot \boldsymbol{\kappa} \in \Gamma$ for all dilatation indicators c

and conversely by the definition symmetry in **3.3**. Notice that the latter of (12), together with **2.4** (G) of [11], implies that the isomorphism $\varsigma'' = t \cdot \boldsymbol{i}_F$ is the only surjection between scalars satisfying (11). ∎

3.5 Missing proofs in Linear Algebra.

(A) The three main definitions of a description and their characterization in **3.0** define what means to say "descriptions are a general universal notion" from a theoretical point of view. Their Segre variant serves more technical purposes: it shows how universal scalars work.

However, from a concrete point of view, also Segre descriptions serve to assess generality. In fact, (B) will show that they are a *formal* extension of the semi-linear transformations of vector spaces, whereas the general isomorphisms were not.

Conversely, one might like to check theoretically that the semi-linear transformations are the most general ones for vector spaces by proving that in vector spaces all descriptions have to be semi-linear transformations. Unfortunately, in spite that in (B) we will give a characterization, one cannot directly use it to prove this. In fact, we will show the lack of the proof of a renaming condition that Linear Algebra considered self-evident. Even the proofs of weaker conditions are missing.

The general condition for semi-linear transformations as in **III.1** of [1] requires that, for any vector-space scalar $s \in \mathrm{F}$ and any vector $v \in A$,

$$\sigma'(sv) = \sigma''(s)\sigma'(v), \tag{13}$$

where $\sigma' : A \twoheadrightarrow B$ denotes an isomorphism between the groups of the vector sums, $\sigma'' : \mathrm{F} \twoheadrightarrow \mathrm{G}$ an isomorphism between the fields concerned and, as usual for vector spaces, the two juxtapositions denote two *different* products of a scalar times a vector.

Semi-linear transformations relate two vector spaces regardless their reference frames. They can also relate their representations, after assuming $X = Y$, since they are projectivities and preserve dimensions, by setting

$$V = \sigma' \cdot U, \tag{14}$$

as we do for renamings by **3.1** (A). To focus this choice of Linear Algebra, we will call such transformations *semi-linear transformation between renamed reference frames.*

In **3.5** (A) of [11] $\mathrm{F} = \mathrm{G}$ was the set of complex numbers and $A = B$, while σ'' was conjugation and $\sigma' = g$ was vector conjugation. Notice that the only difference between (13) and (11) is the notation for the product scalar times vector, which in (13) is juxtaposition *both on the left and on the right.* Again, we have symmetry: σ' and σ'' define a semi-linear transformation iff σ'^{-1} and σ''^{-1} do.

Semi-linearity replaces the simple notion of a vector space as a universal algebra by a *split* one that concerns two algebraic structures, each of which undergoes its transformation. Moreover, the latter structure, the field as a division ring, is not a total (homogeneous) algebra.

On the contrary, our descriptions do not split universal algebras, though **2.4** (F,G) of [11] show that still an auxiliary (total) algebra always rises. This only occurs in the transformations: $\varsigma' = g$ is a Segre description only when there also is some $\varsigma'' = t \cdot \boldsymbol{i}_F$ for a matrix transformation $t = T_g$.

No partial algebra occurs in a Segre description: our scalars in $F \subseteq A^X$ or $G \subseteq B^Y$ merely form Abelian monoids as in **2.6** (B) of [11]. Neither sums (of scalars or vectors) nor their distributivities are needed, as scalars analytically represent certain endomorphisms.

Within Linear Algebra, the assumption (14) is not completely specified, because in $V = t(U)$ one should define what t is. Here, on the contrary, we have such t's by (32) of [11]. Then, through the above characterization **3.4** we can specify (14) as: *"if a Segre description is a semi-linear transformation, then it is between renamed reference frames".*

Yet, a possible proof of such a statement will not fully prove the generality of semi-linear transformations. To save the abstract coordinate-free approach *within Linear Algebra, its birth niche,* we need a stronger statement: *"if a Segre description concerns vector spaces, then it is a renaming".*

In fact, the latter proof could complete the next one in (B). Perhaps, we could get it by proving that *"the preservation of vector subspace dimensions, found at the end of* **3.2**, *implies the renaming condition".*

After counterexample **3.6** (A) of [11] and the uniqueness of **3.1** (C) ibid. the generality of renaming is untenable. Besides, in Linear Algebra, even the proof of the logical independence of the renaming condition is missing and one cannot get it as a new axiom. Once linear transformations (isomorphisms) were discarded, keeping their renaming feature needs some explanation.

(B) We prove that *every semi-linear transformation between renamed reference frames is a Segre description between the two corresponding based vector spaces. Conversely, whenever a Segre description is a renaming between two based vector spaces with dimensions greater than 1, it is a semi-linear transformation between the corresponding renamed reference frames.*

Proof. Since $U : X \to A$ is a vector-space basis, we have its coordinating function $c_U : A \Vdash\!\!\to \mathrm{F}^X$, such that, for each $v \in A$, $v = \sum_x v[x] U_x$, where $v[x]$ denotes the coordinate $(c_U(v))_x$. Given $V : X \to B$, we define $(c_V(v))_x$ likewise. Then, by (13) and (14) $\sigma'(v) = \sum_x \sigma'(v[x]U_x) = \sum_x \sigma''(v[x])\sigma'(U_x) =$

$\sum_x \sigma''(v[x])V_x$, since σ' preserves such (finite-range) sums, namely

$$(15) \qquad (\mathrm{c}_V(\sigma'(v)))_x = \sigma''(v[x]), \quad \text{for all } x \in X.$$

σ' is a centralizer bijection, because for each $e \in \mathcal{E}$ by (14) of [11] and (13) there is $L:X \to A$ such that $\sigma'(e(v)) = \sigma'(\chi_v(L)) = \sigma'(\sum_x v[x]L(x)) = = \sum_x \sigma'(v[x]L(x)) = \sum_x \sigma''(v[x])\sigma'(L(x)) = \sum_x(\mathrm{c}_V(\sigma'(v)))_x(\sigma' \cdot L)(x)) = = \xi_{\sigma'(v)}(M) = f(\sigma'(v))$ for all $v \in A$, where $M = \sigma' \cdot L : X \to B$ by (15) ibid. represents $f \in \mathcal{F}$. Conversely, since $\sigma' : A \twoheadrightarrow B$, given any such an f and M, there is L and $e \in \mathcal{E}$ such that $\sigma' \cdot e = f \cdot \sigma'$ as required by (4).

It preserves the reference flocks, because any $v \in \Phi'_U$ has the form $v = \sum_x c_x U_x$, where $c_x \in \mathrm{F}$ for all $x \in X$ and $\sum_x c_x = 1$. Hence, by (15) $\sigma'(v) = \sum_x \sigma''(c_x)V_x$, where again $\sum_x \sigma''(c_x) = \sigma''(\sum_x c_x) = 1$, since σ'' is a field isomorphism. Conversely, any $v' \in \Phi''_V$ is a $v' = \sigma'(v)$ and we can reverse these implications to get $v \in \Phi'_U$. Then, we can set $\varsigma' = \sigma'$.

Finally, let us define $\varsigma'' : \mathrm{F} \twoheadrightarrow G$. Consider the diagonal matrix isomorphism in **2.5** of [11], $D : \mathrm{F} \twoheadrightarrow F$. Likewise, consider $D' : \mathrm{G} \twoheadrightarrow G$. Then, $\varsigma'' = D' \cdot \sigma'' \cdot D^{-1}$ is a surjection onto G as required. By (13) it also satisfies (11), because for every $S \in F$ there is $s \in \mathrm{F}$ with $S = D_s$ such that by (27) ibid. $\varsigma'(\chi_a(S)) = \sigma'(sa) = \sigma''(s)\sigma'(a) = \xi_{\sigma'(a)}(D'_{\sigma''(s)}) = \xi_{\sigma'(a)}(D'_{\sigma''(D^{-1}(S))}) = \xi_{\varsigma'(a)}(\varsigma''(S))$ for all $a \in A$, since (27) ibid. concerns the latter Menger system too.

(Conversely) As the dimensions are the same by **3.2**, we take $Y = X$. To prove that $\varsigma' : A \twoheadrightarrow B$ is an isomorphism between the groups of the vector sum, we define an endomorphism application that performs sums in our space of dimension greater than 1. Take two different $x, y \in X$, their "sum vector" $u = U_x + U_y$ and, for all $a, b \in A$, the matrices $L : X \to A$ and $M : X \to B$ such that $L(x) = a$, $M(x) = \varsigma'(a)$, $L(y) = b$, $M(y) = \varsigma'(b)$ and $L(z) = \mathbf{0}$, $M(z) = \mathbf{0}$ elsewhere.

Notice that the latter matrix comes from the former by ς'–images, $M = \varsigma' \cdot L$, because ς' commutes with the null endomorphisms. Hence, by **3.4** and the renaming assumption $M = T_{\varsigma'}(L)$.

Then, $\varsigma'(a+b) = \varsigma'(\chi_u(L)) = \xi_{\varsigma'(u)}(\varsigma' \cdot L) = \xi_{\varsigma'(u)}(M)$ by **3.4** and (34) of [11]. Consider two coordinates of $\varsigma'(u)$ with respect to V: $c_x = \mathrm{c}_V(\varsigma'(u)))_x$ and $c_y = \mathrm{c}_V(\varsigma'(u)))_y$. Then, $\varsigma'(a+b) = c_x M_x + c_y M_y$. In particular this holds for $a = U_x$ and $b = \mathbf{0}$ as well as for $a = \mathbf{0}$ and $b = U_y$. Therefore, $c_x = c_y = 1$ by **3.1** (A) and

$$(16) \qquad \varsigma'(a+b) = \varsigma'(a) + \varsigma'(b) \quad , \text{for all } a, b \in A\ .$$

Also, $\sigma' = \varsigma'$ satisfies (13) with $\sigma'' = D'^{-1} \cdot \varsigma'' \cdot D$ as above. In fact, by (27) of [11], (11) and (27) ibid. $\sigma'(sv) = \varsigma'(\chi_v(D_s)) = \xi_{\varsigma'(v)}(\varsigma''(D_s)) =$

$\xi_{\varsigma'(v)}(D'(D'^{-1}(\varsigma''(D_s)))) = \xi_{\varsigma'(v)}(D'_{\sigma''(s)}) = \sigma''(s)\sigma'(v)$ for all $v \in A$ and $s \in \mathrm{F}$.

Such a $\sigma'' : \mathrm{F} \twoheadrightarrow \mathrm{G}$ also is a field isomorphism. In fact, it preserves multiplications, because, in addition to ς'', both D and D'^{-1} trivially do. Moreover, once we have defined sums on F and G by columnwise vector sums, we also easily find both that such sums are (universal) scalars and that D and D'^{-1} preserve sums. Hence, to prove that σ'' preserve sums, we only have to prove that ς'', or also $\varsigma'' \cdot D$, does.

By the bijective representation of scalars as dilatations in **2.4** (G) of [11] this is to prove that, given any $s', s'' \in \mathrm{F}$, for all $b = \varsigma'(a) \in B$, $\xi_b(\varsigma''(D(s' + s''))) = \xi_b(\varsigma''(D(s')) + \varsigma''(D(s'')))$, namely by (11) that $\varsigma'(\chi_a(D(s'+s''))) = \xi_b(\varsigma''(D(s')) + \varsigma''(D(s'')))$. In fact, the field distributivity gets $\varsigma'(\chi_a(D(s' + s''))) = \varsigma'((s' + s'')a) = \varsigma'(s'a + s''a) = \varsigma'(s'a) + \varsigma'(s''a) = \varsigma'(\chi_a(D(s'))) + \varsigma'(\chi_a(D(s''))) = \xi_b(\varsigma''(D(s'))) + \xi_b(\varsigma''(D(s''))) = \varsigma''(D(s'))b + \varsigma''(D(s''))b = (\varsigma''(D(s')) + \varsigma''(D(s'')))b = \xi_b(\varsigma''(D(s')) + \varsigma''(D(s'')))$ by (27) ibid., (16), (27) ibid. again, (11) and (27) ibid. twice again. ∎

3.6 Analytic spaces. Counterexample **3.6** of [11] denies universal generality to the abstract representation-free approach of Algebra. Then, one might look for a *subclass* of algebras, where such an abstract Algebra works.

This subclass is the one where all descriptions are renamings. Safely, it deserves further studies. Yet, its lack of proper descriptions does not save all conventional wisdom. The following example, a sort of converse of **3.6** (B) of [11], shakes the very notion of an algebra as a single concrete object.

Consider an algebra that has singleton bases as well as bases of $n > 1$ reference elements. The one of B. Jónsson & A. Tarski in [5] is the simplest non trivial one. Given a singleton basis $U : 1 \to A$, *any matrix transformation from U has to reach other singleton bases only.* In fact, by characterization **1.6** (A) it has to be a bijection $t : A^1 \twoheadrightarrow A^Y$ that retypes $\boldsymbol{K}$ as in **1.6** of [11]. This is impossible unless Y too is singleton, as shown in **1.7** (B) ibid..

Therefore, by **3.1** (A) any description from U is a renaming, because the reference flocks are singleton. Hence, we stay in our comfortable subclass, e.g. now we can say to have 1 as a dimension. The trouble is that this one-dimensioned space is not the whole algebra: its self-descriptions cannot reach invariant properties that need larger bases to be formalized. *In spite of the common carrier and operations* it cannot sense higher dimensions.

(Besides, to focus on an abstract algebra only is hopeless: even when we disregard representations, abstract representation-free properties can lack invariance. In **6.11** of [6] some well-known abstract properties of Universal Algebra fail after performing mere automorphisms.)

Then, from a concrete point of view, this algebra is not a single mathematical object, but a superposition of this one-dimensioned space with other space(s). Our analytic monoids or Menger systems, which formalize such "analytic spaces" together with the equivalence of **3.4** (D) of [11] or the category of **3.7** ibid., can peer at them. Yet, nothing can melt them to get such a thing as a free "algebra without the choice of a basis".

One might well dismiss our based algebra as a "paradoxical" one. Yet, some preliminary results in [8] show that its one-dimensional space provides both a word catenation monoid and a binary tree algebra with a natural common extension and hint that it can improve one of the best computer memory organization so far known. Its other spaces, as well as the dimensionless space of **3.6** (A) of [11], could likely provide us new methods for memory addressing.

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