DEMONSTRATIO MATHEMATICA
Vol. XLI No 4 2008

A. Batubenge, W. Sasin

AN APPROACH TO HAMILTONIAN MECHANICS ON GLUED SYMPLECTIC PSEUDOMANIFOLDS

Abstract. We define a class of Frölicher spaces locally diffeomorphic to Frölicher subspaces of the Euclidean space $\mathbb{R}^n$ and we call them pseudomanifolds. These differential constructs carry symplectic geometry so that Hamiltonian systems are naturally introduced. When gluing together symplectic pseudomanifolds which intersect transversally, it turns out that up to an equivalence relation, the glued space is symplectic and smooth integral curves extend to singular points.

1. Introduction

This study is an application of Hamiltonian mechanics to a particular class of the so called smooth spaces or Frölicher spaces (also denoted by $\mathbb{F}$-spaces). We refer to a Frölicher structure on a set M as a pair $(\mathcal{C}_M, \mathcal{F}_M)$ of functions $c : \mathbb{R} \to M$ called curves and real-valued functions $f : M \to \mathbb{R}$ such that the following compatibility condition holds:

- $\Gamma\mathcal{F}_M = \{c : \mathbb{R} \to M / f \circ c \in C^\infty(\mathbb{R}) \text{ for all } f \in \mathcal{F}_M\} := \mathcal{C}_M$
- $\Phi\mathcal{C}_M = \{f : M \to \mathbb{R} / f \circ c \in C^\infty(\mathbb{R}) \text{ for all } c \in \mathcal{F}_M\} := \mathcal{F}_M$.

The triplet $(M, \mathcal{C}_M, \mathcal{F}_M)$ is called a Frölicher space. A Frölicher space carries two topologies. One is the initial topology $\tau_{\mathcal{F}}$ generated by the set $\mathcal{F}_M$. This is the weakest topology in which all the functions are continuous. It has subbasis and basis the collections $\{f^{-1}(0,1)\}_{f\in\mathcal{F}_M}$ and $\{f^{-1}(0,\infty)\}_{f\in\mathcal{F}_M}$ respectively. The other is the topology $\tau_{\mathcal{C}}$ generated by the set $\mathcal{C}_M$, the open sets of which are subsets $O \subset M$ such that $c^{-1}(O)$ are open in $\mathbb{R}$. It is easy to see that $\tau_{\mathcal{F}} \subseteq \tau_{\mathcal{C}}$. Except otherwise indicated, Frölicher spaces under consideration in this work are balanced spaces, that is, $\tau_{\mathcal{C}} = \tau_{\mathcal{F}}$ (See [3]). For basics on Frölicher spaces, see [6], [7], [4], [2].

Working with the smooth Frölicher structure is particularly interesting as geometric objects mostly function spaces, or spaces with many singularities or

2000 *Mathematics Subject Classification*: 53D99, 37C05, 37J05, 93C25.
Key words and phrases: Frölicher space, pseudomanifold, constant dimension.

failing to be smooth manifolds can be naturally endowed with this structure. Also we point out that Frölicher smooth functions and curves are globally defined while they have local properties. So the geometry on them stands for a possible generalization of the manifold theory. Moreover, it was proved that the category of Frölicher spaces contains that of convenient spaces (see [7]), that it is a full subcategory of differential spaces in the sense of Sikorski (see [4]) and is embedded in the category of diffeological spaces (see [16]).

The n-dimensional Euclidean space is a natural example of a Frölicher space whose smooth structure is formed by all the C^∞ curves into and all the C^∞ real-valued functions. In this paper we shall deal with those Frölicher spaces which are locally diffeomorphic to subsets of $\mathbb{R}^n$. We call them pseudomanifolds. We show that this class of smooth spaces forms a framework for the modelling of mechanical systems. We will particularly show that when two connected Hausdorff symplectic pseudomanifolds that intersect transversally glue in a smooth way, the resulting Frölicher space has an induced symplectic structure. Then we shall write Hamiltonian mechanics on the generated space. As an application, we will be concerned with the gluing of open manifolds at a point.

2. Smooth maps and diffeomorphisms

DEFINITION 2.1. A map $\varphi : (M, \mathcal{C}_M, \mathcal{F}_M) \to (N, \mathcal{C}_N, \mathcal{F}_N)$ between Frölicher spaces is smooth if $\varphi^*\mathcal{F}_N \subseteq \mathcal{F}_M$. A smooth map with a smooth inverse is called a diffeomorphism.

It is easy to see that φ is smooth if and only if $\varphi_*\mathcal{C}_M \subseteq \mathcal{C}_N$. Combining the above statements yields an additional characterisation of a smooth map as follows

$$f \circ \varphi \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$$

for all $f \in \mathcal{F}_N$ and $c \in \mathcal{C}_M$.

Note that in this work we shall simply say smooth for means of smooth in the Frölicher sense and Frölicher space M for $(M, \mathcal{C}_M, \mathcal{F}_M)$ if there is no fear of confusion. A Frölicher space M is said to be locally diffeomorphic to another Frölicher space N if, for every $x \in M$, there exists a neighborhood U of x diffeomorphic to a subset V of N. In [2] and [8] we show the following two results.

LEMMA 2.1. *Let N be a Frölicher space and M be a set. Let $(\mathcal{C}_M, \mathcal{F}_M)$ be the Frölicher structure induced on the set M via maps $f_i : M \to N, i \in I$. Assume that the map $\varphi : M \to N^I$, given by $\varphi(x) = (f_i(x))_I$, is one-to-one. Then φ is a diffeomorphism onto its range $\varphi(M)$ which is a Frölicher subspace of N^I.*

Proof. Let $c : \mathbb{R} \to M$ be a structure curve on M. Then

$$\varphi \circ c(t) = (f_i \circ c(t))_I$$

for all $t \in \mathbb{R}$. Since the structure on N^I is generated by the family $\{g \circ \pi_i : g \in \mathcal{F}_N,\ i \in I\}$, it follows that $\varphi \circ c : \mathbb{R} \to \varphi(M)$ is a smooth curve on $\varphi(M)$. Hence φ is smooth.

Now, let $(x_i)_I \in \varphi(M)$. It is clear that

$$g \circ f_i \circ \varphi^{-1}((x_i)_I) = g \circ \pi_i \circ \varphi \circ \varphi^{-1}((x_i)_I) = g \circ \pi_i((x_i)_I).$$

Thus φ^{-1} is smooth. ∎

COROLLARY 2.1. *Let M be a set, and let $f_1, \ldots, f_n : X \to \mathbb{R}$ be real-valued functions on M such that the map $\varphi : M \to \mathbb{R}^n$, $\varphi(x) = (f_1(x), \ldots, f_n(x))$, is one-to-one. If $(\mathcal{C}_M, \mathcal{F}_M)$ is a Frölicher structure generated by the family $\{f_1, \ldots, f_n\}$, then φ is a diffeomorphism onto the subspace $\varphi(M)$ of $\mathbb{R}^n$.*

DEFINITION 2.2. A Frölicher space $(M, \mathcal{C}_M, \mathcal{F}_M)$ is called a pseudomanifold if $(M, \mathcal{C}_M, \mathcal{F}_M)$ is locally diffeomorphic to $(\mathbb{R}^n, \mathcal{C}, \mathcal{F})$. That is, for every $x \in M$, there exist a $\tau_{\mathcal{F}_M}$-open neighborhood $\mathcal{U}$ of x and a diffeomorphism φ of $\mathcal{U}$ onto the Frölicher subspace $V := \varphi(\mathcal{U}) \subseteq \mathbb{R}^n$.

Note that Corollary 2.1 above provides various examples of pseudomanifolds. We may observe that the Euclidean Frölicher space $\mathbb{R}^n$ with its canonical structure where curves and functions are C^∞ functions in the usual sense, as well as a smooth manifold, are examples of pseudomanifolds modelled on open sets. In the near future, we shall present different types of pseudomanifolds.

DEFINITION 2.3. Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be a pseudomanifold. Let $x \in M$ be a point, U an open neighborhood of x and φ a diffeomorphism of U onto $V \subset \mathbb{R}^n$. The pair (U, φ) is called a chart on $(M, \mathcal{C}_M, \mathcal{F}_M)$ at x, U is the domain of the chart.

2.1. Remarks

a. The definition of a pseudomanifold stated above does not require n to be a fixed positive integer. If this occurs, we shall call M a pseudomanifold of dimension n or an n-pseudomanifold. Furthermore, if M is locally diffeomorphic to open subsets of $\mathbb{R}^n$ then it is easy to see that the smooth structure under consideration and the manifold structure are coincident. Finally, let us note that for the purposes of symplectic geometry, one may require the modelling subspaces of $\mathbb{R}^n$ to be star-shaped regions of $\mathbb{R}^n$, or closed subspaces of $\mathbb{R}^n$ of constant dimension with nonempty interior. We refer the reader to the literature (see [9]) for more about the concept of a differential space of constant dimension.

b. An operational tangent vector at $x \in M$ is a derivation on $\mathcal{F}_M$, that is a map $v_x : \mathcal{F}_M \to \mathbb{R}$ satisfying Leibniz property. That is, $v_x(fg) = g(x)v_x(f) + f(x)v_x(g)$, where $f, g \in \mathcal{F}_M$. The set of all tangent vectors at $x \in M$ is denoted by T_xM and is called the tangent space on M at x. In [2] and [4], it is shown that T_xM is a linear Frölicher space whose smooth structure is generated by functionals $(df)_x$ defined by setting

$$(df)_x(v) = v(f). \tag{1}$$

A Frölicher space has another tangent structure defined via structure curves as follows. Let $a \in \mathbb{R}$ and $x \in M$. Denote by $C_M^{a,x}$ the set of all smooth curves $c \in \mathcal{C}_M$ such that $c(a) = x$. By a kinematic tangent vector (word borrowed from [7]) to the space M with foot point a we mean

$$X_{c,a}(f) := \frac{d}{dt}(f \circ c)_{|t=a} = df(c(a)), \tag{2}$$

where $c \in C_M^{a,x}$. A tangent cone space at x is the set of all kinematic tangent vectors at x. It is denoted by T_xCM and is not necessarily a linear Frölicher space(see [4]). A straightforward consequence is that $T_xCM \subset T_xM$ for all $x \in M$ and that both operational and kinematic tangent spaces coincide if $dim\ T_xM$ is constant at each $x \in M$. In what follows, a tangent space shall mean the operational one if there is no confusion.

3. Symplectic pseudomanifolds

DEFINITION 3.1. Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be a pseudomanifold of dimension n. A symplectic form ω on M is an exterior form which is closed and nondegenerate. The construct $((M, \mathcal{C}_M, \mathcal{F}_M), \omega)$, where $(M, \mathcal{C}_M, \mathcal{F}_M)$ is a pseudomanifold and ω a symplectic form defined on $(M, \mathcal{C}_M, \mathcal{F}_M)$ is called a symplectic pseudomanifold.

As a symplectic differential space (see [2],[4]), a symplectic n-pseudomanifold (M, ω) is an even dimensional space which inherits the well-known exterior algebra on differential spaces, and the equality dim $T_pM = n$ for all $p \in M$, with $n = 2m, m \in \mathbb{N}, m \neq 0$ holds true.

3.1. Normal form for symplectic forms

In [9] we read

LEMMA 3.1. *Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be a Frölicher space. The following conditions are equivalent:*

(i) *n tangent vectors $v_1, \ldots, v_n \in T_xM$ are linearly independent;*
(ii) *for all smooth functions $f \in \mathcal{F}_M$, the map*

$$\theta : \mathcal{F}_M \to \mathbb{R}^n; \quad f \mapsto \theta(f) = (v_1(f), \ldots, v_n(f))$$

is a surjection;

(iii) *there exist n smooth functions* $f_1, \dots, f_n \in \mathcal{F}_M$ *such that* $v_i(f_j) = \delta_{ij}$, *where* δ_{ij} *is the Kronecker symbol;*
(iv) *there exist n smooth functions* $f_1, \dots, f_n \in \mathcal{F}_M$ *such that* $\det(v_i(f_j)) \neq 0$.

Taking into account (i) and (iii) and using Equation 1 above, one has $\delta_{ij} = v_j(f_i) = (df_i)_x(v_j)$ if and only if $(df_i)_x$ form a basis in the dual space. That is, the forms $(df_i)_x$ are linearly independent.

DEFINITION 3.2. A collection of structure functions $\{f_1, f_2, \dots, f_n\}$ on a pseudomanifold M is called independent at $x \in M$ if $\{(df_1)_x, (df_2)_x, \dots, (df_n)_x\}$ is a linearly independent subset of the cotangent space on M at x.

In the category of differential spaces, it is proved that independent functions $f_1, f_2, \dots, f_n$ at x form an differential basis if any function $g \neq f_i$ in the subalgebra of smooth functions on U is of the form $\omega \circ (f_1, f_2, \dots, f_n)$, where $\omega \in C^\infty(\mathbb{R}^n, \mathbb{R})$. As from [9], one may conclude that on an n-pseudomanifold there are n independent structure functions at each point. That is, there exists a basis of n covectors in the cotangent space at x for all $x \in M$. Equivalently, there is a local basis $\mathcal{B} = \{X_1, X_2, \dots, X_n\}$ of the module of tangent vector fields at x.

THEOREM 3.1. *Let* $((M, \mathcal{C}_M, \mathcal{F}_M), \omega)$ *be a symplectic pseudomanifold of dimension* $2n$. *For every point* $x \in M$ *there exist an open neighborhood* U *of* x *in* M *and* $2n$ *smooth functions* $q^1, \dots, q^n, p^1, \dots, p^n \in \mathcal{F}_x U$, *where* $\mathcal{F}_x U$ *is the subalgebra of smooth functions on* U, *such that*

$$\omega_{|U} = \sum_{i=1}^{n} dq^i \wedge dp^i.$$

The latter form is called the canonical (-normal or Darboux-) form of ω.

Proof. Since $(M, \mathcal{C}_M, \mathcal{F}_M)$ has dimension $2n$, it turns out that for any $x \in M$ there exist an open neighborhood U of x, a local basis $\{W_1, \dots, W_n, V_1, \dots, V_n\} \subseteq \mathfrak{X}(U)$ and smooth functions $e^1, \dots, e^n, e^{n+1}, \dots, e^{2n}$ in the subalgebra $\mathcal{F}_x(U)$ such that

$$W_i(e^j) = \delta_{ij}, \quad W_i(e^{n+j}) = 0, \quad V_i(e^j) = 0, \quad V_i(e^{n+j}) = \delta_{ij},$$

where δ_{ij} is the Kronecker delta symbol. Then

$$de^1, \dots, de^n, de^{n+1}, \dots, de^{2n}$$

form the basis dual to $W_1, \dots, W_n, V_1, \dots, V_n$ in the dual space. So one can choose functions $q^1, \dots, q^n, p^1, \dots, p^n$ in $\mathcal{F}_x(U)$ such that $\{dq^1, \dots, dq^n,$

$dp^1, \ldots, dp^n\}$ form the basis in which $\omega_{|U}$ has the normal form

$$\omega_{|U} = \sum_{i=1}^{n} dq^i \wedge dp^i. \blacksquare$$

LEMMA 3.2. *Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be a Frölicher space and $\{f_1, \ldots, f_n\} \subset \mathcal{F}_x(U)$ be a collection of structure functions such that one of them is injective in a neighborhood U of $x \in M$. Then the map $\psi := (f_1, \ldots, f_n)$ is a diffeomorphism of $(U, \mathcal{C}_U, \mathcal{F}_U)$ onto $(\psi(U), \mathcal{C}_{\psi(U)}, \mathcal{F}_{\psi(U)})$.*

Proof. Let ψ be the map $\psi : U \to \mathbb{R}^n$ defined by

$$\psi(x) := (f_1(x), \ldots, f_n(x)) \quad \forall x \in M,$$

It is clear that ψ is injective. Since the subset $\psi(U) \subsetneq \mathbb{R}^n$ above carries a Frölicher structure, it follows from Lemma 2.1 above that ψ is a diffeomorphism. $\blacksquare$

LEMMA 3.3. *In the conditions of Lemma 3.2, the associated tangent map*

$$\psi_{*x} : T_x M \to T_{\psi(x)} \psi(M)$$

is an isomorphism of linear spaces.

Proof. The map ψ_{*x} is linear. By Lemma 3.2 $\psi_{|U}$ is a diffeomorphism. Then ψ^{-1} exists and $(\psi^{-1})_{*\psi(x)} = (\psi_{*x})^{-1}$. That is, ψ_{*x} is an isomorphismon on U.

Now assume that X is a smooth vector field on M and Y is a vector field on $\psi(M)$. It follows that for all $w \in T_{\psi(x)}\psi(M)$,

$$w = Y(\psi(x)) = \psi_{*x}(X(\psi^{-1}(\psi(x)))) = \psi_{*x}X(x).$$

Taking $v := X(x) \in T_x M$ proves the required result. $\blacksquare$

PROPOSITION 3.1. *Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be a pseudomanifold of dimension n and $N \subseteq M$. If $(N, \mathcal{C}_N, \mathcal{F}_N)$ is a Frölicher n-dimensional subspace imbedded in M, then every local basis of smooth vector fields $\{W_1, \ldots, W_n\}$ on M induces a local basis of smooth vector fields $\{V_1, \ldots, V_n\}$ on N.*

Proof. Let $x \in M$ and U be an open neighborhood of x in M such that $\{W_1, \ldots, W_n\}$ is a local basis over U. Assume without loss of generality that the inclusion map $\iota_N : N \hookrightarrow M$ is smooth and $(\iota_N)_{*x} : T_x N \to T_x M$ ($\iota(x) = x$ for all $x \in N$) is an isomorphism so that ι_N is an embedding. Consider $V_1, \ldots, V_n$ in $\mathfrak{X}(U \cap N)$ as candidates for the local basis on N. We note that

$$V_i(x) = (\iota_N)^{-1}_{*x} W_i(x).$$

Using Equation 1 above, we have

$$\begin{aligned}V_i(x)(f_{|U\cap N}) &= d(f_{|U\cap N})(V_i(x))\\ &= d(f_{|U\cap N})(\iota_N)^{-1}_{*x}W_i(x)\\ &= (\iota_{N\cap U})^*d(f)(\iota_N^{-1})_{*x}W_i(x)\\ &= (W_i f)_{|N\cap U}.\end{aligned}$$

That is, V_i are smooth tangent vector fields. Thus, $V_1,\dots,V_n$ form a local basis on N. ∎

COROLLARY 3.1. *Let $((M,\mathcal{C}_M,\mathcal{F}_M),\omega)$ be a pseudomanifold of dimension $2n$ endowed with symplectic structure ω. Let $N\subseteq M$ a subset of M. If $(N,\mathcal{C}_N,\mathcal{F}_N)$ is a Frölicher subspace of constant maximal dimension then there exists on N a symplectic structure induced by ω.*

Proof. Let $\iota : N \hookrightarrow M$ be the smooth inclusion map. That is, ι is the identity map of M restricted to N and for all $x\in N$ the equality $\dim T_xN = \dim T_xM$ holds and $\iota_{*x} : T_xN \to T_xM$ is an isomorphism of vector spaces. Hence,

$$\dim T_xM = 2n = \dim T_xN$$

for all $x\in N$. Then $\dim N = 2n$. Furthermore, for all $v_1, v_2 \in T_pM$ one has

$$\begin{aligned}\iota^*\omega(v_1,v_2) &= \omega(\iota_{*p}v_1,\iota_{*p}v_2)\\ &= \omega(v_1,v_2).\end{aligned}$$

Hence, the pullback $\iota^*\omega$ is a nondegenerate 2-form on N. One concludes that N together with this pullback is a symplectic Frölicher space, turning ι into a symplectic transformation on M. ∎

A smooth map $\varphi : M_1 \to M_2$ on symplectic pseudomanifolds (M_1,ω_1) and (M_2,ω_2) is said to be symplectic or canonical if $\varphi^*\omega_2=\omega_1$. That is, for all $x\in M_1$ and all $v,w\in T_xM_1$ one has the following identity

$$\omega_{1x}(v,w) = \omega_{2\varphi(x)}(\varphi_{*x}v,\varphi_{*x}w),$$

where ω_{1x} is the evaluation of ω_1 at the point x, $\omega_{2\varphi(x)}$ is the evaluation of ω_2 at the point $\varphi(x)$ and φ_{*x} is the tangent (or derivative) of φ at x.

It follows from the above definition that $\varphi^*\omega_{2|\varphi(U)} = \omega_{1|U}$ always holds for symplectic pseudomanifolds and turns the chart φ into a canonical diffeomorphism. The set of all canonical diffeomorphisms of a symplectic pseudomanifold M is a smooth subgroup of the smooth group Diff(M) of all diffeomorphisms of M with respect to the composition of maps [6]. This subgroup is denoted by $\mathrm{Diff}_{can}(M)$.

The matter discussed above is a framework for a generalized formalism of mechanical systems on smooth spaces. For instance, Hamiltonian systems can be described as follows in the general setting of Frölicher spaces, and restricted to the class of pseudomanifolds.

4. Hamiltonian systems on pseudomanifolds

Let (M, ω) be an n-pseudomanifold. Let $\mathfrak{X}(M)$ and $\Omega^1(M)$ be the $\mathcal{F}_M$-modules of smooth vector fields and 1-forms on M respectively. From the literature on symplectic geometry, it is known that ω induces a vector bundle isomorphism $\omega^\flat : TM \to T^*M$ that corresponds to an $\mathcal{F}_M$-module isomorphism also denoted $\omega^\flat : \mathfrak{X}(M) \to \Omega^1(M)$. But in the next section, we will see that if M is a glued space which may contain singular points, then a nonzero vector field can map to a zero 1-form. Now consider the map

$$\sigma : (\mathcal{F}_M, \{\ ,\ \}) \to (\mathfrak{X}(M), [\ ,\ ])$$

sending $H \in \mathcal{F}_M$ to a vector field $X_H := \sigma(H)$ and satisfying

$$X_H = (\omega^\flat)^{-1}(dH),$$

where $\{\ ,\ \}$ and $[\ ,\ ]$ are Poisson bracket and Lie bracket respectively. The vector field X_H generated in this way is uniquely determined by the equation

$$\omega(X_H, \cdot) = dH(\cdot).$$

Recall that the field X_H surely exists if the associated map $\omega^\flat$ is bijective, or on a restricted domain in the general case and the uniqueness follows from the nondegeneracy of ω. The vector field X_H attached to a function $H \in \mathcal{F}_M$ such that $i_{X_H}\omega = dH$ is called the global Hamiltonian vector field and H is the energy function for the mechanics, while X_H is considered as the Hamiltonian system whose Hamiltonian function H carries the total energy of the system. The triplet $((M, \mathcal{C}_M, \mathcal{F}_M), \omega, H)$ is said to be a dynamical (-Hamiltonian in this case) system. The 1-form generated by the symplectic form ω and the function H is dH. That is,

$$i_{X_H}\omega = \sum_{i=1}^{n} (\partial_{V_j} H.dp_j + \partial_{W_j} H.dq^j)$$

if $X_H = \sum_{i=1}^{n} (r_i W_i + s_i V_i)$ in the basis $\{W_1, \dots, W_n, V_1, \dots, V_n\}$.

PROPOSITION 4.1. *Let (M, ω) be a symplectic $2n$-pseudomanifold. The Hamiltonian vector field X_H associated with the Hamiltonian function $H : M \to \mathbb{R}$ can be written as*

$$X_H = \sum_{i=1}^{n} (-\partial_{W_i} H \cdot V_i + \partial_{V_i} H \cdot W_i)$$

with respect to a local basis $\{W_1, \ldots, W_n, V_1, \ldots, V_n\}$ *on* $\mathfrak{X}(U)$*, where* U *is an open neighborhood of a point* $p \in M$*.*

Proof. In the basis $\{W_1, \ldots, W_n, V_1, \ldots, V_n\}$ the vector field X_H can be written as

$$X_H = \sum_{i=1}^{n} (r_i W_i + s_i V_i).$$

Since

$$i_{X_H}\omega = \sum_{j=1}^{n} (\partial_{V_j} H \cdot dp_j + \partial_{W_j} H \cdot dq^j),$$

one has the identification

$$(dq^j \wedge dp_j)(r_j W_j + s_j V_j, \cdot) = \partial_{V_j} H \cdot dp_j + \partial_{W_j} H \cdot dq^j.$$

Expanding the left-hand side and using the duality between W_j and V_j, dq^j and dp_j, we have

$$\begin{aligned}(r_j W_j + s_j V_j)(dq^j \wedge dp_j) &= (r_j W_j(dq_j)) \wedge dp_j - (r_j W_j(dp_j)) \wedge dq^j \\ &\quad + (s_j V_j(dq^j)) \wedge dp_j - (s_j V_j(dp_j)) \wedge dq^j \\ &= r_j dp_j - s_j dq^j.\end{aligned}$$

It follows from the equation above that

$$r_j dp_j - s_j dq^j = \partial_{V_j} H \cdot dp_j + \partial_{W_j} H \cdot dq^j$$

so that

$$r_j = \partial_{V_j} H, \qquad s_j = -\partial_{W_j} H$$

which proves the result. ∎

Now every integral curve c of the Hamiltonian vector field X_H should have $2n$ components $q^1(t), \ldots, q^n(t), p_1(t), \ldots, p_n(t)$ satisfying the identities

$$\dot{q}_j = \partial_{V_j} H \circ c \quad \text{and} \quad \dot{p}_j = -\partial_{W_j} H \circ c$$

with respect to the local basis $\{W_1, \ldots, W_n, V_1, \ldots, V_n\}$. A vector field on a symplectic pseudomanifold (M, ω) is said to be locally Hamiltonian if at every point x of M there is an open neighborhood $U \ni x$ such that X restricted to U is Hamiltonian. Hence $X = X_H$ and H is the Hamiltonian function associated with X_H. That is,

$$i_{X|U}\omega = dH_{|U}.$$

A smooth function f on a pseudomanifold is said to be a first integral of a vector field $X_h = \{h, \cdot\}$ if $\{h, f\} = 0$. The standard results in the theory of symplectic manifolds hold true in the setting of symplectic n-pseudomanifolds. They emphasize the conservative properties of Hamiltonian vector fields in this smooth setting. It is easily verified that a Hamiltonian function H is constant along the trajectories of the flow of $X = X_H$ and the energy is conserved in the system. That is, H is a first integral of X_H as we can write

$$X_H(H) = 0.$$

5. Gluing symplectic pseudomanifolds

In what follows, we consider two connected Hausdorff symplectic $2n$-pseudomanifolds $(M_1, \mathcal{C}_1, \mathcal{F}_1)$ and $(M_2, \mathcal{C}_2, \mathcal{F}_2)$ that glue in the sense of Sasin (see [12] and [13]). The gluing diffeomorphism $h : M_{01} \to M_{02}$ maps points of a subset of $(M_{01}, \mathcal{C}_{01}, \mathcal{F}_{01})$ onto those of a subset $(M_{01}, \mathcal{C}_{02}, \mathcal{F}_{02})$ in such a way as to obtain a subset $\Delta \equiv \pi_{\rho_h}(M_{01}) = \pi_{\rho_h}(M_{02})$, where π_{ρ_h} is the quotient map $(M_1, \mathcal{C}_1, \mathcal{F}_1) \sqcup (M_1, \mathcal{C}_2, \mathcal{F}_2) \to (M_1 \cup_h M_2, \mathcal{C}_1 \cup_h \mathcal{C}_2, \mathcal{F}_1 \cup_h \mathcal{F}_2)$ identifying every $x \in M_{01}$ with $h(x) \in M_{02}$ and leaving the other points fixed. Recall that M_{01} and M_{02} are initial objects and $M_1 \cup_h M_2$ is a final object in the category of Frölicher spaces. The Frölicher space $M := (M_1 \cup_h M_2, \mathcal{C}_1 \cup_h \mathcal{C}_2, \mathcal{F}_1 \cup_h \mathcal{F}_2)$ provided with the final smooth structure obtained by means of the quotient map π_{ρ_h} is called the glued Frölicher space of M_1 and M_2 along the diffeomorphism h. More on gluing differential spaces can be found in [12], [13], and [14]. Let us then note the following:

1. $f \in \mathcal{F}_M$ if and only if $f|M_1 \in \mathcal{F}_{M_1}$ or $f|M_2 \in \mathcal{F}_{M_2}$.
2. $T_pM = (\iota_1)_{*p}T_pM_1 \oplus (\iota_2)_{*p}T_pM_2$ for $p \in \Delta$.
3. $(\iota_\Delta)_{*p}T_p\Delta = (\iota_1)_{*p}T_pM_1 \cap (\iota_2)_{*p}T_pM_2$, where $\iota_\Delta : \Delta \hookrightarrow M$ is the inclusion map.

DEFINITION 5.1. Let $f_1 \in \mathcal{F}_{M_1}$ and $f_2 \in \mathcal{F}_{M_2}$ such that $f_1|\Delta = f_2|\Delta$, the smooth map $f_1 \sqcup f_2 : M_1 \cup_h M_2 \to \mathbb{R}$ defined by

$$f_1 \sqcup f_2|M_i = f_i \quad i = 1, 2 \tag{3}$$

is called a conjunction map.

Clearly, $f_1 \sqcup f_2$ is smooth by construction as f_1 and f_2 are smooth by assumption. Also, it is easy to observe that $f_1 \sqcup f_2 \in \Phi\bar{\mathcal{C}}_0 = \mathcal{F}_M$.

DEFINITION 5.2. A vector field $X \in \mathfrak{X}(M)$ is said to be tangent to the subspace Δ if for any point $p \in \Delta$ there is a tangent vector $v \in T_pM$ such that

$$X(p) = (\iota_\Delta)_{*p}v.$$

We shall denote by $\mathfrak{X}_\Delta(M)$ the set of all smooth vector fields tangent to M which are also tangent to Δ.

LEMMA 5.1. *Let $X \in \mathfrak{X}_\Delta(M)$. Let $Y : \Delta \to T\Delta$ be a vector field defined by*

$$(\iota_\Delta)_{*p} Y(p) = X(p), \quad p \in \Delta, \tag{4}$$

then Y is smooth as a tangent vector field on Δ and Y is unique.

Proof. Observe that since X is smooth, it follows by construction that Y is also smooth. Moreover, one can see that ι_Δ is an embedding, so Y is unique since $(\iota_\Delta)_{*p}$ is an isomorphism of linear Frölicher spaces. ∎

DEFINITION 5.3. The vector field Y defined in Equation (4) above is called the restriction of $X \in \mathfrak{X}_\Delta(M)$ to the subspace Δ and is denoted by $X|\Delta$.

PROPOSITION 5.1. *If $(M, \mathcal{C}_M, \mathcal{F}_M)$ is a glued pseudomanifold of M_1 and M_2 along h, then*

$$\mathfrak{X}(M) = \mathfrak{X}_\Delta(M). \tag{5}$$

Proof. The inclusion $\mathfrak{X}_\Delta(M) \subseteq \mathfrak{X}(M)$ is obvious. We need only show the reverse inclusion. Let $X \in \mathfrak{X}(M)$, then $X \in \mathfrak{X}_{M_j \backslash \Delta}(M)$, for $j = 1, 2$. This follows from the assumption that Δ is closed as boundary, making $M_j \backslash \Delta$ an open set. Hence $X \in \mathfrak{X}_{cl(M_j \backslash \Delta)}(M)$. That is, $X \in \mathfrak{X}_{M_j}(M)$. It follows that $X(p) \in (\iota_j)_{*p} T_p M_j$ whenever $p \in \Delta$, $j = 1, 2$. So

$$X(p) \in (\iota_1)_{*p} T_p M_1 \cap (\iota_2)_{*p} T_p M_2,$$

which is equivalent to

$$X(p) \in (\iota_\Delta)_{*p} T_p \Delta.$$

Thus $X \in \mathfrak{X}_\Delta(M)$, which proves the reverse inclusion. ∎

DEFINITION 5.4. A pair (X_1, X_2) of vector fields $X_1 \in \mathfrak{X}_\Delta(M_1)$ and $X_2 \in \mathfrak{X}_\Delta(M_2)$ is said to be consistent on Δ if $X_1|\Delta = X_2|\Delta$. The unique vector field denoted by $X_1 \sqcup X_2$ such that

$$X_1 \sqcup X_2 | M_i = X_i, \quad i = 1, 2$$

is called the conjunction of vector fields X_1 and X_2.

PROPOSITION 5.2. *Let $\mathfrak{X}_\Delta(M_{1,2}) = \{(X_1, X_2) \in \mathfrak{X}_\Delta(M_1) \times \mathfrak{X}_\Delta(M_2)\}$ be the set of all pairs of vector fields $X_1 \in \mathfrak{X}_\Delta(M_1)$ and $X_2 \in \mathfrak{X}_\Delta(M_2)$ which are consistent on Δ. Then the correspondence*

$$\mathfrak{X}(M) \to \mathfrak{X}_\Delta(M_{1,2})$$

is bijective.

Proof. The proof is a straightforward consequence of Proposition 5.1 and Definition 5.2 above. ∎

Proposition 5.3. *Let $(M = M_1 \cup_h M_2, \mathcal{C}_M, \mathcal{F}_M)$ be the pseudomanifold following a transversal intersection along h. Let $c : \mathbb{R} \to M$ be a smooth curve on M such that $c(t)$ lies in M_1 for $t < 0$, $c(t)$ lies in M_2 for $t > 0$ and $c(0) \in \Delta$. Then*

$$c'(0) \in (\iota_\Delta)_{*c(0)}(T_{c(0)}\Delta).$$

Proof. Let c_- denote the restriction of c to $(-\infty, 0]$ and c_+ the restriction of c to $[0, +\infty)$. Since c is assumed smooth, it turns out that

$$c'(0) = (\iota_1)_{*c(0)}c'_-(0) = (\iota_2)_{*c(0)}c'_+(0).$$

It follows that

$$c'(0) \in (\iota_1)_{*c(0)}T_{c(0)}M_1 \;\cap\; (\iota_2)_{*c(0)}T_{c(0)}M_2 = (\iota_\Delta)_{*c(0)}(T_{c(0)}\Delta). \qquad (6)$$ ∎

Corollary 5.1. *For every smooth vector field $X \in \mathfrak{X}(M_1 \cup_h M_2)$ there is an integral curve at singular points.*

Proof. We only observe that a piecewise curve defined by

$$c(t) = \begin{cases} c_1(t) & \text{for } t \in (-\infty, t_0] \\ c_2(t) & \text{for } t \in [t_0, +\infty) \end{cases}$$

such that $c_1 : [t_0, +\infty) \to M_1$ is an integral curve for $X_1 \in \mathfrak{X}_\Delta(M_{1,2})$. On the other hand $c_2 : (-\infty, t_0] \to M_2$ is an integral curve for $X_2 \in \mathfrak{X}_\Delta(M_{1,2})$. Then $c_1(t_0) = c_2(t_0) \in \Delta$ is a smooth integral curve for $X \in \mathfrak{X}(M)$. ∎

Definition 5.5. Two k-forms $\omega_1 \in \Omega^k(M_1)$ and $\omega_2 \in \Omega^k(M_2)$ are said to be consistent on Δ if

$$\iota_{1\Delta}^*\omega_1 = \iota_{2\Delta}^*\omega_2, \qquad (7)$$

where $\iota_{1\Delta}$ (resp. $\iota_{2\Delta}$) is the inclusion map of Δ into M_1 (resp. M_2).

Let $\bar{\Omega}^k(M) = \{\omega_1 \sqcup \omega_2 : \omega_1 \in \Omega^k(M_1),\ \omega_2 \in \Omega^k(M_2);\ \iota_{1\Delta}^*\omega_1 = \iota_{2\Delta}^*\omega_2\}$ denote the set of all Δ-consistent k-forms on M.

Definition 5.6. The conjunction of Δ-consistent k-forms ω_1 and ω_2 is the k-form defined by

$$\omega_1 \sqcup \omega_2 : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \mathcal{F}_M,$$

such that

$$(\omega_1 \sqcup \omega_2)(X_1 \sqcup Y_1, \ldots, X_k \sqcup Y_k) := \omega_1(X_1, \ldots, X_k) \sqcup \omega_2(Y_1, \ldots, Y_k) \qquad (8)$$

where $X_1 \sqcup Y_1, \ldots, X_k \sqcup Y_k \in \mathfrak{X}(M)$.

Proposition 5.4. *If ω_1 and ω_2 are Δ-consistent k-forms on $M = M_1 \cup_h M_2$ then $d\omega_1$ and $d\omega_2$ are Δ-consistent $(k+1)$-forms on M.*

Proof. Clearly one has $\iota_{1\Delta}^* d\omega_1 = \iota_{2\Delta}^* d\omega_2$ which follows from $\iota_{1\Delta}^*\omega_1 = \iota_{2\Delta}^*\omega_2$. ∎

DEFINITION 5.7. The $(k+1)$-form $d\omega = d\omega_1 \sqcup d\omega_2$ obtained from $\omega = \omega_1 \sqcup \omega_2$ is called the exterior derivative of the k-form ω.

THEOREM 5.1. *Let (M_1, ω_1) and (M_2, ω_2) be symplectic pseudomanifolds. Then $\omega_1 \sqcup \omega_2$ is a symplectic form on $M = M_1 \cup_h M_2$ and $\omega_1 \sqcup \omega_2$ assigns to any tangent vector field $X = X_1 \sqcup X_2$ a unique 1-form $\alpha = \alpha_1 \sqcup \alpha_2$ on M, where α_1, α_2 are Δ-consistent 1-forms on (M_1, ω_1) and (M, ω_2) respectively, corresponding to X_1 and X_2.*

Proof. Observe that $\omega_1 \sqcup \omega_2$ is symplectic by construction. We need to show that $\omega_1 \sqcup \omega_2$ maps $X_1 \sqcup X_2$ into $\alpha_1 \sqcup \alpha_2$, where $X_i \in \mathfrak{X}(M_i)$, $\alpha_i \in \Omega^1(M_i)$, $i = 1, 2$. In fact, in the usual way we know that

$$\omega : X \mapsto i_X \omega = \omega(X, \cdot) = \alpha$$

uniquely yields a 1-form since ω is nondegenerate, and

$$\omega(X, \cdot) = \omega_1 \sqcup \omega_2(X_1 \sqcup X_2, \cdot)$$

by definition. It follows that

$$\omega_1(X_1, \cdot) \sqcup \omega_2(X_2, \cdot) = \alpha_1 \sqcup \alpha_2. \tag{9}$$

It remains to show the correctness of the definition of $\alpha_1 \sqcup \alpha_2$ with respect to ω_1 and ω_2. That is, we show that α_1 and α_2, images of ω_1 and ω_2, are Δ-consistent. It is easy to see that

$$(\iota_{1\Delta}^* \alpha_1)(v) = (\iota_{2\Delta}^* \alpha_2)(v),$$

where $v \in T_pM$, $p \in M$ and $u \in T_p\Delta$ satisfies $(\iota_{1\Delta})_{*p} u = X_1(p)$ and according to Lemma 5.1 above, $(\iota_{2\Delta})_{*p} u = X_2(p)$. ∎

COROLLARY 5.2. *The correspondance $X_H \mapsto dH$ is not bijective.*

6. Application: Gluing at a point and related mechanics

6.1. The spaces are symplectic. Let (M_1, ω_1) and (M_2, ω_2) be symplectic pseudomanifolds of constant dimension. Let us consider that these spaces are glued at a point so that $M = M_1 \cup_h M_2$ and $\Delta = \{p\}$, $p \in M$. The $\mathcal{F}_M$-module $\mathfrak{X}_\Delta(M)$ is isomorphic with the $\mathbb{R}$-module $\{\alpha \in \Omega^1(M) : \alpha(p) = 0\}$ vanishing at p. It is easy to see that $\mathfrak{X}_\Delta(M) = \{X \in \mathfrak{X}(M) : X(p) = 0\}$. Then for any $X \in \mathfrak{X}_\Delta(M)$, the 1-form $\alpha = i_X \omega$ satisfies the condition

$$\alpha_p = 0.$$

Therefore, $\mathfrak{X}_\Delta(M) = \{(X_1, X_2) \in \mathfrak{X}(M_1) \times \mathfrak{X}(M_2) : X_1(p) = 0, X_2(p) = 0\}$ and clearly ω_1 and ω_2 are consistent since

$$\iota_{1\Delta}^* \omega_1 = 0, \quad \iota_{2\Delta}^* \omega_2 = 0.$$

Hence, we obtain the glued symplectic space $(M_1 \cup_h M_2, \omega_1 \sqcup \omega_2)$. A vector field $X_1 \sqcup X_2 \in (X)_\Delta(M_{1,2})$ corresponds to a 1-form $\alpha_1 \sqcup \alpha_2$, where $\alpha_1 = i_{X_1}\omega_1$, $\alpha_2 = i_{X_2}\omega_2$. Clearly $\alpha_i(p) = 0$, $i = 1, 2$.

EXAMPLE: Hamiltonian systems glued at a point

Consider the canonical pseudomanifold $(\mathbb{R}^{12}, \mathcal{C}, \mathcal{F})$ and two pseudomanifolds

$$M_1 = \{(q^1, q^2, q^3, p_1, p_2, p_3, 0, 0, 0, 0, 0, 0); q^i, p_i \in \mathbb{R},\ i = 1, 2, 3\} \subseteq \mathbb{R}^{12}$$
$$M_2 = \{(0, 0, 0, 0, 0, 0, \bar{q}^1, \bar{q}^2, \bar{q}^3, \bar{p}_1, \bar{p}_2, \bar{p}_3); \bar{q}^i, \bar{p}_i \in \mathbb{R},\ i = 1, 2, 3\} \subseteq \mathbb{R}^{12}.$$

Then M_1 and M_2 are obviously Frölicher subspaces of $\mathbb{R}^{12}$ which are considered as configuration spaces for two mechanical systems with 6 degrees of freedom. The configuration coordinates (q^1, q^2, q^3) or $(\bar{q}^1, \bar{q}^2, \bar{q}^3)$ and the momenta (p_1, p_2, p_3) or $(\bar{p}_1, \bar{p}_2, \bar{p}_3)$ determine together the instantaneous states. Then $\mathbb{R}^{12}$ can be considered as the phase space of the system.

In the Hamiltonian formulation the equations of the motion for such a classical system are written in terms of first order differential equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}; \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} \quad \text{for } i = 1, 2, 3. \tag{10}$$

A Hamiltonian function $H(q, p)$ defining the system in case of absence of constraining forces and time dependence is the total energy of the system, that is, the kinetic plus the potential energies. According to some observation made by Eledrisi [5] on structured spaces, we note that on a pseudomanifold, the set of singular points lying in the transversal intersection is

$\Delta = M_1 \cap M_2 = \{0\}$. That is, $p_0 \in \Delta$ if and only if $p_0 = (0, ..., 0) \in \mathbb{R}^{12}$.

Assume that (M_1, ω_1) and (M_2, ω_2) are symplectic with symplectic Δ-consistent forms given by

$$\omega_1 = \sum_{i=1}^{3} dq^i \wedge dp_i, \quad \omega_2 = \sum_{i=1}^{3} d\bar{q}^i \wedge d\bar{p}_i.$$

Consider two potential functions $V_1 : M_1 \to \mathbb{R}$, $V_2 : M_2 \to \mathbb{R}$ such that $V_1(p_0) = V_2(p_0)$ and two Hamiltonian functions given by

$$H_1 = \frac{1}{2}\sum_{i=1}^{3} \frac{p_i^2}{m} + V_1,$$
$$H_2 = \frac{1}{2}\sum_{i=1}^{3} \frac{\bar{p}_i^2}{m} + V_2$$

where m designates the mass of material points. We need to calculate the

Hamiltonian vector field in each case. Then we shall obtain the corresponding integral curves in M_1 and in M_2.

We obtain

$$X_{H_1} = \sum_{i=1}^{3} \frac{p_i}{m}\frac{\partial}{\partial q^i} - \frac{\partial V_1}{\partial q^i}\frac{\partial}{\partial p_i}, \tag{11}$$

$$X_{H_2} = \sum_{i=1}^{3} \frac{\bar{p}_i}{m}\frac{\partial}{\partial \bar{q}^i} - \frac{\partial V_1}{\partial \bar{q}^i}\frac{\partial}{\partial \bar{p}_i}. \tag{12}$$

The integral curves for X_{H_1} satisfy the (Hamilton-Jacobi) equations

$$\frac{dq^i}{dt} = \frac{p_i}{m}, \qquad \frac{dp_i}{dt} = -\frac{\partial V_1}{\partial q^i}. \tag{13}$$

Hence, assuming without loss of generality that $\frac{\partial V_j}{\partial q^i}$ =constant we have

$$\gamma(t) = \left(\sum_{i=1}^{3} \frac{1}{m} p_i t, -\sum_{i=1}^{3} \frac{\partial V_1}{\partial q^i} t\right).$$

Similarly for X_{H_2} one has

$$\bar{\gamma}(t) = \left(\sum_{i=1}^{3} \frac{1}{m} \bar{p}_i t, -\sum_{i=1}^{3} \frac{\partial V_2}{\partial \bar{q}^i} t\right),$$

which we can glue for the mechanics on $M = M_1 \sqcup M_2$.

6.2. General case. Let (M_1, g_1) and (M_2, g_2) be pseudo-Riemannian open manifolds which have a transversal intersection at a point p. That is,

$$M_1 \sqcup M_2 = M, \qquad M_1 \cap M_2 = \{p\} := \Delta.$$

Let $0_p \in T_pM$ be the zero tangent vector on M at $p \in M$, and $T_{0_p}(TM_i)$ the tangent space to TM_i at 0_p. Then

$$TM_1 \cap TM_2 = \{0_p\} \quad \text{and} \quad T_{0_p}(TM_1) \cap T_{0_p}(TM_2) = \{0_{0_p}\},$$

where we denoted by 0_{0_p} the tangent vector to TM_i at the vector 0_p. Now,

$\tilde{T}(M) := TM_1 \sqcup TM_2$, $\tilde{T}^*M := T^*M_1 \sqcup T^*M_2$, with $T^*M_1 \cap T^*M_2 = \{0_p^*\}$.

The projections of the glued bundles are

$$\pi := \pi_1 \sqcup \pi_2 : \tilde{T}M \to M, \text{where } \pi_i : TM_i \to M_i,$$

$$\tau := \tau_1 \sqcup \tau_2 : \tilde{T}^*M \to M, \text{ where } \tau_i : T^*M_i \to M_i,$$

for $i = 1, 2$. Let us recall that a function f is smooth on M if $f|M_i \in \mathcal{F}_{M_i}$. Therefore, $g_1 \sqcup g_2$ is a smooth function on M and it is clear that $g_1 \sqcup g_2$ is a Riemannian metric on M.

Moreover, the Riemannian metrics g_i are consistent on Δ since

$$\iota_{i\Delta}^* g_i = 0, \qquad i = 1, 2.$$

Next, consider a Lagrangian function on the glued space.

$$L : (M_1 \cup_h M_2, g_1 \sqcup g_2) \to \mathbb{R}, \qquad L(v) = \frac{1}{2} g(v, v),\ v \in \tilde{T}M.$$

That is,

$$L := L_1 \sqcup L_2, \quad \text{with} \quad L_i : TM_i \to \mathbb{R} \quad (i = 1, 2)$$

the Lagrangians on TM_i given by

$$L_i(v) = \frac{1}{2} g_i(v, v), \quad v \in TM_i, \quad L_1(0_p) = L_2(0_p).$$

It is clear that if L_1 and L_2 are hyperregular, then one obtains a glued Legendre transformation $\mathcal{L} : \tilde{T}M \to \tilde{T}^*M$, with

$$\mathcal{L} := \mathcal{L}_1 \sqcup \mathcal{L}_2, \qquad \mathcal{L}_i : TM_i \to T^*M_i$$

mapping tangent vectors in the glued tangent bundle onto tangent covectors in the glued cotangent bundle. Since $\mathcal{L}_i$ are diffeomorphisms by assumption, so is $\mathcal{L}$. Of course,

$$\mathcal{L}_i(0_p) = 0 \quad \text{and} \quad \mathcal{L}_i(v)(w) = g_i(v, w),$$

where $i = 1, 2$ and $v, w \in TM_i$. Let us consider the canonical 1-forms $\theta_i : T_\alpha TM_i \to \mathbb{R}$, $\alpha \in T^*M_i$, i.e. $\theta_i \in \Omega^1(T^*M_i)$, and $\theta_i(0_{0_p}) = 0$ by linearity. Now, set

$$\theta = \theta_1 \sqcup \theta_2 \quad \text{and} \quad \omega = \omega_1 \sqcup \omega_2,$$

where $\omega_i = d\theta_i \in \Omega^2(M_i)$ are canonical symplectic 2-forms on T^*M_i. Again,

$$\iota_{i\Delta}^* \omega_i = 0,$$

that means that the forms collapse at the gluing point. It follows that

$$\omega_L := \mathcal{L}_1^* \omega_1 \sqcup \mathcal{L}_2^* \omega_2 = \omega_{1L_1} \sqcup \omega_{2L_2}.$$

The above ingredients are enough for us to write a mechanical system on a glued space. If $E : \tilde{T}M \to \mathbb{R}$ denotes the energy function with the associated vector field X_E such that $i_{X_E}\omega_L = dE$, then the geodesics on the glued space are base integral curves of the vector fields X_E constructed on the glued tangent bundle. It can be observed that some geodesics are piecewise smooth curves with bifurcation.

Now we state

PROPOSITION 6.1. *Let (M, g) be the glued Riemannian space, where $M = M_1 \cup_h M_2$ and M_i are smooth manifolds whose transversal intersection con-*

sists of a singleton $\{p\}$, and $g = g_1 \sqcup g_2$ the glued metric. Then the Lagrangian $L : \tilde{T}M \to \mathbb{R}$ given by

$$L(v) = \frac{1}{2}g(v,v), \qquad v \in \tilde{T}M$$

is hyperregular and the associated Legendre transformation satisfies

$$\mathcal{L}(v)(w) = g(v,w), \qquad v,w \in \tilde{T}M.$$

The energy function is

$$E = L = \frac{1}{2}g(v,v), \qquad v \in \tilde{T}M$$

and the associated Hamiltonian is given by

$$H(\alpha) = \frac{1}{2}\langle \mathcal{L}^{-1}(\alpha), \mathcal{L}^{-1}(\alpha)\rangle, \qquad \alpha \in \tilde{T}^*M.$$

Proof. Define a function $\Phi_{v,w} : \mathbb{R} \to \mathbb{R}$ by setting

$$\Phi_{v,w}(t) = L(v + tw), \qquad v,w \in \tilde{T}M.$$

From the given Lagrangian we have

$$\begin{aligned}\Phi_{v,w}(t) &= \frac{1}{2}g(v+tw, v+tw)\\ &= \frac{1}{2}g(v,v) + tg(v,w) + \frac{1}{2}t^2 g(w,w).\end{aligned}$$

Hence

$$\frac{d}{dt}\Phi_{v,w}(t)|_{t=0} = g(v,w) = \mathcal{L}(v)(w). \tag{14}$$

Now we need to show that $\mathcal{L}$ is a diffeomorphism. Since it is bijective according to the properties of g and smooth by definition, we only show that $\mathcal{L}^{-1}$ is smooth. Note that the canonical smooth structure on $\tilde{T}M$ is generated by the set $\{\alpha \circ \pi\} \cup \{d\alpha\}$ and on $\tilde{T}^*M$ by the set $\{\alpha \circ \tau\} \cup \{d\alpha\}$, where $\alpha \in \mathcal{F}_M$, π and τ are bundle projections respectively. So it is enough to show that the composition of $\mathcal{L}^{-1}$ with the generators in $\tilde{T}^*(M)$ is smooth.

1. We have

   $$(\alpha \circ \pi) \circ \mathcal{L}^{-1} = \alpha \circ (\pi \circ \mathcal{L}^{-1}) = \alpha \circ \tau.$$

   This is a smooth function on $\tilde{T}^*M$ as a composite of smooth functions.
2. Next, we show that $d\alpha \circ \mathcal{L}^{-1}$ is smooth. Let $p \in M$ and U be an open neighborhood of $p \in M$. Let $\{W_1, \ldots, W_n\}$ be a local basis over U. That is, there exist $W_1^*, \ldots, W_n^*$ smooth functions such that $W_i^*(p)(W_j(p)) = \delta_{ij}$, where $i,j = 1, \ldots, n$ according to Lemma 3.1.

Consider the diffeomorphism $\psi : U \times \mathbb{R}^n \to T^*U$ given by

$$\psi(p, r_1, \ldots, r_n) = \sum_{i=1}^{n} r_i W_i^*(p),$$

where $p \in U$, $r \in \mathbb{R}^n$. Then from the equivalent definition of a smooth map (see Definition 2.1), $d\alpha \circ \mathcal{L}^{-1}$ is smooth if and only if $d\alpha \circ \mathcal{L}^{-1} \circ \psi$ is smooth. According to the properties of the 2-form g which is a nondegenerate function, there exist unique vector fields $A_1, \ldots, A_n \in \mathfrak{X}(U)$ such that

$$W_i^*(X) = g(A_i, X) \tag{15}$$

where $X \in \mathfrak{X}(U)$, $i = 1, \ldots, n$. Note that $\mathcal{L}(u) = i_u(g) = g(u, \cdot)$. Then using Equation 14 to solve Equation (15) gives the identity

$$\mathcal{L}^{-1}(W_i^*(p)) = A_i(p),$$

where $p \in U$. It follows that for all $(p, r) \in U \times \mathbb{R}^n$ Equation 1

$$\begin{aligned} d\alpha(\mathcal{L}^{-1} \circ \psi)(p, r) &= \mathcal{L}^{-1} \circ \psi(p, r)(\alpha) \\ &= \mathcal{L}^{-1}\Big(\sum_{i=1}^{n} r_i W_i^*(p)\Big)(\alpha) \\ &= \sum_{i=1}^{n} r_i \mathcal{L}^{-1}(W_i^*(p))(\alpha) \\ &= \sum_{i=1}^{n} r_i A_i(p)(\alpha). \end{aligned}$$

The latter is a finite sum of smooth maps $A_i(p)$ as tangent vectors at p, for each $i = 1, \ldots, n$ and α is a structure function on U. Thus, $d\alpha \circ \mathcal{L}^{-1} \circ \psi$ is smooth. But $d\alpha \in \mathcal{F}_{TU}$ is a generating function, therefore $\mathcal{L}^{-1}$ is a smooth map.

Next, note that from $H = E \circ \mathcal{L}^{-1}$, and $E(v) = \langle \mathcal{L}(v), v \rangle - L(v)$, we have

$$E(v) = g(v, v) - \frac{1}{2} g(v, v) = \frac{1}{2} g(v, v) = L.$$

Thus

$$H(\alpha) = (E \circ \mathcal{L}^{-1})(\alpha) = E(\mathcal{L}^{-1}(\alpha)) = \frac{1}{2} g(\mathcal{L}^{-1}(\alpha), \mathcal{L}^{-1}(\alpha)). \blacksquare$$

Now we may consider a mechanical system with potential on the glued space.

PROPOSITION 6.2. *In the conditions just described in the proposition above, let* $V : M \to \mathbb{R}$ *be a smooth function and* $L : \tilde{T}M \to \mathbb{R}$ *be the Lagrangian* *defined by*

$$L(v) = \frac{1}{2}g(v,v) - (V \circ \pi)(v), \tag{16}$$

where $v \in \tilde{T}M$. *Then the Legendre transformation is the symplectomorphism*

$$\mathcal{L}(v)(w) = g(v,w)$$

and the energy function $E : \tilde{T}M \to \mathbb{R}$ *is given by*

$$E(v) = \frac{1}{2}g(v,v) + (V \circ \pi)(v). \tag{17}$$

The associated Hamiltonian $H : \tilde{T}^*M \to \mathbb{R}$ *is given by*

$$H = L \circ \mathcal{L}^{-1} + 2V \circ \tau, \tag{18}$$

where $\tau : \tilde{T}^*M \to M$ *is the canonical projection.*

Proof. A similar construction leads to $\mathcal{L}(v)(w) = g(v,w)$, turning $\mathcal{L}$ into a diffeomorphism. Therefore, L is hyperregular. Thus,

$$E(v) = \frac{1}{2}g(v,v) + (V \circ \pi)(v).$$

Notice that $\omega_L = \mathcal{L}^*\omega_0$ is closed on $\tilde{T}M$ since

$$\mathcal{L}^*\omega_0 = \mathcal{L}^*(d\theta_0) = d(\mathcal{L}^*(\theta_0))$$

is exact, where θ_0 and ω_0 are the canonical 1-form and 2-form on $\tilde{T}^*M$ respectively. Also it is a nondegenerate 2-form since $\mathcal{L}$ is a diffeomorphism. One concludes that ω_L is a symplectic form. Therefore, $\mathcal{L}$ is a symplectomorphism.

Next, consider

$$E(v) = \frac{1}{2}g(v,v) + (V \circ \pi)(v),$$

and let $v \in \mathcal{L}^{-1}(\alpha)$, for $\alpha \in \tilde{T}^*M$. Then

$$\begin{aligned} H(\alpha) &= E(\mathcal{L}^{-1}(\alpha)) = \tfrac{1}{2}g(\mathcal{L}^{-1}(\alpha), \mathcal{L}^{-1}(\alpha)) + (V \circ \pi)(\mathcal{L}^{-1}(\alpha)) \\ &= \tfrac{1}{2}g(v,v) + (V \circ \pi)(v) = \tfrac{1}{2}g(v,v) - L(v)) + L(v) + (V \circ \pi)(v) \\ &= L(v) + 2(V \circ \pi)(v) = L \circ \mathcal{L}^{-1}(\alpha) + 2(V \circ \pi)(\mathcal{L}^{-1}(\alpha)) \\ &= L \circ \mathcal{L}^{-1}(\alpha) + 2(V \circ \tau)(\alpha) \ \blacksquare \end{aligned}$$

References

[1] R. Abraham, G. Marsden, T. Ratiu, *Manifolds, Tensor Analysis, and Applications*, 2nd ed., Appl. Math. Sci. Springer, New York 1988.

[2] A. Batubenge, *Symplectic Frölicher Spaces of Constant Dimension*, Univ. Cape Town, 2005 (unpublished).
[3] A. Cap, *K-Theory for Convenient Algebra*, manuscript, 1993.
[4] P. Cherenack, *Frölicher Versus Differential Spaces: A prelude to Cosmology*, Kluwer Academic Publishers (2000), 391–413.
[5] Y. I. Eledrisi, *Geometrical Properties of Gluing of Structured Spaces*, Doctoral thesis, Warsaw University of Technology, Warsaw 1998.
[6] A. Frölicher, A. Kriegl, *Linear Spaces and Differentiation Theory*, J. Wiley and Sons, New York 1988.
[7] A. Kriegl, P. Michor, *Convenient settings of global analysis*, Amer. Math. Soc., 1997.
[8] P. Ntumba, A. Batubenge, *On the way to Frölicher-Lie Groups*, Quaestiones Math. 28 (2005), 73–74.
[9] W. Sasin, P. Multarzynski, *Algebraic characterization of the dimension of differential spaces*, Proceedings of the Winter School on Geometry and Physics **2**(1989), no. 22, 193–199.
[10] W. Sasin, *De Rham cohomology of differential spaces*, Demonstratio Math. 22 (1989), 249–270.
[11] W. Sasin, *Geometrical properties of gluing of differential spaces*, Demonstratio Math. 24 (1991), 635–656.
[12] W. Sasin, *On equivalence relation on a differential space*, Praha, Czechoslovakia CMUC 29 3 (1998), 529–539.
[13] W. Sasin, *Gluing of differential spaces*, Demonstratio Math. 25 (1992), 361–384.
[14] W. Sasin, P. Multarzynski, *The Legendre transformation in differential spaces*, Supplemento ai Rendiconti del Circolo Matematico di Palermo, Serie II, 30 (1993), 35–46.
[15] W. Sasin, K. Spallek, *Gluing of differentiable spaces and applications*, Math. Ann. 292 (1992), 85–102.
[16] C. Tore, *A Tangent Bundle on Diffeological Spaces*, manuscript, Escuela de Matematica, Universidad de Costa Rica, San Jose 1998.

A. Batubenge
SCHOOL OF MATHEMATICS
WITS UNIVERSITY, JOHANNESBURG
PRIVATE BAG 3, WITS 2050 SOUTH AFRICA
e-mail: Tshidibi.Batubenge@wits.ac.za

W. Sasin
FACULTY OF MATHEMATICS AND INFORMATION SCIENCE
WARSAW UNIVERSITY OF TECHNOLOGY
Pl. Politechniki 1, 00-661 WARSAW, POLAND
e-mail: wsasin@mini.pw.edu.pl

Received March 20, 2008.